SOME ASPECTS IN THE USE OF THE KARHUNEN-LOÈVE DECOMPOSITION FOR ANALYZING VIBRATING SYSTEMS

Rubens Sampaio
Department of Mechanical Engineering, Pontifícia Universidade Católica do Rio de Janeiro
rsampaio@mec.puc-rio.br

Sergio Bellizzi
CNRS-Laboratoire de Mécanique et Acoustique de Marseille
bellizzi@lma.cnrs-mrs.fr

Abstract. *The Karhunen-Loève, KL, decomposition establishes that a second order random process can be expanded as a series involving a sequence of deterministic orthogonal functions with orthogonal random coefficients. The KL decomposition, that originally appeared in the mathematic litterature, has been used in Mechanics as a tool to obtain reduced models and is a strong candidate to extend modal analysis to non-linear systems . The KL method can be viewed as a statistical procedure. One initially supposes that the observed system dynamics can be modelled as a second-order ergodic stochastic process. The method consists then in constructing a spatial autocorrelation tensor from data obtained through numerical or physical experiments and performing its spectral decomposition. The autocorrelation tensor is by definition Hermitian and positive semi-definite. Therefore, its decomposition provides a set of orthogonal eigenfunctions (called proper orthogonal modes, POMs). These POMs can then be used as a basis for the dynamics projection and in the construction of a reduced-order model by truncation. The purpose of this paper is to analyse the efficiency of the method in the case of transient response. A mean operator involving time and ensemble average is used to construct the autocorrelation tensor. The associated POMs are compared with the POMs given by the classical approach applied for one time history.*

Keywords: *Karhunen-Loève decomposition, modal analysis, transient problems*

1. Introduction

The Karhunen-Loève (KL) expansion or decomposition initially appeared in the signal processing literature, where it is customary called Principal Components Analysis (PCA). It is a powerful statistical tool to perform data analysis and compression. It soon found many useful applications such as voice and image recognition. In Mechanical Engineering, where it is also known as the Proper Orthogonal Decomposition (POD), it was firstly employed to uncover coherent structures in turbulent flow fields [8]. Coherent structures can be defined as recurrent spatial forms that are energy dominant [6]. Since then, it has been consistently developed and applied to many fluid dynamics problems [11, 12].

Interestingly enough, only recently has the KL expansion attracted the attention of structural dynamicists seeking an alternative approach to obtaining reduced-order models of linear and nonlinear dynamical systems [7, 9, 13]. The objective of reducing a mathematical model is to obtain a simpler one that not only still has a good degree of predictive capability but also is suitable for its intended application. This can be, for instance, the design of a real-time feedback controller that is not too computationally intensive, or simply the performance of a parameter analysis.

The KL method is primarily a statistical procedure. One initially supposes that the observed system dynamics can be modelled as a second-order ergodic stochastic process. The method consists then in constructing a spatial autocorrelation tensor from data obtained through numerical or physical experiments and performing its spectral decomposition. Since it deals only with data, there is no distinction between linear or nonlinear systems and it can even be implemented, in the physically obtained data case, without any previous knowledge about the mechanical characteristics of the system. The autocorrelation tensor is by definition Hermitian and positive semi-definite. Therefore, its decomposition provides a set of orthogonal eigenfunctions (called proper orthogonal modes, POMs, or empirical eigenmodes) and nonnegative real eigenvalues (or proper orthogonal values, POVs, or empirical eigenvalues). These POMs can then be used as a basis for the dynamics projection and in the construction of a reduced-order model through the retention of a finite number of them. An important property of the expansion is that the magnitude of a POV is a measure of the energy contained in the respective POM. Furthermore, the expansion is in a sense optimal, meaning that no other linear decomposition can better reproduce the particular dynamics which generated the POMs with the same number of modes.

The purpose of this paper is to analyse the efficiency of the method in the case of transient response. A mean operator involving time and ensemble average is used to construct the autocorrelation tensor. The associated POMs are compared with the POMs given by the classical approach applied for one time history.

2. Karhunen-Loève expansion

Let $\mathcal{D}$ be a compact subset of $\mathbb{R}^l$ and $\{X(z)\}_{z\in\mathcal{D}}$ a stochastic field defined on a probability space $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}^n$. This random field is a l-parameter family on real valued vector, $X(z,\omega)$ for $(z,\omega) \in \mathcal{D}\times\Omega$.

Assumption I: We will assume that $\{X(z)\}_{z\in\mathcal{D}}$ is a second-order random field i.e.

$$E(\|X(z)\|^2_{<>}) = E(< X(z), X(z) >) < \infty, \forall z \in \mathcal{D}$$

where $E(.)$ denotes the mean or ensemble average and $\|.\|_{<>}$ the norm associated to the Euclidian inner product $< ., . >$ in $\mathbb{R}^n$.

Assumption **I** implies that $\forall z \in \mathcal{D}$, $X(z) \in L^2(\Omega, \mathbb{R}^n)$ where $L^2(\Omega,\mathbb{R}^n)$ denotes Hilbert space of the second order random variables with the inner product defined by $\prec Y, Z \succ = E(< Y, Z >)$ and the associated norm denoted by $\|.\|_{\prec\succ}$.

