On measures of nonlinearities for dynamical systems with uncertainties

Rubens Sampaio
Departamento de Engenharia Mecânica, Pontifícia Universidade Católica do Rio de Janeiro, Rua Marquês de São Vicente 225, Gávea, 22453-900, Rio de Janeiro, RJ, Brazil
rsampaio@mec.puc-rio.br

Christian Soize
Laboratoire de Mécanique, Université de Marne-la-Vallèe; 5, Bd Descartes, 77454 Marne-la-Vallée, France
soize@univ-mlv.fr

Abstract. *This paper studies the transient dynamics of a linear dynamical system with elastic barriers excited by a deterministic transient force whose Fourier Transform has a bounded frequency narrow band. The system is then non-linear. In order to measure the degree of non-linearity of the system, one looks for the mechanical energy transferred outside the frequency band of excitation as a function of the parameter* η *defined by* $\frac{\epsilon}{a}$*, in which* ϵ *is the size of the barrier gap and* a *is the amplitude of the excitation force. The mechanical energy transferred outside the frequency band of excitation can potentially be a source of excitation for other subsystems. Consequently, a quantification of this energy transfer is important for the understanding of the non-linear dynamical system. In addition, it is well known that this type of non-linear dynamical system is very sensitive to uncertainties. For this reason one studies the system as being deterministic, and also stochastic in order to take into account random uncertainties. The proposed analysis is then applied to a Timoshenko beam having its motion constrained by a symmetric elastic barrier at its free end. In particular, one shows the confidence region of the random mechanical energy transferred outside the excitation band as a function of* η *for several levels of model and data uncertainties. This type of results allows the robustness of the predictions to be analyzed with respect to model and data uncertainties.*

Keywords: *Nonlinear Dynamics, Vibroimpacts, System Uncertainties*

1. Introduction

The non-linear dynamics of linear dynamical systems with barriers inducing impacts have received a considerable attention in the last two decades. Although there are some engineering systems where impacts are part of the project, most of the time this phenomenon is related to wear, fatigue and noise as, for example, in the case of gear boxes. The interest in vibroimpact systems arises due to their intrinsic non-linear characteristic which prevents their study through more traditional methods such as modal analysis. Actually, systems of this kind have an extremely complex dynamic behaviour, sometimes even chaotic. Therefore, they are normally studied with bifurcation diagrams and Poincaré maps. However, most of the vibroimpact systems investigated so far consists of simple ones with a single degree of freedom. It is expected that the flexibility of a structure will play an important role in its impact response, specially through the excitations of many of its degrees of freedom. Also one expects some exchange of energy among modes due to impacts. A lot of works have been published concerning one single degree of freedom and multi-degrees of freedom deterministic systems excited by deterministic harmonic signals or by narrow- or wide-band stochastic processes (see for instance Babitsky and Birkett [1]). A review of such works can be found in the recent paper by Dimentberg and Iourtchenko [2]. It should be noted that deterministic continuous systems with impacts have received less attention probably due to the difficulties of such non-linear dynamical problems which are very hard to analyzed with analytical tools or with numerical methods. However, some recent representative works of this type can be found in Refs. [4].

Some of the features of this work are:

- It is not about a single, nor multiple, degree of freedom system, but deals with a continuous system. Nevertheless, in order to simplify the presentation and also to show that the methodology applies to a general dynamical system, we start with a discretization of the continuous system, say by using the finite element method.
- The excitation is neither narrow- nor broad-band stochastic process (including white noise modelling) nor deterministic harmonic signal. In this paper the excitation will be modelled by a deterministic narrow-band signal. This choice is important because it gives some robustness to the excitation. One centers the band around one of the natural frequencies of the linear system(without impact) and the width of the band is chosen in order to allow modifications of the system to be taken into account (nonlinearities and uncertainties).
- It deals with the deterministic and also stochastic modelling of the continous system. The stochastic aspects being induced by the uncertainties in the data and in the model (the matrices that represent the linear continuous system are random).

- Measures of nonlinearities are proposed. In order to analyze the degree of non-linearity of the system, one looks for the mechanical energy transferred outside the frequency band of excitation as a function of the parameter η, defined by $\frac{\epsilon}{a}$, in which ϵ is the size of the barrier gap and a is the amplitude of the excitation force. When ϵ is zero or infinity, there are no impacts. When it is between this two bounds the continuous system-barrier behaves non-linearly for amplitude a sufficiently high. It turns out that the non-linearity depends on η. The interest of measuring the amount of energy that is transferred outside the band of excitation is to evaluate the dangerous consequences like exciting sensitive subsystems whose lowest eigenfrequency is outside the band of excitation.
- Stochastic systems are considered in order to evaluate the robustness of the numerical prediction of the energy transferred with respect to data and model uncertainties.

This paper is divided of four parts. Section 2is devoted to the modelling and analysis of the deterministic non-linear dynamical system. In Section 3one presents the stochastic modelling of the system in order to take into account data and model uncertainties. Section 4deals with numerical applications. We take a Timoshenko beam with an elastic barrier. Finally, general analysis and conclusions are presented in Section 5.

2. Modelling and analysis of the deterministic non-linear dynamical system

In this section one presents the mean model of the dynamical system with excitation, the reduced mean model obtained by using the elastic modes of the linear mean dynamical system and finally, one describes the different energies one needs to analyze the energy transferred outside the excitation band.

