Finite Element Formulation Applied to Linear Stability Analysis of Viscous Flows

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Abstract. *Linear hydrodynamic stability of a laminar flow may be determined by slightly disturbing the flow and tracking the fate of that disturbance. It may die away, persist as a disturbance of similar magnitude, or grow indefinitely leading to a different laminar flow state, to a transient flow, or even to a turbulent flow. Linear stability formulation leads to an eigenproblem where the eigenvalues correspond to the rate of growth of the disturbances and the eigenfunction to the amplitude of the perturbation. Most of the literature on linear stability analysis uses stream function formulation and spectral methods, with domain spanning basis functions, to descritize the governing equations. Therefore, the results are restricted to simple domains. In this work, linear stability analysis is formulated in terms of primitive variables (velocity and pressure) and the finite element method is used to solve the equations that describe the evolution of the perturbation. This approach leads to a large, complex, non-Hermitian and ill-conditioned generalized eigenvalue problem. It is important to recognize the eigenvalues that are related to physical disturbances and those that are related to features introduced in the matrix eigenproblem by the discretization process. Plane Couette flow was used as a prototype problem to analyze the behavior of the spectrum as a function of the flow parameters, method to solve the equations and the level of discretization.*

Keywords: *stability analysis, eigenproblem, finite element method, Couette flow.*

1. Introduction

Linear stability analysis is the usual first step taken to investigate the physical realizability of a laminar flow, although nonlinear effects will generally lead to a discrepancy of linear theory and experimental observations of instability in real systems. Plane Couette flow is always stable according to linear theory (see Drazin[2]), although transition occurs in experiments, as discussed by Tillmark and Alfredsson 1992 [8].

The hydrodynamic stability theory applied on pure shear flows of viscous, Newtonian incompressible fluids leads to the Orr-Sommerfeld equation. The base flow is the solution of the steady state Navier Stokes equations. The disturbances imposed on the base flow have the form of travelling waves (with wave numbers α and β) whose amplification with respect to time is investigated in the framework of linearized equations. The task is to determine the growth rate σ of these disturbances, because the real part of the temporal variation of the disturbances is written as $e^{\mathcal{R}(\sigma)t}$. If $\mathcal{R}(\sigma) > 0$, the perturbations grow with time and the flow is unstable. Both analytical and numerical approaches have been used to solve Orr-Sommerfeld equation, for exemple Orszag 1971[6], Dongarra *et.al* 1991[3].

In this work, instead of using stream function formulation (Orr-Sommerfeld equation) as done in literature, the stability analysis is formulated in terms of primitive variables (velocity and pressure). In the literature, the most common methods to discretize this equation are spectral methods, for example Chebyshev-τ. Therefore, the results are restricted to simple domains, where domain spanning functions can be easily defined. Here, finite element method is used to discretize the governing equations. The main advantages that this approach can be extended to complex domains.

Both discretizations lead to a generalized eigenvalue problem (GEP). There are computational advantages in using finite element formulation, since the matrices $\mathbf{J}$ and $\mathbf{M}$ of the GEP ($\mathbf{J}\mathbf{x} = \sigma\mathbf{M}\mathbf{x}$) are sparses; they are fully occupied matrices when spectral method is used. QZ algorithm of Moler and Stewart 1973[4] was chosen to solve the GEP.

Dongarra *et.al* 1996[3], that used Chebyshev-τ method to discretize the Orr-Sommerfeld equation and QZ method to solve the GEP, reported that the singularity of $\mathbf{M}$ might account for the appearance of the spurious eigenvalues. It is important to recognize the eigenvalues that are related to physical disturbances and those that are related to features introduced in the matrix eigenproblem by the discretization process. This work shows that the eigenvectors may give indications of the trustworthiness of their related eigenvalues.