Assumption II: We will also assume that $\{X(z)\}_{z\in\mathcal{D}}$ is continuous in quadratic mean i.e.

$$\|X(z+h) - X(z)\|^2_{\prec\succ} \to 0 \text{ as } h \to 0.$$

Karhunen-Loève expansion: Under assumptions **I, II**, the auto-covariance function $C_X(z_1, z_2)$ of the field $\{X(z)\}_{z\in\mathcal{D}}$ defines a continuous self-adjoint Hilbert-Schmidt operator Q on the Hilbert space $L^2(\mathcal{D},\mathbb{R}^n)$:

$$(Q\psi)(z) = \int_{\mathcal{D}} C_X(z, z')\psi(z')dz', \text{ for } \psi \in L^2(\mathcal{D},\mathbb{R}^n).$$

This operator has a countable number of eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \cdots$, the associated eigenfunctions are the solutions of the integral equation $(Q\psi_n)(z) = \lambda_n\psi_n(z)$ and constitute an orthonormal basis $\{\psi_n\}_{n\geq 1}$ of $L^2(\mathcal{D},\mathbb{R}^n)$ i.e.

$$\ll \psi_n, \psi_m \gg = \int_{\mathcal{D}} < \psi_n(z), \psi_m(z) > dz = \delta_{n,m}$$

where $\ll ., . \gg$ denotes the inner product in $L^2(\mathcal{D},\mathbb{R}^n)$ with the associated norm $\|.\|_{\ll\gg}$.

The random field $\{X(z)\}_{z\in\mathcal{D}}$ can be expanded in terms of the eigenfunctions, ψ_n, as

$$\forall z \in \mathcal{D}, X(z) - m_X(z) = \sum_{n=1}^{\infty} \xi_n\psi_n(z) \tag{1}$$

where the equality is achieved in $L^2(\Omega,\mathbb{R}^n)$, $m_X(z)$ denotes the mean function of the random field and $\xi_1, \xi_2, \cdots, \xi_n, \cdots$ are scalar uncorrelated random variables given by

$$\xi_n = \int_{\mathcal{D}} < X(z) - m_X(z), \psi_m(z) > dz \text{ with } E(\xi_n^2) = \lambda_n.$$

Finally the eigenvalues, λ_n, are related to the mean "energy" of the random field according to the following relation

$$E(\|X - m_X\|^2_{\ll\gg}) = \sum_{n=1}^{\infty} \lambda_n.$$

In the literature, expansion (1) is called KL expansion of the random field $\{X(z)\}_{z\in\mathcal{D}}$ and the set $\{\psi_n\}_{n\geq 1}$ is referred as the coherent structure.

Optimallity property: The KL expansion satisfies the following optimallity property: For any arbitrary orthogonal basis, $(\tilde{\psi}_n)$, of $L^2(\mathcal{D},\mathbb{R}^n)$ and for every positive integer p,

$$E(\|X(z) - m_X(z) - \sum_{n=1}^{p} \xi_n\psi_n\|^2) \leq E(\|X(z) - m_X(z) - \sum_{n=1}^{p} \tilde{\xi}_n\tilde{\psi}_n\|^2) \tag{2}$$

where $\tilde{\xi}_1, \tilde{\xi}_2, \cdots, \tilde{\xi}_n, \cdots$ are scalar random variables given by

$$\tilde{\xi}_n = \int_{\mathcal{D}} < X(z) - m_X(z), \tilde{\psi}_n(z) > dz.$$

3. Application to random vibrations

In random vibrations of continuous structures, the displacement field is assumed to be a stochastic field $\{u(z)\}_{z\in\mathcal{D}}$ where the domain $\mathcal{D} = \mathcal{D}_t \times \mathcal{D}_\mathbf{x} \subset \mathbb{R}\times\mathbb{R}^p$ (with $p = 1, 2,$ or 3). Usually, $\mathcal{D}_t$ defines the time interval of interest and without lost of generality, we assume in the sequel that $\mathcal{D}_t = [0, T]$ where $T \in \mathbb{R}+$. In dynamics problems, the objective is to expand the displacement field as a series in separate variables

$$u(t,\mathbf{x}) = \sum_{k=1}^{\infty} a_k(t)\phi_k(\mathbf{x}) \tag{3}$$

where ϕ_k are deterministic $\mathbb{R}^n$-valued functions, and $\{a_k(t)\}_{t\in\mathcal{D}_t}$ are scalar random processes.

The KL expansion given by the classical KL theory (as described in the previous section) can not give straight answer to this question (series (1) is not in separate variables). Another way to use the KL theory is to apply it, for fixed $t \in \mathcal{D}_t$, to the random field $\{u(t,\mathbf{x})\}_{\mathbf{x}\in\mathcal{D}_\mathbf{x}}$. That gives the following series (in $\mathcal{L}^2(\Omega,\mathbb{R}^n)$)

$$u(t,\mathbf{x}) - m_u(t,\mathbf{x}) = \sum_{k=1}^{\infty}\xi_k(t)\psi_k(t,\mathbf{x}) \ \text{ where } \psi_k \text{ solve } \int_{\mathcal{D}_\mathbf{x}} C_u(t,t,\mathbf{x},\mathbf{x}')\psi_k(t,\mathbf{x}')d\mathbf{x}' = \lambda_k\psi_k(t,\mathbf{x}). \tag{4}$$

Again the ψ_k depend on the time variable and hence expansion (4) is not in the variable separated form. Note that if the covariance function $C_u(t,t,\mathbf{x},\mathbf{x}')$ does not depend on t then the functions ψ_k also do not depend on t and expansion (4) is now a series in separate variables and it is optimal to represent the random field $\{u(t,\mathbf{x})\}_{\mathbf{x}\in\mathcal{D}_\mathbf{x}}$ for fixed $t \in \mathcal{D}_t$ in the sense that the error term

$$E(\|u(t,.) - m_u(t,.) - \sum_{k=1}^{p}\xi_k(t)\psi_k(.)\|^2_{\ll\gg}) \tag{5}$$

is minimum for each integer p. This situation corresponds to the time stationary case.

We focus now on a new approach based on the KL theory to expand the random field as a series in separate variables (3). This approach differs altogether from the approaches presented above.