2.1 Finite element model of the mean non-linear dynamical system

The main interest of the paper is to study a linear continuous system with elastic barriers that induce through impact non-linearities. However the methodology one presents is general and can be applied to a larger class of problems, as for example those related to a linear system interacting with a subsystem that originates non-linearities, as the case of an elastic barrier. Systems having large deformations are, of course, not included. In order to focus in the methodology one starts with a finite dimensional system that could be the result of a discretization process. This system, referred as the *mean model*, is described by the following matrix equation in $\mathbb{R}^m$,

$$[\mathbb{M}]\,\ddot{\mathbf{y}}(t) + [\mathbb{D}]\,\dot{\mathbf{y}}(t) + [\mathbb{K}]\,\mathbf{y}(t) + \mathbf{f}_{\rm NL}(\mathbf{y}(t),\dot{\mathbf{y}}(t)) = \mathbf{f}(t) \quad , \tag{1}$$

where $[\mathbb{M}]$, $[\mathbb{D}]$, $[\mathbb{K}]$ are the mass, damping and stiffness matrices, that are supposed to be symmetric and positive-definite real matrices, and $\mathbf{y}(t)$ is the displacement vector, $\mathbf{f}_{\rm NL}(\mathbf{y}(t),\dot{\mathbf{y}}(t))$ describes the nonlinear vector forces, $\mathbf{f}(t)$ the applied vector load. The non-linear mapping $(\mathbf{y},\mathbf{z}) \mapsto \mathbf{f}_{\rm NL}(\mathbf{y},\mathbf{z})$ is assumed to be such that $\mathbf{f}_{\rm NL}(0,0) = 0$. The vector load $\mathbf{f}(t)$ is written as

$$\mathbf{f}(t) = a\,g(t)\,\mathbf{f}_0 \quad , \tag{2}$$

in which a is the amplitude and $\mathbf{f}_0$ is a normalized vector describing the position of the applied forces. The impulse $t \mapsto g(t)$ is a square integrable real-valued function on $\mathbb{R}$ whose Fourier Transform $\omega \mapsto \widehat{g}(\omega) = \int_{\mathbb{R}} e^{-i\omega t} g(t)\,dt$ has a bounded support $\underline{B}_2 \cup B_2$ with

$$B_2 = [\omega_{\min}, \omega_{\max}] \quad , \quad \underline{B}_2 = [-\omega_{\max}, -\omega_{\min}] \quad . \tag{3}$$

The notation B_2 will be explained in Section 2.3In addition it is assumed that $\max_{\omega \in B_2} |\widehat{g}(\omega)| = 1$.

2.2 Reduced mean model

Let $\{\phi_1, \ldots, \phi_m\}$ be an algebraic basis of $\mathbb{R}^m$. The reduced mean model of the dynamic system whose mean finite element model is defined by Eq. (1) is obtained by projection of Eq. (1) on the subspace V_n of $\mathbb{R}^m$ spanned by $\{\phi_1, \ldots, \phi_n\}$ with $n \ll m$. Let $[\,\Phi_n]$ be the $(m \times n)$ real matrix whose columns are vectors $\{\phi_1, \ldots, \phi_n\}$. The generalized applied force $\mathbf{F}^n(t)$ is an $\mathbb{R}^n$-vector such that $\mathbf{F}^n(t) = [\,\Phi_n]^T\,\mathbf{f}(t)$. The generalized mass, damping and stiffness matrices $[\,\underline{M}_n]$, $[\,\underline{D}_n]$ and $[\,\underline{K}_n]$ are positive-definite symmetric $(n \times n)$ real matrices such that $[\,\underline{M}_n] = [\,\Phi_n]^T[\,\mathbb{M}\,][\,\Phi_n]$, $[\,\underline{D}_n] = [\,\Phi_n]^T[\,\mathbb{D}\,][\,\Phi_n]$, and $[\,\underline{K}_n] = [\,\Phi_n]^T[\,\mathbb{K}\,][\,\Phi_n]$. Consequently, the reduced mean model of the nonlinear dynamic system, written as the projection $\mathbf{y}^n$ of $\mathbf{y}$ on V_n, can be written as

$$\mathbf{y}^n(t) = [\,\Phi_n]\,\mathbf{q}^n(t) \quad , \tag{4}$$

in which the vector $\mathbf{q}^n(t) \in \mathbb{R}^n$ of the generalized coordinates verifies the mean nonlinear differential equation,

$$[\,\underline{M}_n]\,\ddot{\mathbf{q}}^n(t) + [\,\underline{D}_n]\,\dot{\mathbf{q}}^n(t) + [\,\underline{K}_n]\,\mathbf{q}^n(t) + \mathbf{F}^n_{\rm NL}(\mathbf{q}^n(t),\dot{\mathbf{q}}^n(t)) = \mathbf{F}^n(t) \quad , \tag{5}$$

where, for all $\mathbf{q}$ and $\mathbf{p}$ in $\mathbb{R}^n$,

$$\mathbf{F}^n_{\text{NL}}(\mathbf{q},\mathbf{p}) = [\,\Phi_n]^T\,\mathbf{f}_{\text{NL}}([\,\Phi_n]\,\mathbf{q},[\,\Phi_n]\,\mathbf{p}) \quad . \tag{6}$$

2.3 Quantification of the transferred energies outside the excitation band

The objective of this section is to quantify the mechanical energy transferred outside the excitation band. It is assumed that Eq. (1) has a unique solution $t \mapsto \mathbf{y}(t)$ such that $\mathbf{y}$ and $\dot{\mathbf{y}}$ are square integrable vector-valued functions on $\mathbb{R}$. An approximation of this solution is computed using the reduced mean model defined by Eqs. (4)-(6). The positive frequency band $\mathbb{R}^+ = [0,+\infty[$ is then written as

$$\mathbb{R}^+ = [0,+\infty[= B_1 \cup B_2 \cup B_3 \quad , \tag{7}$$

in which $B_1 = [0,\omega_{\min}[$ and $B_3 =]\omega_{\max},+\infty[$. The sets B_1 and B_3 are the bands outside the frequency band of excitation B_2. The total mechanical energy, denoted by $\widetilde{e}$, of the non-linear dynamical system corresponding to the solution mentioned above is written as,

$$\widetilde{e} = \int_{\mathbb{R}} (\frac{1}{2} < [\mathbb{M}]\,\dot{\mathbf{y}}(t)\,,\dot{\mathbf{y}}(t) > + \frac{1}{2} < [\mathbb{K}]\,\mathbf{y}(t)\,,\mathbf{y}(t) >)\,dt \quad . \tag{8}$$