2. Mathematical Formulation

2.1 Steady State Flow and Linear Stability Analysis

The flow considered in this work is the flow between two infinite parallel plates located at $y = \pm 1$ which are moving in the x-direction with velocities $u = \pm 1$. To analyze the stability of any laminar flow its necessary to calculate the pressure p_0 and velocity $\mathbf{u}_0$ fields that define the base flow, which is steady.

$$Re\,[(\mathbf{u_0}\cdot\nabla)\mathbf{u_0}] = -\nabla p_0 + \nabla\cdot\mathbf{T_0} \quad \text{and} \quad \nabla\cdot\mathbf{u_0} = 0, \tag{1}$$

where $Re = \frac{UL\rho}{\mu}$ is the Reynolds number, the dimensionless parameter that characterizes the ratio of inertial to viscous forces. U and L are suitable units of velocity and length, and $\mathbf{T_0} = (\nabla\mathbf{u_0}) + (\nabla\mathbf{u_0})^T$ is the viscous stress tensor.

Boundary conditions of the base flow are: $u_0(\pm1) = \pm1,\ v_0(\pm1) = 0$, where $\mathbf{u_0} = (u_0, v_0, 0)$.

Plane Couette flow is the trivial solution:

$$\mathbf{u_0} = (y, 0, 0), \qquad \text{and} \qquad p_0 = 0. \tag{2}$$

The stability of the flow can be studied after introducing infinitesimal perturbations to the base flow. The equations may be linearized for sufficiently small disturbances and because they are small enough they can be represented as a linear combination of a complete basis set of linearly independent normal modes. Most convenient, in the present case, is a set of Fourier modes. Thus, the disturbed fields are written as the sum of the base state and an infinitesimal perturbation:

$$\begin{aligned}
\mathbf{u}(\mathbf{x},t) &= \mathbf{u_0} + \epsilon\,\mathbf{u}'(y)\,e^{i(\alpha x+\beta z)+\sigma t},\\
p(\mathbf{x},t) &= p_0 + \epsilon\,p'(y)\,e^{i(\alpha x+\beta z)+\sigma t}.
\end{aligned}$$

$\mathbf{u_0}$ and p_0 are the velocity and pressure of the base flow, which is known. $\alpha, \beta \in R$ are the wavenumbers in the x and z directions respectively ($\beta = 2\pi/\lambda$, where λ is the wavelength of the perturbation in z direction); $\sigma \in C$ is the growth factor. If $\mathcal{R}(\sigma) > 0$, the disturbances grow with time and the flow is said to be unstable.

On substituting these expressions into the equations that describe the transient flow and neglecting $\mathcal{O}(\epsilon^2)$ terms, the linearized equations that describe the amplitude of perturbations are obtained:

$$i\alpha u' + \frac{dv'}{dy} + i\beta w' = 0. \tag{3}$$

$$Re\,[(\sigma + i\alpha y)u' + v'] = -i\alpha p' + \frac{d^2u'}{dy^2} - (\alpha^2+\beta^2)u',$$

$$Re\,[(\sigma + i\alpha y)v'] = -\frac{dp'}{dy} + \frac{d^2v'}{dy^2} - (\alpha^2+\beta^2)v',$$

$$Re\,[(\sigma + i\alpha y)w'] = -i\beta p' + \frac{d^2w'}{dy^2} - (\alpha^2+\beta^2)w',$$

note that it is linear in the unknown fields: $\mathbf{u}'(y) = (u'(y), v'(y), w'(y))$ and $p'(y)$.

The boundary conditions are:

$$\mathbf{u}'(y = \pm1) = \mathbf{0}. \tag{4}$$

The perturbation fields ($\mathbf{u}' = (u'(y), v'(y), w'(y)), p'$) and the rate of growth (σ) can be found by applying Galerkin's weighted residual method to Eqs.(3) - **Formulation 3D**.

However this three-dimensional problem can be reduced to an equivalent two-dimensional problem using the transformation first introduced by Squire,1933[7]:

$$(\alpha^*)^2 = \alpha^2 + \beta^2, \quad \sigma^* = \sigma, \quad \frac{p'^*}{\alpha^*} = \frac{p'}{\alpha}$$

$$\alpha^* u'^* = \alpha u' + \beta w', \quad v'^* = v', \quad \alpha^* Re^* = \alpha Re.$$

Then using the transformation in the equations (3):

$$i\alpha^* u'^* + \frac{dv'^*}{dy} = 0, \tag{5}$$

$$Re^* \left[(\sigma^* + i\alpha^* y) u'^* + v'^* \right] = -i\alpha^* p'^* + \frac{d^2 u'^*}{dy^2} - (\alpha^*)^2 u'^*, \tag{6}$$

$$Re^* \left[(\sigma^* + i\alpha^* y) v'^* \right] = -\frac{dp'}{dy} + \frac{d^2 v'^*}{dy^2} - (\alpha^*)^2 v'^*. \tag{7}$$

with the same boundary conditions (4).