First, we consider the Hilbert space $\mathcal{L}^2_T(\mathcal{D}_t\times\Omega,\mathbb{R}^n)$ with the inner product given by $\prec Y, Z \succ = \mathcal{E}(< Y, Z >)$ with $\mathcal{E}(.) = \frac{1}{T}\int_0^T E(.)dt$. We can next define the mean and covariance functions of the random field $\{u(.,\mathbf{x})\}_{x\in\mathcal{D}_x}$ as $\mathbf{m}_u(\mathbf{x}) = \mathcal{E}(u(.,\mathbf{x}))$, and $\mathcal{C}_u(\mathbf{x},\mathbf{x}') = \mathcal{E}((u(.,\mathbf{x}) - \mathbf{m}_u(\mathbf{x})) \otimes (u(.,\mathbf{x}') - \mathbf{m}_u(\mathbf{x}')))$ which are only spatial variable dependant. Finally, assuming that assumptions I and II, the random field $\{u(.,\mathbf{x})\}_{\mathbf{x}\in\mathcal{D}x}$ can be expanded (see section 1) as (the equality is achieved in $\mathcal{L}^2_T(\mathcal{D}_t\times\Omega,\mathbb{R}^n)$)

$$\forall x \in \mathcal{D}_x, u(t,\mathbf{x}) - \mathbf{m}_u(\mathbf{x}) = \sum_{k=1}^{\infty}\xi_k(t)\psi_k(\mathbf{x}) \ \text{ where } \psi_k \text{ solve } \int_{\mathcal{D}_\mathbf{x}} \mathcal{C}_u(\mathbf{x},\mathbf{x}')\psi_k(\mathbf{x}')d\mathbf{x}' = \lambda_k\psi_k(\mathbf{x}) \tag{6}$$

and $\{\xi_1(t)\}_{t\in\mathcal{D}_t}, \{\xi_2\}_{t\in\mathcal{D}_t}, \cdots, \{\xi_m\}_{t\in\mathcal{D}_t}, \cdots$ are scalar random processes given by

$$\xi_k(t) = \int_{\mathcal{D}_x} < u(t,\mathbf{x}) - m_u(\mathbf{x}), \psi_k(\mathbf{x}) > d\mathbf{x} \tag{7}$$

with the following orthogonal properties

$$\mathcal{E}(\xi_{k_1}\xi_{k_2}) = 0 \text{ if } k_1 \neq k2 \text{ and } \mathcal{E}(\xi_k^2) = \lambda_k. \tag{8}$$

As in the classical case, the eigenvalues, λ_k, are related to the mean “energy” of the random field according to the following relation

$$\mathcal{E}(\|u - \mathbf{m}_u\|^2_{\ll\gg}) = \sum_{k=1}^{\infty}\lambda_k. \tag{9}$$

The functions ψ_k will be called T-mean KL modes.

Some remarks

1- The existence of the KL expansion as described in equations (6)(7) does not require any assumption on stationarity and ergodicity properties. These properties are only required in pratical construction of POMs.
2- The KL expansion as described in equations (6)(7) usually depends on the time parameter T. So, if the random field under examination is not stationary, the influence of T has to be carefully examined.
3- If the random field $\{u(t,\mathbf{x})\}_{(t,\mathbf{x})\in\mathcal{D}_t\times\mathcal{D}_x}$ is weakly stationary with respect to the time variable (i.e. if $m_u(t,\mathbf{x}) = m_u(\mathbf{x})$ and $C_u(t,t',\mathbf{x},\mathbf{x}') = C_u(t-t',\mathbf{x},\mathbf{x}')$) then $\mathbf{m}_u(\mathbf{x}) = m_u(\mathbf{x}), \mathcal{C}_u(\mathbf{x},\mathbf{x}') = C_u(0,\mathbf{x},\mathbf{x}')$ and hence the expansion (6) coincides with the expansion (4). Moreover, the expansion does not depend on the parameter T.

4. Examples

4.1 Linear case

Consider a discrete mechanical system with n degrees of freedom. Let $\mathbf{U}(t)$ be the displacement vector. We assume that $\mathbf{U}(t)$ satisfies the equation of motion

$$\mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{C}\dot{\mathbf{U}}(t) + \mathbf{K}\mathbf{U}(t) \quad = \quad \mathbf{F}(t),\ t > 0 \tag{10}$$

$$\mathbf{U}(0) = \mathbf{U}_0,\ \dot{\mathbf{U}}(0) = \dot{\mathbf{U}}_0 \tag{11}$$

in which $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are $n \times n$ matrices, $\mathbf{F}(t)$ is a stochastic excitation, and the vectors $\mathbf{U}_0$ and $\dot{\mathbf{U}}_0$ define the initial conditions of the motion.

Under the assumption that $\mathbf{F}(t)$ is a white-noise excitation process with the covariance matrix $\mathbf{C}_F(\tau) = \mathbb{E}(\mathbf{F}(t + \tau)\mathbf{F}^T(t)) = \mathbf{S}_F\delta(\tau)$ where $\mathbf{S}_F$ is a constant symmetric matrix and that the initial conditions are deterministic (or at least uncorrelated with the excitation), the evolution of covariance matrix, $\mathbf{R}_{\mathcal{U}}(t) = \mathbb{E}(\boldsymbol{\mathcal{U}}(t)\boldsymbol{\mathcal{U}}^T(t))$ of the random vector $\boldsymbol{\mathcal{U}}(t) = (\mathbf{U}^T(t), \dot{\mathbf{U}}^T(t))^T$ satisfies (see for example [1]) the differential equation

$$\dot{\mathbf{R}}_{\mathcal{U}}(t) \quad = \quad \mathbf{A}_{\mathcal{U}}\mathbf{R}_{\mathcal{U}}(t) + \mathbf{R}_{\mathcal{U}}(t)\mathbf{A}_{\mathcal{U}}^T + \mathbf{D}_{\mathcal{U}},\ t > 0,\ \text{with } \mathbf{R}_{\mathcal{U}}(0) = \mathbf{R}_{\mathcal{U}_0} \tag{12}$$

where $\mathbf{A}_{\mathcal{U}} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{pmatrix}$, $\mathbf{D}_{\mathcal{U}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{-1}\mathbf{S}_F\mathbf{M}^{-T} \end{pmatrix}$ and $\mathbf{R}_{\mathcal{U}_0}$ is the covariance matrix of the random vector $\boldsymbol{\mathcal{U}}(0) = (\mathbf{U}_0^T, \dot{\mathbf{U}}_0^T)^T$ which is equal to the null matrix if the initial conditions are deterministic.