Let $\widehat{\mathbf{y}}(\omega) = \int_{\mathbb{R}} e^{-i\omega t}\mathbf{y}(t)\,dt$ be the Fourier Transform of $\mathbf{y}$. Using the Parseval formula, Eq. (8) yields

$$\widetilde{e} = \int_{\mathbb{R}} \underline{h}(\omega)\,d\omega = 2\int_{\mathbb{R}^+} \underline{h}(\omega)\,d\omega \quad , \tag{9}$$

in which $\underline{h}(\omega)$ is the density of the mechanical energy in the frequency domain which is written as

$$\underline{h}(\omega) = \frac{1}{2\pi}\{\frac{1}{2} < \omega^2[\mathbb{M}]\,\widehat{\mathbf{y}}(\omega)\,,\overline{\widehat{\mathbf{y}}(\omega)} > + \frac{1}{2} < [\mathbb{K}]\,\widehat{\mathbf{y}}(\omega)\,,\overline{\widehat{\mathbf{y}}(\omega)} >\} \quad . \tag{10}$$

From Eqs. (7) and (9), it can deduced that

$$\widetilde{e} = \widetilde{e}_1 + \widetilde{e}_2 + \widetilde{e}_3 \quad , \tag{11}$$

in which

$$\widetilde{e}_j = 2\int_{B_j} \underline{h}(\omega)\,d\omega \quad , \quad j = 1,2,3 \quad . \tag{12}$$

The transferred mechanical energy outside the excitation band B_2 is denoted by $\widetilde{e}_{13}$ which is defined by

$$\widetilde{e}_{13} = \widetilde{e}_1 + \widetilde{e}_3 \quad . \tag{13}$$

Using the reduced mean model defined by Eqs. (4)-(6), the approximation $\underline{h}^n(\omega)$ of $\underline{h}(\omega)$ defined by Eq. (10) can be written as

$$\underline{h}^n(\omega) = \frac{1}{2\pi}\{\frac{1}{2} < \omega^2[\underline{M}_n]\,\widehat{\mathbf{q}}^n(\omega)\,,\overline{\widehat{\mathbf{q}}^n(\omega)} > + \frac{1}{2} < [\underline{K}_n]\,\widehat{\mathbf{q}}^n(\omega)\,,\overline{\widehat{\mathbf{q}}^n(\omega)} >\} \quad , \tag{14}$$

in which $\widehat{\mathbf{q}}^n(\omega) = \int_{\mathbb{R}} e^{-i\omega t}\mathbf{q}^n(t)\,dt$ is the Fourier Transform of $\mathbf{q}^n$. The corresponding energies computed with this approximation are denoted by $\widetilde{e}^{\,n}$, $\widetilde{e}^{\,n}_1$, $\widetilde{e}^{\,n}_2$, $\widetilde{e}^{\,n}_3$, $\widetilde{e}^{\,n}_{13}$. In order to explore the results in a non-dimensional way one introduces the following parameters,

$$e^n_1 = \frac{\widetilde{e}^{\,n}_1}{\widetilde{e}^{\,n}} \quad , \quad e^n_2 = \frac{\widetilde{e}^{\,n}_2}{\widetilde{e}^{\,n}} \quad , \quad e^n_3 = \frac{\widetilde{e}^{\,n}_3}{\widetilde{e}^{\,n}} \quad , \quad e^n_{13} = \frac{\widetilde{e}^{\,n}_{13}}{\widetilde{e}^{\,n}} \quad , \tag{15}$$

3. Modelling and analysis of the non-linear dynamical system with random uncertainties

The first source of uncertainties in this type of problem is due to the mathematical-mechanical modelling process leading to the boundary value problem. This type of uncertainty is structural, and cannot be represented as, simply, the usual variation of parameters[5, 6]. This uncertainties are called *the model uncertainties*. Concerning the second source of uncertainties, they come from the parameters such as geometry, material properties, boundary and initial conditions, etc, related to the boundary value problem. The uncertainties in these parameters are called *data uncertainties*. It is worthwhile to remark that the errors related to the construction of an approximation of the solution of the boundary value problem, that have to be controlled in order to meet the specifications of the numerical approximation, are not uncertainties. For the class of systems one studies the sources of uncertainties are in the data related to the non-linear term and in the data and model related to the linear part.

3.1 Probabilistic modelling of uncertainties

From this point one constructs the probability model of uncertainties from the mean reduced model defined by Eqs. (4)-(6). All the random variables are defined in a probability space $(\Theta, \mathcal{F}, \mathcal{P})$

(A) *Parametric probabilistic model of data uncertainties for the non-linear term.* Usually, data uncertainties are modelled by using parametric probabilistic approach consisting in modelling each uncertain parameter by a random variable whose probability distribution has to be constructed using the available information. The non-linear term $\mathbf{F}^n_{\rm NL}(\mathbf{q}^n(t), \dot{\mathbf{q}}^n(t))$ in Eq. (5) is rewritten as $\mathbf{F}^n_{\rm NL}(\mathbf{q}^n(t), \dot{\mathbf{q}}^n(t); \mathbf{s})$ in which $\mathbf{s}$ is an $\mathbb{R}^\nu$-vector of uncertain parameters. The probabilistic modelling of vector $\mathbf{s}$ is as an $\mathbb{R}^\nu$-valued random variable whose probability distribution on $\mathbb{R}^\nu$ is denoted by $P_{\mathbf{S}}(d\mathbf{s})$. The available information for constructing $P_{\mathbf{S}}(d\mathbf{s})$ depends on the nature of the parameters constituting vector $\mathbf{s}$ (for instance positivity, boundedness of components, etc). When this information is defined the probability distribution can be constructed using the maximum entropy principle with the constraints defined by the available information.