Again the unknown perturbation fields ($\mathbf{u}'^* = (u'^*(y), v'^*(y)), p'^*$) and the rate of growth (σ^*) can be found by applying Galerkin's weighted residual method to Eqs. (5) to (7) - **Formulation 2D**.

These equations define the equivalent two-dimensional problem. In particular, since $\alpha^* \geq \alpha$ it immediately follows that $Re^* \leq Re$ and hence there is:

Squire Theorem. To obtain the minimum critical Reynolds number it is sufficient to consider only two dimensional disturbances.

Using continuity ($u'^* = \frac{i}{\alpha} v'^*$) and manipulating Eqs.(6) - (7):

$$Re^* \left(\frac{i\sigma^*}{\alpha^*} - y \right) \left[\frac{d^2 v'^*}{dy^2} - {\alpha^*}^2 v'^* \right] - i{\alpha^*}^3 v'^* + 2i\alpha^* \frac{d^2 v'^*}{dy^2} - \frac{i}{\alpha^*} \frac{d^4 v'^*}{dy^4} = 0. \tag{8}$$

In this context, it is natural to introduce a stream function - ψ, such that

$\widehat{u} = \frac{\partial \psi}{\partial y}$ and $\widehat{v} = -\frac{\partial \psi}{\partial x}$, where, $\widehat{\mathbf{u}} = \mathbf{u}'(y) e^{i(\alpha x + \beta z) + \sigma t}$ and $\psi = \phi(y) e^{i(\alpha x + \beta z) + \sigma t}$,

so $u' = \frac{d\phi}{dy}$; $v' = -i\alpha\phi$.

It follows that $\phi(y)$ satisfies the well known **Orr-Sommerfeld equation**:

$$(i\alpha y + \sigma)(\frac{d^2}{dy^2} - \alpha^2)\phi = \frac{1}{Re}(\frac{d^2}{dy^2} - \alpha^2)^2 \phi. \tag{9}$$

Most of the literature, which studies linear stability analysis of parallel flows, discretizes Orr-Sommerfeld equation using spectral methods. Therefore, the results are restricted to simple domains beyond others difficulties. For example, operating at high Reynolds number the problem mandates the use of a large spectral orders to guarantee scale resolution and reliable approximations of the eigenvalues.

In the present work the **formulations 2D** and **3D** are used instead of the Orr-Sommerfeld equation and, as it has already been mentioned, Galerkin's weighted residual method is applied to descretize the resulting differential equations.

2.2 Galerkin's Finite Element Formulation

To apply Galerkin's weighted residual method, the weighting functions used for the momentum and continuity equations are the same basis function used to expand the velocity and pressure perturbed fields, respectively. They are $\phi_j(y)$ and $\chi_j(y)$, which are a piecewise quadratic and a linear discontinuous polynomial functions.

For the case of **Formulation 3D** , the weighting residual equations of continuity and momentum are

$$R_c^k = \int_{-1}^{1} (i\alpha u'_h + \frac{dv'_h}{dy} + i\beta w'_h) \, \chi_k \, dy = 0,$$

$$R_{mx}^k = \int_{-1}^{1} \left(Re \left[(\sigma + i\alpha y) u'_h + v'_h \right] + i\alpha p'_h - \frac{d^2 u'_h}{dy^2} + (\alpha^2 + \beta^2) u'_h \right) \, \phi_k \, dy = 0,$$

$$R_{my}^k = \int_{-1}^{1} \left(Re \left[(\sigma + i\alpha y) v'_h \right] + \frac{dp'_h}{dy} - \frac{d^2 v'_h}{dy^2} + (\alpha^2 + \beta^2) v'_h \right) \, \phi_k \, dy = 0,$$

$$R_{mz}^k = \int_{-1}^{1} \left(Re \left[(\sigma + i\alpha y) w'_h \right] + i\beta p'_h - \frac{d^2 w'_h}{dy^2} + (\alpha^2 + \beta^2) w'_h \right) \, \phi_k \, dy = 0.$$