When A is a stability matrix (this is the case when the matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are symmetrical and positive definite), the matrix function $\mathbf{R}_{\mathcal{U}}(t)$ tends to a constant symmetrical matrix $\hat{\mathbf{R}}_{\mathcal{U}}$ which solves the following Lyapounov equation

$$\mathbf{0} \quad = \quad \mathbf{A}_{\mathcal{U}}\hat{\mathbf{R}}_{\mathcal{U}} + \hat{\mathbf{R}}_{\mathcal{U}}\mathbf{A}_{\mathcal{U}}^T + \mathbf{D}_{\mathcal{U}}. \tag{13}$$

Recalling that the covariance matrix $\mathbf{R}_U(t)$ of the displacement vector $\mathbf{U}(t)$ is equal to the first block of dimension $n \times n$ of the matrix $\mathbf{R}_{\mathcal{U}}(t)$, we can now use the equations (12) and (13) to analyse the behaviour of the KL decomposition from transient to stationary responses of the system (10)(11). The transient response is characterized by the differential equation (12) whereas the stationary response is characterized by the algebraic equation (13).

4.1.1 Relation between KL modes and Linear Normal Modes

We focus on a linear system (10)(11) with proportional damping. This discussion complements the results presented in [2] and [3]. The Linear Normal Modes (LNM) are classically defined from the free responses of the undamped system as $\mathbf{K\Phi} = \mathbf{M\Phi\Omega^2}$ where $\mathbf{\Phi} = [\mathbf{\Phi}_1 \cdots \mathbf{\Phi}_i \cdots \mathbf{\Phi}_n]$ denotes the modal matrix with the normalisation condition $\mathbf{\Phi}^T\mathbf{M\Phi} = \mathbf{I}$ which imply that $\mathbf{\Phi}^T\mathbf{K\Phi} = \mathbf{\Omega^2} = \text{diag}(\omega_i^2)$, and ω_i^2 and $\mathbf{\Phi}_i$ denotes the square resonance frequencies and the associated normal mode vectors. Note that, by assumption, the matrix $\mathbf{\Phi}^T\mathbf{C\Phi}$ is also diagonal. The objective of this section is to establish when the Karhunen-Loève decomposition (which is built from forced responses) can give access to the LNM.

Identity mass matrix case

We first consider the case $\mathbf{M} = \mathbf{I}$. Using the normal mode vectors as the representation base, the vector $\mathbf{X}(t)$ defined by

$$\mathbf{U}(t) = \mathbf{\Phi}X(t) = \sum_{i=1}^{n} \mathbf{\Phi}_i X_i(t) \tag{14}$$

satisfy the following second order differential equation

$$\ddot{\mathbf{X}}(t) + \mathbf{\Theta}\dot{\mathbf{X}}(t) + \mathbf{\Omega}^2\mathbf{X}(t) \quad = \quad \mathbf{\Phi}^T\mathbf{F}(t), \tag{15}$$

where $\mathbf{\Theta} = \mathbf{\Phi}^T\mathbf{C\Phi} = \text{diag}(2\tau_i\omega_i)$ is diagonal. The evolution of the covariance matrix, $\mathbf{R}_{\mathcal{X}}(t) = \mathbb{E}(\boldsymbol{\mathcal{X}}(t)\boldsymbol{\mathcal{X}}^T(t))$, of $\boldsymbol{\mathcal{X}}(t) = (\mathbf{X}^T(t), \dot{\mathbf{X}}^T(t))^T$ is given by

$$\dot{\mathbf{R}}_{\mathcal{X}}(t) \quad = \quad \mathbf{A}_{\mathcal{X}}\mathbf{R}_{\mathcal{X}}(t) + \mathbf{R}_{\mathcal{X}}(t)\mathbf{A}_{\mathcal{X}}^T + \mathbf{D}_{\mathcal{X}}, \tag{16}$$

$$\mathbf{R}_{\mathcal{X}}(0) \quad = \quad \mathbf{R}_{\mathcal{X}_0} \tag{17}$$

where $\mathbf{A}_{\mathcal{X}} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{\Theta} & -\mathbf{\Omega}^2 \end{pmatrix}$, $\mathbf{D}_{\mathcal{X}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Phi}^T\mathbf{S}_F\mathbf{\Phi} \end{pmatrix}$ and $\mathbf{R}_{\mathcal{X}0}$ is easily deduced from $\mathbf{R}_{\mathcal{U}_0}$. The covariance $\hat{\boldsymbol{R}}_{\mathcal{X}}$ of the stationary response is given by

$$0 \quad = \quad \mathbf{A}_{\mathcal{X}}\hat{\boldsymbol{R}}_{\mathcal{X}} + \hat{\boldsymbol{R}}_{\mathcal{X}}\mathbf{A}_{\mathcal{X}}^T + \mathbf{D}_{\mathcal{X}}. \tag{18}$$

If the matrix $\mathbf{\Phi}^T\mathbf{S}_F\mathbf{\Phi}$ is diagonal (i.e. when the modal excitation terms $\mathbf{\Phi}_i^T\mathbf{F}(t)$ are uncorrelated), it is easy to show from equation (18) that the stationary covariance matrix $\hat{\boldsymbol{R}}_{\mathcal{X}}$ and the stationary covariance matrix $\hat{\boldsymbol{R}}_X$ are also diagonal. Now, recalling the change of variables, we obtain the relation

$$\hat{\boldsymbol{R}}_U = \mathbf{\Phi}\hat{\boldsymbol{R}}_X\mathbf{\Phi}^T \tag{19}$$

from which we can deduce (recalling $\mathbf{\Phi}^T\mathbf{\Phi} = \boldsymbol{I}$) that the Karhunen-Loève decomposition of the stationary response $\mathbf{U}(t)$ coincides with the modal expansion (14).