(B) *Non-parametric probabilistic model of model and data uncertainties for the linear part.* Model uncertainties cannot be taken into account using the parametric probabilistic approach. A non-parametric probabilistic approach can be used to take into account model uncertainties and data uncertainties [5, 6]. The principle of construction of such non-parametric probabilistic approach of uncertainties for the linear part of the non-linear dynamical system whose reduced mean model is defined by Eqs. (4)-(6) consists in substituting the generalized mass, damping and stiffness matrices in Eq. (5) by random matrices $[\mathbf{M_n}]$, $[\mathbf{D_n}]$ and $[\mathbf{K_n}]$ whose probability distributions have been constructed using the maximum entropy principle with an adapted available information. The explicit form of the probability distributions of the random matrices $[\mathbf{M_n}]$, $[\mathbf{D_n}]$ and $[\mathbf{K_n}]$ are given in Refs. [5, 6].

(C) *Stochastic reduced model.* The stochastic transient response of the nonlinear dynamic system with a nonparametric probabilistic approach of model and data uncertainties is the stochastic process $\mathbf{Y}^n(t)$, indexed by $\mathbb{R}$, with values in $\mathbb{R}^m$, such that

$$\mathbf{Y}^n(t) = [\,\Phi_n]\,\mathbf{Q}^n(t) \quad , \tag{16}$$

in which the stochastic process $\mathbf{Q}^n$, defined in the probability space $(\Theta, \mathcal{F}, \mathcal{P})$, indexed by $\mathbb{R}$, with values $\mathbb{R}^n$, is such that

$$[\mathbf{M}_n]\,\ddot{\mathbf{Q}}^n(t) + [\mathbf{D}_n]\,\dot{\mathbf{Q}}^n(t) + [\mathbf{K}_n]\,\mathbf{Q}^n(t) + \mathbf{F}^n_{NL}(\mathbf{Q}^n(t), \dot{\mathbf{Q}}^n(t); \mathbf{S}) = \mathbf{F}^n(t) \quad . \tag{17}$$

Let $|||\mathbf{Q}^n|||$ be the norm such that

$$|||\mathbf{Q}^n|||^2 = \mathcal{E}\{\int_{\mathbb{R}} ||\mathbf{Q}^n(t)||^2\,dt\} \quad , \tag{18}$$

in which $\mathcal{E}$ is the mathematical expectation and where $||\mathbf{u}||^2 = u_1^2 + \ldots + u_n^2$ is a square of the Euclidean norm of $\mathbf{u}$ in $\mathbb{R}^n$. It is assumed that the non-linear term is such that Eq. (17) has a unique second-order mean-square solution such that

$$|||\mathbf{Q}^n||| < +\infty \quad , \quad |||\dot{\mathbf{Q}}^n||| < +\infty \quad , \tag{19}$$

3.2 Probabilistic quantification of the transferred energies outside the excitation band for the uncertain system

The objective of this section is to adapt Section 3.1to the reduced stochastic system defined by Eqs. (16) and (17). The random total mechanical energy associated with $\widetilde{e}^{\,n}$ is denoted by $\widetilde{E}^{\,n}$ and is such that

$$\widetilde{E}^{\,n} = \int_{\mathbb{R}} (\frac{1}{2} < [\mathbb{M}]\,\dot{\mathbf{Y}}^n(t)\,, \dot{\mathbf{Y}}^n(t) > + \frac{1}{2} < [\mathbb{K}]\,\mathbf{Y}^n(t)\,, \mathbf{Y}^n(t) >)\,dt \quad . \tag{20}$$

The density of the random mechanical energy in the frequency domain associated with $\underline{h}^n(\omega)$ defined by Eq. (14) is denoted by $H^n(\omega)$ and can be written as

$$H^n(\omega) = \frac{1}{2\pi}\{\frac{1}{2} < \omega^2[\underline{M}_n]\,\widehat{\mathbf{Q}}^n(\omega)\,, \overline{\widehat{\mathbf{Q}}^n(\omega)} > + \frac{1}{2} < [\underline{K}_n]\,\widehat{\mathbf{Q}}^n(\omega)\,, \overline{\widehat{\mathbf{Q}}^n(\omega)} >\} \quad , \tag{21}$$

in which $\widehat{\mathbf{Q}}^n(\omega) = \int_{\mathbb{R}} e^{-i\omega t}\mathbf{Q}^n(t)\,dt$ is the Fourier Transform of $\mathbf{Q}^n$.

Let $H^n_{\rm dB}(\omega)$ be the density of the random mechanical energy in dB normalized with respect to the total mechanical energy $\widetilde{e}_{\rm lin}$ of the linear mean system. One then has

$$H^n_{\rm dB}(\omega) = \log_{10}(H^n(\omega)/\widetilde{e}_{\rm lin}) \quad . \tag{22}$$

Let $\widetilde{E}^n_1$, $\widetilde{E}^n_2$, $\widetilde{E}^n_3$ and $\widetilde{E}^n_{13}$ be the random energies associated with $\widetilde{e}^n_1$, $\widetilde{e}^n_2$, $\widetilde{e}^n_3$ and $\widetilde{e}^n_{13}$ such that

$$\widetilde{E}^n_j = 2\int_{B_j} H^n(\omega)\,d\omega \quad , \quad j = 1, 2, 3 \quad , \quad \widetilde{E}^n_{13} = \widetilde{E}^n_1 + \widetilde{E}^n_3 \quad . \tag{23}$$