Where, ${\mathbf{u}'}_h(\mathbf{x}) = \sum_{j=1}^{M} \mathbf{U}_j \phi_j$ and $p'_h(\mathbf{x}) = \sum_{j=1}^{N} P_j \chi_j$.

As an example, the continuity and the x-component of the perturbed momentum weighted residual after substituting the expansions for the velocity and pressure fields are shown:

$$\bullet\ R_c^k = \textstyle\sum_j U_j \underbrace{\left[\int_{-1}^{1} i\alpha\phi_j\, \chi_k\, dy\right]}_{\frac{\partial R_c^k}{\partial U_j}} + \sum_j V_j \underbrace{\left[\int_{-1}^{1} \frac{d\phi_j}{dy}\, \chi_k\, dy\right]}_{\frac{\partial R_c^k}{\partial V_j}} + \sum_j W_j \underbrace{\left[\int_{-1}^{1} i\beta\phi_j\, \chi_k\, dy\right]}_{\frac{\partial R_c^k}{\partial W_j}};$$

$$\bullet\ R_{mx}^k = \textstyle\sum_j U_j \underbrace{\overset{\#}{\left[\sigma \int_{-1}^{1} Re\phi_j\, \phi_k\, dy\right]}}_{\frac{\partial R^k{}_{mx}}{\partial U_j}} + \sum_j U_j \underbrace{\overset{\#}{\left[\int_{-1}^{1} \left((Rei\alpha y + \alpha^2 + \beta^2)\phi_j\, \phi_k + \frac{d\phi_j}{dy}\frac{d\phi_k}{dy}\right)\, dy\right]}}_{\frac{\partial R^k{}_{mx}}{\partial U_j}} +$$

$$+ \textstyle\sum_j V_j \underbrace{\left[\int_{-1}^{1} Re\phi_j\, \phi_k\, dy\right]}_{\frac{\partial R_{mx}^k}{\partial V_j}} + \sum_j P_j \underbrace{\left[\int_{-1}^{1} i\alpha\chi_j\, \phi_k\, dy\right]}_{\frac{\partial R_{mx}^k}{\partial P_j}} = 0.$$

note that the temporal contribution, where the growth factor appears, can be separated.

In matrix vector form, the set of algebraic equations that governs the perturbations is

$$(\sigma\mathbf{M} + \mathbf{J})\,\mathbf{c}' = 0$$

where $\mathbf{c}'$ is the column vector of coefficients of the finite element expansions of the perturbed velocity and pressure. $\mathbf{M}$, which contains the temporal contribution, is called mass matrix and $\mathbf{J}$ is the jacobian matrix. They are the matrices of sensitivities of the weighted residuals with respect to the unknown coefficient of the perturbations.

It follows that the discretization of the perturbed equations give rise to a generalized, non-Hermitian and ill-conditioned eigenproblem.

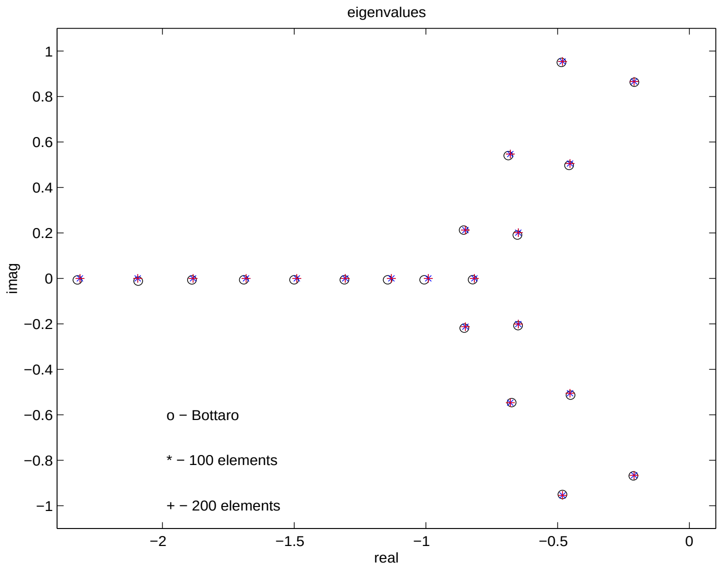
Repeating the same procedure for the **Formulation 2D** a smaller, but also generalized, non-Hermitian and ill-conditioned eigenproblem is also obtained.