If the matrices $\mathbf{\Phi}^T\mathbf{S}_F\mathbf{\Phi}$ and $\mathbf{R}_{\mathcal{X}_0}$ are diagonal, the same property holds for the transient response over $[0,T]$ for arbitrary T. Indeed, from equation (16) we can show that the covariance matrix function, $\mathbf{R}_{\mathcal{X}}(t)$, is 2×2-block matrix with diagonal blocks of the same size. For each t, the matrix $\boldsymbol{R}_X(t)$ is diagonal, and integrating over $[0,T]$ the relation $\boldsymbol{R}_U(t) = \mathbf{\Phi}^T\boldsymbol{R}_X(t)\mathbf{\Phi}$ gives

$$\boldsymbol{\mathcal{R}}_U = \mathbf{\Phi}\boldsymbol{\mathcal{R}}_X\mathbf{\Phi}^T. \tag{20}$$

where $\boldsymbol{\mathcal{R}}_X = \mathcal{E}(\mathbf{X}(.)\mathbf{X}^T(.)) = \frac{1}{T}\int_0^T \boldsymbol{R}_X(t)dt$ and $\boldsymbol{\mathcal{R}}_U = \mathcal{E}(\mathbf{U}(.)\mathbf{U}^T(.))$. Hence, the modal expansion (14) coincides with the Karhunen-Loève decomposition of the transient process with respect to the inner product $\mathcal{E}(.) = \frac{1}{T}\int_0^T \mathbb{E}(.)dt$ (i.e the T-mean KL modes are equal to the LNM).

General mass matrix case

When $\mathbf{M} \neq \mathbf{I}$, introducing the square root $\mathbf{M}^{\frac{1}{2}}$ of the matrix $\mathbf{M}$ (i.e. $\mathbf{M} = \mathbf{M}^{\frac{1}{2}}\mathbf{M}^{\frac{1}{2}}$) and using the change of variable $\mathbf{V} = \mathbf{M}^{\frac{1}{2}}\mathbf{U}$, the equation of motion (10) reads

$$\ddot{\mathbf{V}}(t) + \mathbf{C}_M\dot{\mathbf{V}}(t) + \mathbf{K}_M\mathbf{V}(t) = \mathbf{M}^{-\frac{1}{2}}\mathbf{F}(t). \tag{21}$$

where the new damping and stiffness matrices, $\mathbf{C}_M = \mathbf{M}^{-\frac{1}{2}}\mathbf{C}\mathbf{M}^{-\frac{1}{2}}$ and $\mathbf{K}_M = \mathbf{M}^{-\frac{1}{2}}\mathbf{K}\mathbf{M}^{-\frac{1}{2}}$, are still symmetrical. The previous results can now be applied to the new variable $\mathbf{V}$. Let $\mathbf{\Psi}_i$, for $i = 1,\cdots,n$, be the normal mode vectors with the normalisation condition $\mathbf{\Psi}^T\mathbf{\Psi} = \mathbf{I}$ (where $\mathbf{\Psi}$ denotes the modal matrix i.e $\mathbf{\Psi} = [\mathbf{\Psi}_1\cdots\mathbf{\Psi}_i\cdots\mathbf{\Psi}_n]$).

If the matrix $\mathbf{\Psi}^T\mathbf{M}^{-\frac{1}{2}}\mathbf{S}_F\mathbf{M}^{-\frac{1}{2}T}\mathbf{\Psi}$ is diagonal, then the the Karhunen-Loève decomposition of the stationary response $V(t)$ coincides with the modal expansion

$$\mathbf{V}(t) = \mathbf{\Psi}\tilde{\mathbf{X}}(t) = \sum_{i=1}^{n}\mathbf{\Psi}_i\tilde{X}_i(t). \tag{22}$$

Knowing the mass matrix $\boldsymbol{M}$, it is possible to extract the LNM from $\mathbf{\Psi}$ using the relation

$$\mathbf{\Phi} = \mathbf{M}^{-\frac{1}{2}}\mathbf{\Psi}. \tag{23}$$

When $\mathbf{M} \neq \mathbf{I}$, the Karhunen-Loève decomposition of the stationary response $\mathbf{U}(t)$ do not coincide with the modal expansion and the LNMs can only be obtained from the variable $\mathbf{V}$. Note that the condition can be written using the LNM as

$$\mathbf{\Psi}^T\mathbf{M}^{-\frac{1}{2}}\mathbf{S}_F\mathbf{M}^{-\frac{1}{2}T}\mathbf{\Psi} = \mathbf{\Phi}^T\mathbf{R}_F\mathbf{\Phi}. \tag{24}$$

The same result can be obtained for the transient response but it will not be discussed here.

We have shown that the Karhunen-Loève decomposition can give access to the LNM when the modal excitation terms $\mathbf{\Phi}_i^T\mathbf{F}(t)$ are uncorrelated. In the next section, we will discuss the influence of the correlation coefficient among modal excitation terms.