Similarly to Section , this random energies are normalized as follows

$$E_1^n = \frac{\widetilde{E}_1^n}{\widetilde{E}^n} \quad , \quad E_2^n = \frac{\widetilde{E}_2^n}{\widetilde{E}^n} \quad , \quad E_3^n = \frac{\widetilde{E}_3^n}{\widetilde{E}^n} \quad , \quad E_{13}^n = \frac{\widetilde{E}_{13}^n}{\widetilde{E}^n} \quad , \tag{24}$$

3.3 Stochastic solver and convergence

In this section, one introduces the stochastic solver that is used and one analyses the stochastic convergence. The Monte Carlo numerical simulation and mathematical statistics are used for solving the stochastic equations defined by Eqs. (16) and (17). Let $\mathbf{S}(\theta)$ and $[\mathbf{M}_n(\theta)]$, $[\mathbf{D}_n(\theta)]$, $[\mathbf{K}_n(\theta)]$ be independent realizations of random variable $\mathbf{S}$ and random matrices $[\mathbf{M}_n]$, $[\mathbf{D}_n]$, $[\mathbf{K}_n]$ for $\theta \in \Theta$.

(A) *Construction of realizations of random variable* $\mathbf{S}$. Each realization $\mathbf{S}(\theta)$ of random variable $\mathbf{S}$ is usually constructed using random generator associated with the probability distribution $P_{\mathbf{S}}(d\mathbf{s})$. Because the generation is standard it will not be detailed here.

(B) *Construction of realizations of random matrix variables* $[\mathbf{M}_n]$, $[\mathbf{D}_n]$, $[\mathbf{K}_n]$. Let $[\mathbf{A}_n]$ be anyone of the three random matrices above and let $[\underline{\mathbf{A}}_n]$ be its mean value which is a positive-definite matrix. Its Cholesky factorization yields $[\underline{\mathbf{A}}_n] = [\underline{L}_n]^T[\underline{L}_n]$. Each realization $[\mathbf{A}_n(\theta)]$ can be generated using the following algebraic representation [5, 6],

$$[\mathbf{A}_n] = [\underline{L}_n]^T\,[\mathbf{G}_n]\,[\underline{L}_n] \quad , \tag{25}$$

In which the positive-definite random matrix $[\mathbf{G}_n]$ is written as

$$[\mathbf{G}_n] = [\mathbf{L}_n]^T\,[\mathbf{L}_n] \quad . \tag{26}$$

In Eq. (25), $[\mathbf{L}_n]$ is an upper triangular random matrix with values in $\mathbb{M}_n(\mathbb{R})$ such that:

(1) Random variables $\{[\mathbf{L}_n]_{jj'}, j \leq j'\}$ are independent.

(2) For $j < j'$, real-valued random variable $[\mathbf{L}_n]_{jj'}$ can be written as $[\mathbf{L}_n]_{jj'} = \sigma_n U_{jj'}$ in which $\sigma_n = \delta(n+1)^{-1/2}$ and where $U_{jj'}$ is a real-valued Gaussian random variable with zero mean and variance equal to 1.

(3) For $j = j'$, positive-valued random variable $[\mathbf{L}_n]_{jj}$ can be written as $[\mathbf{L}_n]_{jj} = \sigma_n\sqrt{2V_j}$ in which σ_n is defined above and where V_j is a positive-valued gamma random variable whose probability density function $p_{V_j}(v)$ with respect to dv is written as

$$p_{V_j}(v) = \mathbf{1}_{\mathbb{R}^+}(v)\frac{1}{\Gamma\left(\frac{n+1}{2\delta^2} + \frac{1-j}{2}\right)}\; v^{\frac{n+1}{2\delta^2} - \frac{1+j}{2}}\; e^{-v} \quad , \tag{27}$$

in which $\mathbf{1}_{\mathbb{R}^+}(v) = 1$ if $v \in \mathbb{R}^+$ and $= 0$ if not, and where Γ is the usual Gamma function. This algebraic representation exhibits δ which is the positive parameter allowing the dispersion of random matrix $[\mathbf{A}_n]$ to be controlled. This parameter has to be given for each random matrix and controls the level of uncertainties. In special it controls the uncertainties of mass, damping or stiffness of the linear continuous system of the non-linear dynamical system.

(C)*Construction of realizations of the solution of the stochastic reduced system.* The realization $\mathbf{Y}^n(t,\theta)$ for $\theta \in \Theta$ of $\mathbf{Y}^n(t)$ defined by Eq. (23) is given by

$$\mathbf{Y}^n(t,\theta) = [\,\Phi_n]\,\mathbf{Q}^n(t,\theta) \quad , \tag{28}$$

in which the realization $\{\mathbf{Q}^n(t,\theta), t \in \mathbb{R}\}$ of the stochastic process $\{\mathbf{Q}^n(t), t \in \mathbb{R}\}$, is the solution of the following deterministic non-linear reduced equation,

$$[\mathbf{M}_n(\theta)]\,\ddot{\mathbf{Q}}^n(t,\theta) + [\mathbf{D}_n(\theta)]\,\dot{\mathbf{Q}}^n(t,\theta) + [\mathbf{K}_n(\theta)]\,\mathbf{Q}^n(t,\theta) + \mathbf{F}_{NL}^n(\mathbf{Q}^n(t,\theta), \dot{\mathbf{Q}}^n(t,\theta); \mathbf{S}(\theta)) = \mathbf{F}^n(t,\theta) \quad . \tag{29}$$

This equation is solved by using an implicit unconditionnally stable scheme such as Newmark algorithm. At each time step, the non-linear algebraic equation coming from the scheme is solved by iteration.