The eigenproblem resulting from the most popular spectral discretization (Chebyshev-τ method) has the unfortunate property that the matrices $\mathbf{M}$ and $\mathbf{J}$ are full, in contrast to Galerkin residual method that leads to sparse matrices.

One approach to solve these generalized eigenvalue problems is the $\mathbf{QZ}$ algorithm of Moler and Stewart 1933 [4]. This algorithm relies on the fact that there exist unitary matrices $\mathbf{Q}$ and $\mathbf{Z}$ such that $\mathbf{QJZ}$ and $\mathbf{QMZ}$ are both upper triangular. The eigenvalues σ_i are obtained from the relation $\sigma_i = j_i/m_i$, where j_i and m_i are the diagonal elements of $\mathbf{QJZ}$ and $\mathbf{QMZ}$. As well the $\mathbf{QZ}$ algorithm yields the coefficient os eigenvectors, which can be used as an useful convergence indicator, as we will see in the results discussion.

3. Results and Discussions

A recent work from Bottaro 2003 [1] has been chosen to validate the results from the present work. The predictions were obtained using **formulation 2D** with the same parameters of the Bottaro's work: Reynolds number $Re = 500$, wavenumber $\alpha = 1.5$. In this case, the Reynolds number is not in a problematic range, the literature reports difficulties in calculating the spectrum for Reynolds number above 2000 [3]. All spectrum shown by Bottaro almost coincides with the one predicted here, as shown in Figure 1.



present work's eigenvalues
-2.77819312 0.00000000i
-2.54037680 -0.00000000i
-2.31287712 -0.00000000i
-2.09355520 -0.00000000i
-1.88392581 0.00000001i
-1.68262023 -0.00000002i
-1.49019099 0.00000004i
-1.30610088 -0.00000006i
-1.13087541 - 0.00000001i
-0.99098559 + 0.00000000i
-0.85016395 ± 0.21248353i
-0.81598348 + 0.00000001i
-0.67961571 ± 0.54693083i
-0.64959440 ± 0.20097982i
-0.48117767 ± 0.95374072i
-0.45305206 ± 0.50507735i
-0.20983820 ± 0.86608633i

Figure 1. Bottaro's eigenvalues × present work

A mesh convergence analysis is also shown in Figure 1. The eigenvalues computed with 100 and 200 elements are virtually the same, at least for the leading part of spectrum.

To check the performance of finite element formulation in a high Reynolds number range, the **formulation 2D** was also used and the parameters were chosen in order to compare with Dongarra's work [3]: Reynolds number $Re = 13000$, wavenumber $\alpha = 1$. Table 1 presents a comparison between the results obtained in this analysis and that of Dongarra's. The agreement is, again, excellent. The finite element formulation with primitive variables was able to accurately compute the eigenvalues even at hight Reynolds number.

Table 1. The first 14 pares eigenvalues compared with Dongarra's results [3].

present work		Dongarra [3]	
-0.04751538	± 0.82761528 i	-0.04751548439	± 0.8276152337 i
-0.10918556	± 0.73181689 i	-0.1091860424	± 0.7318167785 i
-0.12791484	± 0.86944857 i	-0.1279149536	± 0.8694486153 i
-0.15939916	± 0.65168062 i	-0.1594003003	± 0.6516804277 i
-0.18051597	± 0.76711854 i	-0.180516493	± 0.7671186628 i
-0.20355520	± 0.58015697 i	-0.203557283	± 0.5801567166 i
-0.22517343	± 0.68283696 i	-0.2251746419	± 0.6828371673 i
-0.24376429	± 0.51439983 i	-0.2437675825	± 0.5143995235 i
-0.26534591	± 0.60824053 i	-0.2653481107	± 0.6082408213 i
-0.28111986	± 0.45289403 i	-0.2811241939	± 0.4528935800 i
-0.30246438	± 0.54002158 i	-0.3024678732	± 0.5400219613 i
-0.31631320	± 0.39466744 i	-0.3162828159	± 0.3947096982 i
-0.33730415	± 0.47644893 i	-0.3373108149	± 0.4764491821 i