4.1.2 Influence of the correlation coefficient between modal excitation terms

We consider here a 2 DOF linear system (10)-(**??**) with proportional damping and identity mass matrix. We assume that the matrix $\mathbf{\Phi}^T\mathbf{R}_F\mathbf{\Phi}$ is not diagonal. Let

$$\mathbf{\Phi}^T\mathbf{S}_F\mathbf{\Phi} = \begin{pmatrix} \sigma_{11} & \rho\sqrt{\sigma_{11}\sigma_{22}} \\ \rho\sqrt{\sigma_{11}\sigma_{22}} & \sigma_{11} \end{pmatrix} \tag{25}$$

where σ_{11} and σ_{22} denote the modal input level and ρ the associated correlation coefficient. Solving equation (18), the stationary covariance matrix $\hat{\boldsymbol{R}}_X$ has components

$$\hat{R}_{X_{11}} = \frac{\sigma_{11}}{4\tau_1\omega_1^3},\ \hat{R}_{X_{22}} = \frac{\sigma_{22}}{4\tau_2\omega_2^3} \text{ and } \hat{R}_{X_{12}} = \rho_{X_{12}}\sqrt{\hat{R}_{X_{11}}\hat{R}_{X_{22}}} \tag{26}$$

with

$$\rho_{X_{12}} = \rho \frac{8\tau_1^2 r_\omega \sqrt{r_\tau r_\omega}(1 + r_\tau r_\omega)}{(1 - r_\omega^2)^2 + 4\tau_1^2(1 + r_\tau r_\omega)(r_\tau + r_\omega)r_\omega}, \; r_\tau = \frac{\tau_2}{\tau_1} \text{ and } r_\omega = \frac{\omega_2}{\omega_1} \tag{27}$$

where ω_i and τ_i denote the resonance frequencies and the associated damping ratios (see equation (15)) of the two modal components considered. Introducing the ratio $r_\sigma = \frac{\sigma_{22}}{\sigma_{11}}$, the stationary covariance matrix takes the form

$$\hat{\boldsymbol{R}}_X = \frac{\sigma_{11}}{4\tau_1\omega_1^3} \begin{pmatrix} 1 & \rho_{X_{12}}\sqrt{\frac{r_\sigma}{r_\tau r_\omega^3}} \\ \rho_{X_{12}}\sqrt{\frac{r_\sigma}{r_\tau r_\omega^3}} & \frac{r_\sigma}{r_\tau r_\omega^3} \end{pmatrix} \tag{28}$$

showing that its eigenvectors only depend on modal damping coefficients (τ_1, τ_2), modal frequency ratio r_ω, modal input level ratio r_σ and correlation coefficient ρ. Note that the eigenvectors do not depend on the absolute values of the modal frequencies.

Figure 1 shows the euclidian norm of the error vector between the canonical vector $e_1 = (1, 0)^T$ and the normalized eigenvector of $\hat{\boldsymbol{R}}_X$ (i.e stationary KL mode) versus the correlation coefficient ρ for $\tau = \tau_1 = \tau_2 = 0.01$ and $\tau = 0.1$. The difference between the KL decomposition and the modal decomposition increases with ρ and decreases when the modal frequency ratio increases. The dotted lines shows the error vector between the canonical vector $e_1 = (1, 0)^T$ and the normalized eigenvector of $\boldsymbol{\mathcal{R}}_X = \frac{1}{T}\int_0^T \boldsymbol{R}_X(t)dt$ (i.e T-mean KL mode) where the matrix function $\boldsymbol{R}_X(t)$ has been obtained by solving equation (16)numerically over $[0, T]$ with $T = 200$ ($\tau = 0.1$) and $T = 100$ ($\tau = 0.01$). Due to the large value of T, the results are close to the previous one.

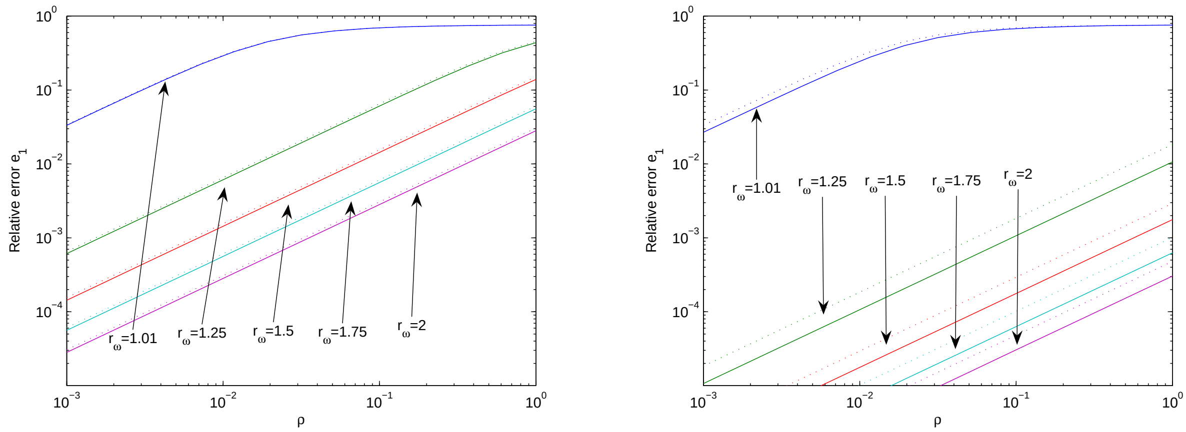

Figure 1. Euclidian norm of the error vector between the canonical vector $e_1 = (1, 0)^T$ and the stationary KL mode and the T-mean KL mode versus the correlation coefficient ρ with $\tau = 0.1$ (left) and $\tau = 0.01$ (right).