(D)*Stochastic convergence* The mean-square convergence of the second-order stochastic solution of Eq. (17) with respect to dimension n of the stochastic reduced model and to the number n_s of realizations used in the Monte Carlo numerical simulations is controlled by the norm $|||\mathbf{Q}^n|||$ defined by Eq. (18). Using the usual estimation of the mathematical expectation operator $\mathcal{E}$, convergence with respect to n and n_s is studied by constructing the function $(n_s, n) \mapsto \mathrm{conv}(n_s, n)$ defined by

$$\mathrm{conv}(n_s, n) = \left\{\frac{1}{n_s}\sum_{k=1}^{n_s}\int_{\mathbb{R}} \|\mathbf{Q}^n(t,\theta_k)\|^2\, dt\right\}^{1/2} \quad , \tag{30}$$

in which $\mathbf{Q}^n(t,\theta_1),\ldots,\mathbf{Q}^n(t,\theta_{n_s})$ are n_s independent realizations of $\mathbf{Q}^n(t)$.

(E) *Statistical estimations of the random energies.* One is interested in constructing statistical estimations for the stochastic process $\{H^n(\omega),\omega \in \mathbb{R}\}$ defined by Eq. (21) and for the random variables E_1^n, E_2^n, E_3^n, E_{13}^n defined by Eq. (23), whose realizations are directly deduced from the realizations of $\mathbf{Q^n}$. Let X be the positive-valued random variable representing either $H^n(\omega)$ for ω fixed in $\mathbb{R}$ or any of the random variables E_1^n, E_2^n, E_3^n, E_{13}^n. The mean value $m_X = \mathcal{E}\{X\}$ is estimated by

$$\widetilde{m}_X = \frac{1}{n_s}\sum_{k=1}^{n_s} X(\theta_k) \quad , \tag{31}$$

in which $X(\theta_1),\ldots,X(\theta_{n_s})$ are n_s independent realizations of X. The confidence region of random variable X is constructed by using the quantiles. Let F_X be the cumulative distribution function (continuous from the right) of random variable X such that $F_X(x) = P(X \leq x)$. For $0 < p < 1$, the pth quantile or fractile of F_X is defined by

$$\zeta(p) = \inf\{x : F_X(x) \geq p\} \quad . \tag{32}$$

Then the upper envelope x^+ and the lower envelope x^- of the confidence region with probability level P_c are defined by

$$x^+ = \zeta\, P_c \quad , \quad x^- = \zeta(1-P_c) \quad . \tag{33}$$

The estimation of x^+ and x^- is performed by using the sample quantiles. Let $x_1 = X(\theta_1),\ldots,x_{n_s} = X(\theta_{n_s})$. Let $\widetilde{x}_1 < \ldots < \widetilde{x}_{n_s}$ be the order statistics associated with $x_1 < \ldots < x_{n_s}$. Therefore, one has the following estimation,

$$x^+ \simeq \widetilde{x}_{j^+} \quad , \quad j^+ = \mathrm{fix}(n_s\, P_c) \quad , \tag{34}$$

$$x^- \simeq \widetilde{x}_{j^-} \quad , \quad j^- = \mathrm{fix}(n_s(1-P_c)) \quad , \tag{35}$$

in which $\mathrm{fix}(z)$ is the integer part of the real number z.

4. Application to a Timoshenko beam with an elastic barrier

This section deals with the application of the theory developed in the previous sections. The linear part of the continuous system is a Timoshenko beam with added dissipation. The non-linear force is due to a symmetrical linear elastic barrier.

The geometrical properties of the beam are length $1\,m$, width $0.1\,m$, height $0.1\,m$. The boundary conditions are of a cantilever beam, with the free end having its motion limited by an elastic barrier distant of ϵ, in both sides of the beam. The gap ϵ is considered as a parameter. The beam homogeneous, isotropic, whose material properties are density $7500\,kg/m^3$, Young' s modulus $2.1\times10^{10}\ N/m^2$, Poisson's coefficient 0.3, shearing correction factor $5/6$. The damping model is introduced by the model damping rate which is 0.02 for the first three modes, 0.01 for the fourth mode and 0.005 for the others. The elasticity constant of the barrier is $\underline{k}_b = 10^7\ N/m$.

(A) *Mean finite element model*. The mean finite element model of the cantilever beam is constituted of 100 2-nodes Timoshenko beam elements. The first six computed eigenfrequencies are 26.9, 162.7, 432.9, 794.1, 1219.2 and 1685.3 Hz.

(B) *Description of excitation force*. The vector load is defined by Eq. (2). The amplitude a is considered as a parameter. The force is a point force applied at the middle point of the beam. The impulse function g is such that

$$g(t) = \frac{1}{\pi\, t}\{sin(t(\Omega_c + \Delta\Omega/2)) - sin(t(\Omega_c - \Delta\Omega/2))\} \quad , \tag{36}$$

whose Fourier Transform is $\widehat{g}(\omega) = \mathbf{1}_{\underline{B}_2\cup B_2}$. The frequency band B_2 is defined by Eq. (3) with $\omega_{\min} = 2\pi f_{\min}$ and $\omega_{\max} = 2\pi f_{\max}$ with $f_{\min} = 148$ Hz and $f_{\max} = 178$ Hz. The corresponding bandwidth $\Delta\Omega = 2\pi\Delta f$ is then such that $\Delta f = 30$ Hz and the central frequency $\Omega_c = 2\pi\, f_c$ is such that $f_c = 163$ Hz. Consequently, the frequency band of excitation is central in the second eigenfrequency of the linear system.

(C) *Reduced mean model*. The numerical presented in this section are computed with $n = 40$, and the modes were calculated with the finite element model. This valued was chosen to assure good convergence for the deterministic and the stochastic solutions.

(D) *Parametric probabilistic model of the barrier*. Since the gap is taken as a parameter of the problem it is not considered as uncertain. On the other hand, the stiffness of the barrier is uncertain and modelled by a positive-valued random variable K_b whose mean value is $\underline{k}_b$, for which the coefficient of variation δ_b is 0 (no uncertainty) or 0.05 (uncertainty) and whose probability distribution is the Gamma law.

(E) *Nonparametric probabilistic model of the beam*. As explained in Section 3.3(B), the uncertainty levels for the mass, damping and stiffness of the linear system are controlled by the dispersion parameters δ_M, δ_D and δ_K, respectively.

In order to simplify the presentation, one only consider the cases $\delta_M = \delta_D = \delta_K$. The common valued will be denoted by $\delta_{\rm m}$. Two values are considered $\delta_{\rm m} = 0$ (no uncertainty) and $\delta_{\rm m} = 0.1$(uncertainty).

In this paper we show only results for $\delta_{\rm m} = 0.1$ and $\delta_b = 0.05$.

The integration time step is taken as $\Delta t = 1/(2 f_{\rm max})$ and the time integration $T = n_{\rm time}\,\Delta t$ with $n_{\rm time} = 8192$. The integration in $\mathbb{R}$ is approximated by an integration over the interval $[t_0, t_1]$ in which $t_0 = -T/2$ and $t_1 = T/2 - \Delta t$. The sampling time points are $t_k = t_0 + k\Delta t, k = 0, \ldots, n_{\rm time} - 1$. To compute the Fourier Transform by FFT algorithm, the integration frequency step is taken as $\Delta\omega = 2\omega_{\rm max}/n_{\rm freq}$ with $n_{\rm freq} = n_{\rm time}$. The sampling frequency points are $\omega_k = -\omega_{\rm max} + k\Delta\omega, k = 0, \ldots, n_{\rm freq} - 1$. Equation (29) is integrated over $[t_0, t_1]$ with zero initial conditions at t_0. The given choice of the parameters are such that $\mathcal{E}\{\|\mathbf{Q}(t_1, \theta)\|^2\}$ is negligible at the final time t_1.

Figure 1(left) is relative to the random fraction function $\eta \mapsto E_1^n(\eta)$, defined by Eq. (23) and related to the random mechanical energy transferred to band B_1. Three quantities are represented in this figure: the graph $\eta \mapsto e_1^n(\eta)$ for the mean system, defined by Eq. (15), the graph $\eta \mapsto \mathcal{E}\{E_1^n(\eta)\}$ of the mean value for the stochastic system, and finally, the confidence region of the random fraction function $\eta \mapsto E_1^n(\eta)$. Figure 1 (right) is relative to the random fraction function $\eta \mapsto E_2^n(\eta)$, defined by Eq. (23) and related to the random mechanical energy in band B_2. Three quantities are represented in this figure: the graph $\eta \mapsto e_2^n(\eta)$ for the mean system, defined by Eq. (15), the graph $\eta \mapsto \mathcal{E}\{E_2^n(\eta)\}$ of the mean value for the stochastic system, and finally, the confidence region of the random fraction function $\eta \mapsto E_2^n(\eta)$. Figure 2 (left) is similar to Fig. 4 but for random energy E_3 transferred in band B_3 Figures 2 (right) is relative to the cumulative distribution functions of the random energies for a fixed value of η such that $\log_{10}\eta = -6.0$. It displays the graph of $\zeta^1 \mapsto {\rm Proba}\{E_1^n(\eta) \leq \zeta^1\}$ of the random variable $E_1^n(\eta)$ related to the random mechanical energy transferred to band B_1.

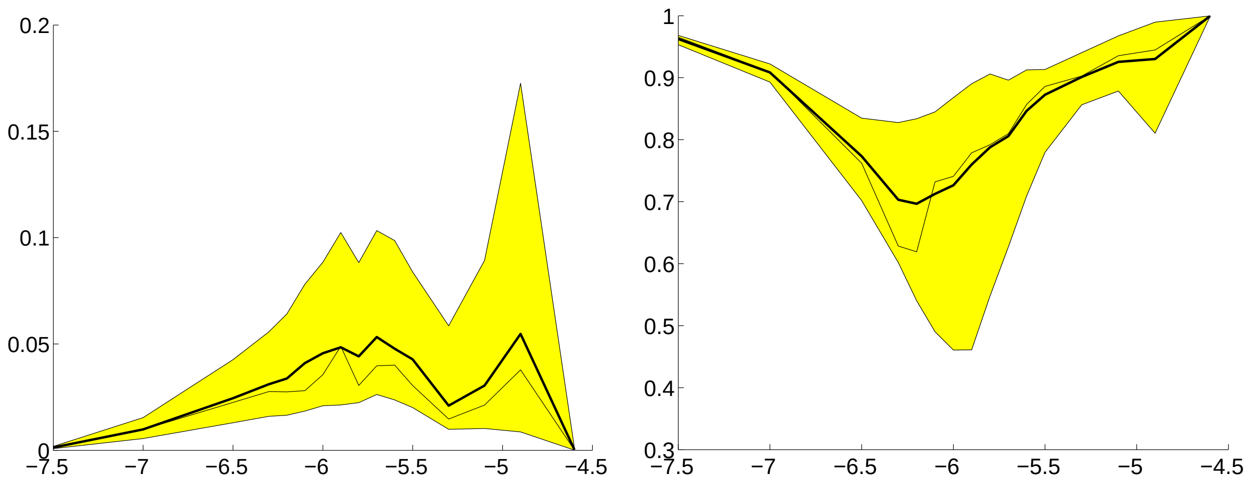

Figure 1. Confidence region of the random fraction function $E_1^n(\eta)$ (left) and $E_2^n(\eta)$ related to the random mechanical energy transferred to band B_1 and B_2, respectively. Horizontal axis $\log_{10}\eta$. Vertical axis fraction in linear scale, adimensional.