4.2 Continuous linear case

We limit the discussion to a beam. Let $\varphi_i(z)$ be the modal functions (with $\int_0^L \varphi_i(z)\varphi_j(z)dz = \delta_{ij}$ where L denotes the length of the beam). The displacements of the beam can be expanded in a truncated series

$$u(t, z) = \sum_{i=1}^{n} \varphi_i(z)x_i(t) \tag{29}$$

where the modal components can be modelled as

$$\ddot{\mathbf{X}}(t) + \boldsymbol{\Theta}\dot{\mathbf{X}}(t) + \boldsymbol{\Omega}^2\mathbf{X}(t) \quad = \quad \mathbf{G}(t) \tag{30}$$

where $\boldsymbol{\Omega}^2 = \mathrm{diag}(\omega_i^2)$ and $\boldsymbol{\Theta} = \mathrm{diag}(2\tau_i\omega_i)$. The ω_i denote the modal frequencies and τ_i the associated modal damping ratios. The component of the modal excitation vector $\mathbf{G}(t)$ can be related to the physical excitation $F(t, z)$ by

$$g_i(t) = \int_0^L \varphi_i(z)F(t, z)dz.$$

In our case, $F(t, z)$ is a random process and we will assume that $G(t)$ is a white-noise random process.

4.2.1 Relation between KL modes and Linear Normal Modes

The objective of this section is to compare the KL decomposition and the modal expansion. Using the same arguments as in section 4.1.1 it is possible to show that if the covariance matrix of the modal excitation, $\mathbf{G}(t)$, is diagonal then the KL decomposition coincides with the modal expansion (29). Unfortunately this is not generally the case and hence the KL decomposition can not give access to the true normal mode. Qualitative errors can be obtained from (26) (see figure 1).

Another source of perturbation is introduced by the spatial sampling. Let $z_k = k\Delta z$ for $k = 1, \cdots, m$ with $\Delta z = L/m$. The covariance matrix of the discrete response $\mathbf{U}(t) = (u(t, z_k))$ is related to the covariance matrix of the modal components $X(t)$ by $\mathbf{R}_U = \mathbf{\Phi}\mathbf{R}_X\mathbf{\Phi}^T$ where $\mathbf{\Phi}$ is a $m \times n$ matrix with components $\phi_{ki} = (\varphi_i(z_k))$.

If $\mathbf{\Phi}^T\mathbf{\Phi} = \boldsymbol{I}$, then $\mathbf{R}_U\mathbf{\Phi} = \mathbf{\Phi}\mathbf{R}_X$ and the eigenvectors of $\mathbf{R}_X$ are given by the eigenvectors of $\mathbf{R}_U$ multiplied by the matrix $\mathbf{\Phi}$. Note that qualitative errors given by (26) are not modified (the weight matrix $\mathbf{\Phi}^T\mathbf{\Phi} = \boldsymbol{I}$).

In practice we have never $\mathbf{\Phi}^T\mathbf{\Phi} = \boldsymbol{I}$, and hence qualitative errors given by (26) are not valid and we except that the difference between the KL-modes and the LNM increase when m decreases. Figures 2 compares the exact first, second, fifth and sixth modal functions of clamped-free beam with the associated KL modes obtained from the stationary covariance matrix (equation (18)) and from the T-mean covariance matrix (equation (16) with $\mathbf{R}_{\mathcal{X}_0} = 0$). A localized excitation force $F(x,t) = \delta(x - x_f)f(t)$ has been used where $f(t)$ is a scalar random process with covariance function $C_f(\tau) = \mathbb{E}(f(t+\tau)f^T(t)) = S_f\delta(\tau)$. The parameters values was: $L = 0.6$, $EI = 1.4$, $\rho S = 0.1620$, $n = 10$, $\tau_i = \tau = 0.01$, $x_f = 0.05$ (all modes were excited and the correlation coefficient between pairs of modal components were equal to 1), $S_f = 1$, $T = 1$ (which correspond to approximately four fundamental periods of the smaller resonance frequency). We can observe that, as expected, the KL modes differ from the LNM. The difference decreases when m increases. Due to the ratio between successive resonance frequencies, the difference also decreases when the mode increases. Finally a difference between stationary KL modes and T-mean KL modes can be observed on the first two modes.

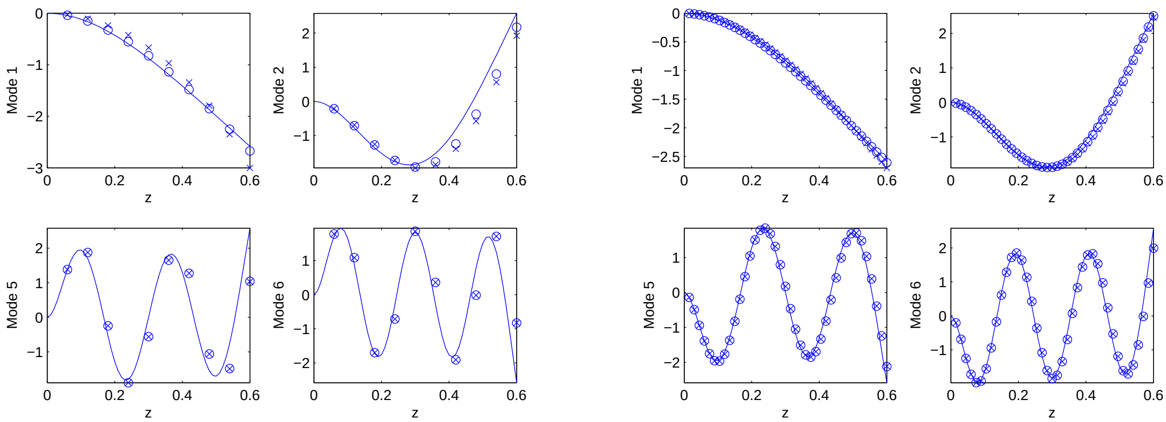

Figure 2. Modal functions (solid line) of clamped-free beam and corresponding KL modes obtained from the stationary covariance matrix $\mathbf{R}_U$ ($\circ$), and the T-mean covariance matrix $\boldsymbol{\mathcal{R}}_U$ ($\times$) with $m = 10$ (left) and $m = 40$ (right).

4.2.2 Relation with classical POM modes

The objective of this section is to compare the Karhunen-Loeve decomposition and the POM as described for example in [3] [4]. To compute the POMs we need a displacement history. The displacement histories were obtained from excitation histories by solving equation (30) numerically using the Newmark algorithm. The excitation histories were simulated using the procedure described in [5]. The same parameters values were used with the time discretization parameter $\Delta t = 0.00025$. Figure 3 compares the first and second modes of the clamped-free beam, the corresponding KL modes obtained from the stationary covariance and the T-mean covariance (as in figure 2), and the corresponding POM modes obtained with several independent sampled trajectories. The POMs depend clearly on the excitation histories.

5. Conclusion

A KL expansion based on a mean operator involving time and ensemble average has been proposed. This approach can be used to analyse transient responses. It gives the same result in the time stationary case as the classical approach. The relation between KL modes and mode shapes for general vibrating structural systems described by second-order ordinary

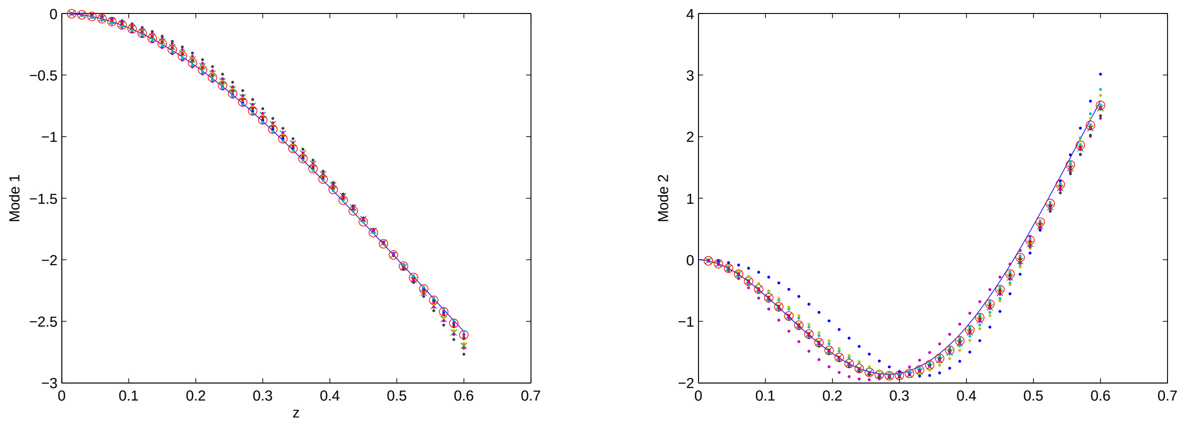

Figure 3. The first (left) and second (right) modes (solid line) of clamped-free beam, the corresponding KL mode obtained from the stationary covariance matrix $\mathbf{R}_U$ ($\circ$) and the T-mean covariance matrix $\boldsymbol{\mathcal{R}}_U$ ($\times$), and the corresponding POM modes obtained with several sampled trajectories ($\cdot$).

differential equations has been derived.

6. Acknowledgements

The authors thank the support of the Project Capes-Cofecub 476/04.

7. References

Arnold L., *Stochastic Differential Equations: Theory and Applications*. New York: Wiley InterScience, 1973.

Kerschen, G. and Golinval, J.C. *Physical interpretation of the proper orthogonal modes using singular value decomposition*. Journal of Sound and Vibration, **249**(5):849–865, 2002.

Feeny, B.F. and Liang, Y. *Interpreting proper orthogonal modes of randomly excited vibration systems*. Journal of Sound and Vibration, **265**:953–966, 2003.

Trindade, M.A., Wolter, C. and Sampaio, R. *Karhunen-Loève decomposition of coupled axial/bending vibrations of beams subject to impacts*. Journal of Sound and Vibration, **279**:1015-1036, 2005.

Poirion, F. and Soize, C. *Simulation numérique des champs stochastiques gaussiens homogènes et non homogènes*. La recherche Aérospatiale, **1**:41–61, 1989.

Holmes, P., Lumley, J.L., and Berkooz, G. *Turbulence, coherent structures, dynamical systems and symmetry*. Cambridge University Press, 1996.

Kreuzer, E. and Kust, O. *Analysis of long torsional strings by proper orthogonal decomposition*. Archive of Applied Mechanics, **67**:68–80, 1996.

Lumley, J.L. *Stochastic tools in turbulence*. Academic Press, 1971.

Ma, X. and Vakakis, A.F. *Karhunen-Loève decomposition of the transient dynamics of a multibay truss*. AIAA Journal, **37**(8):939–946, 1999.

Papoulis A., 1965 "Probability, Random Variables and Stochastic Process", McGraw-Hill, Inc.

Sirovich, L. *Turbulence and the dynamics of coherent structures part I: coherent structures*. Quarterly of Applied Mathematics, **45**(3):561–571, 1987.

Sirovich, L. *Turbulence and the dynamics of coherent structures part II: symmetries and transformations*. Quarterly of Applied Mathematics, **45**(3):573–582, 1987.

Steindl, A., Troger, H., and Zemann, J.V. *Nonlinear Galerkin methods applied in the dimension reduction of vibrating fluid conveying tubes*. In Paidoussis, M.P., Bajaj, A.K., Corke, T.C., Farabee, T.M., and Williams, F. Hara D.R., eds., ASME Symposium on Fluid-Structure Interaction, Aeroelasticity, Flow-Induced Vibration and Noise, ASME, Dallas, EUA, vol. **AD53-1**, pp. 273–280, 1997.

Wolter, C., Trindade, M., Sampaio, R. Obtaining mode shapes through the Karhunen-Loève expansion for distributed-parameter linear systems, Shock and Vibration, **9**(4–5): 177–192, 2002.

8. Responsibility notice

The author(s) is (are) the only responsible for the printed material included in this paper