5. Analysis of the results and conclusions

5.1 Maximum of non-linearity effects as a function of η

The measures of non-linearity is given by the fraction of energy that is transferred outside the band of excitation. Figures 1 to 2 (left) show that a maximum of non-linearity effect is obtained for mid-value of $\eta = \frac{\epsilon}{a}$ and not for the extremes, near zero or very large. Near zero means that the gap is very small with respect to the displacement, that is there are a large number of impacts with low energy (small gap). This case is frequent, for instance, in Robotics (looseness). Very large means that the gap is sufficiently big with respect to the displacement such that the number of impacts is small and with low energy. In the medium range, the impacts are more frequent and also more energetic. It is worthwhile to insist that as η is the ratio of the gap and the amplitude of the excitation, even for very small gaps the effect of η can be large depending on the force. For example for $\epsilon = 2 \times 10^{-6}\, m$ a force of $1\, N$ corresponds to a numerical value of $\eta = 2 \times 10^{-6}$, and Fig. 2 (left) shows that for this value there is a transfer of energy of 30 to 50 percent outside the band of excitation.

5.2 Relation of the non-linearities with the spectral density

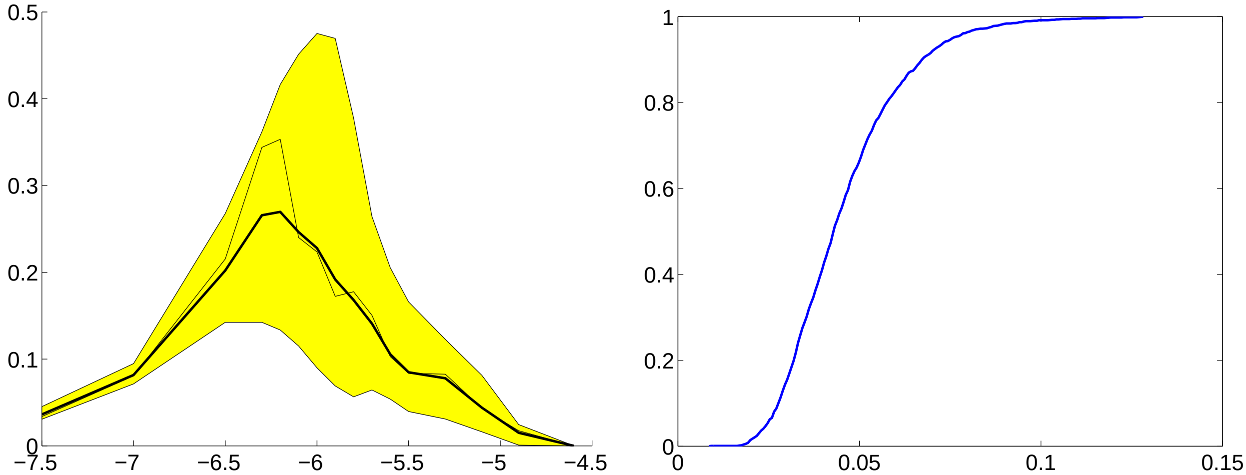

Figure 2. Left figure are data for E_3. On right is the cumulative distribution function $\zeta^1 \mapsto \mathrm{Proba}\{E_1^n(\eta) \leq \zeta^1\}$ of the random variable $E_1^n(\eta)$ for $\log_{10}\eta = -5.5$ related to the random mechanical energy transferred to band B_1. Horizontal axis fraction ζ^1, adimensional. Vertical axis probability level in linear scale.

It can be shown that even when there is a small amount of energy transferred outside the band of excitation the effect of this transfer in the spectral density is very large causing the response to become a broad-band signal.

5.3 Uncertainties effects

The Figs. 1 to 2 show that the point of maximum of non-linearities is also the point of less robustness with respect to uncertainties. On the other hand the two limit cases, near zero and very large η are relatively robust with respect to uncertainties. It is useful to discuss the effect of two types of uncertainties: barrier uncertainty and model uncertainties for the linear system. The non-linearity effects are less robust for model uncertainties than that for barrier uncertainty. On the other hand the frequency response in the frequency band of excitation is robust with respect to uncertainties. Finally, Figs. 2(right) allows the probability of the random energies to be estimated.

6. Acknowledgments

This work was founded by the Brazilian Agency Conselho Nacional de Desenvolvimento Científico e Tecnológico(CNPq) and the CAPES-COFECUB Project $n^o 476/04$.

7. References

V.I. Babitsky, N. Birkett, Theory of Vibroimpact Systems and Applications, Springer-Verlag, 1998.
M.F. Dimentberg, D.V. Iourtchenko, Random vibrations with impacts: A review, Nonlinear Dynamics, 36 (2004) 229-254.
C. Wolter, M.A. Trindade, R. Sampaio, Reduced-order model for an impacting beam using the Karhunen-Loève expansion, Tendências em Matemática Aplicada e Computacional, TEMA 3(2) (2002) 217-226.
M.A. Trindade, C. Wolter, R. Sampaio, Karhunen-Loève decomposition of coupled axial-bending vibrations of beams subject to impacts, Journal of Sound and Vibration 279 (3-5) (2004) 1015-1036.
C. Soize, A nonparametric model of random uncertainties for reduced matrix model in structural dynamics, Probabilistic Engineering Mechanics 15(30 (2000) 277-294.
C. Soize, Random matrix theory for modeling uncertainties in computational mechanics, Computer Methods in Applied Mechanics and Engineering 194 (12-16) (2005) 1333-1366.

8. Responsibility notice

The author(s) is (are) the only responsible for the printed material included in this paper