Studying the eigenvectors, it was noticed that the oscillation frequency of the eigenvectors rises as the eigenvalues move away from the imaginary axis, as indicated in Figure 2.

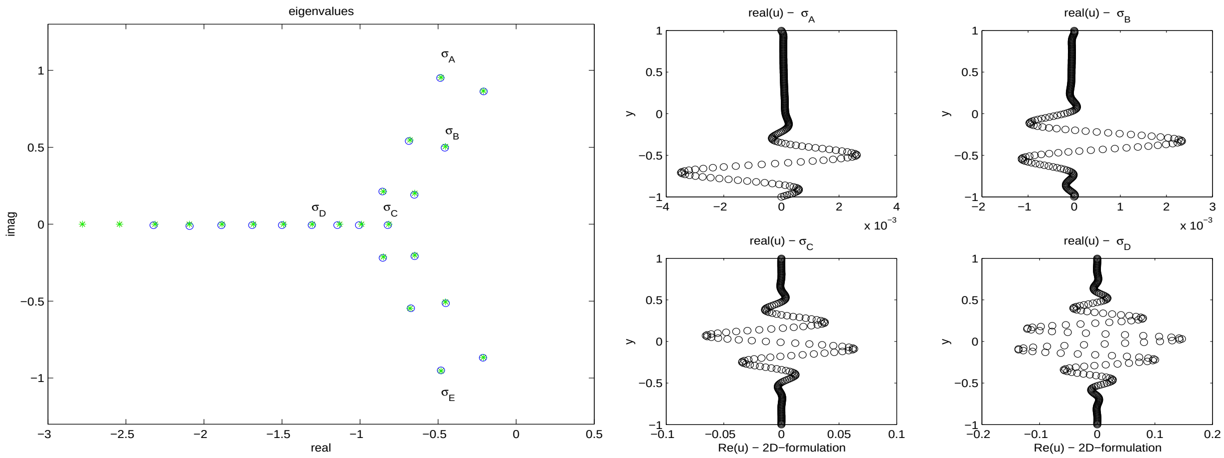


Figure 2. eigenvalues and eigenvectors

When the eigenvector oscillation frequency becomes very hight, the discretization may not be enough to describe the eigenfunction. In this case the computed eigenvalues are not correct and they are a function of the discretization. Figure 3 shows the eigenvalues far from the imaginary axis. It is clear that in this part of the spectrum the solution is not converged with the discretization. However, this is not important to determine the stability of the flow, which is determined by the leading eigenvalues of the spectrum.

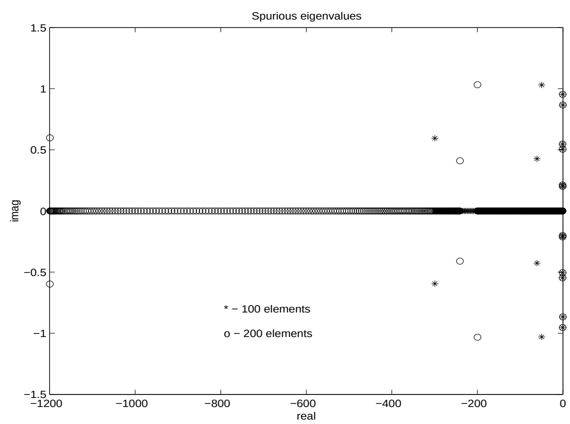


Figure 3. Spurious eigenvalues

For each eigenvalue with imaginary part different from zero, its conjugate is still an eigenvalue and the eigenvectors associated are a reflection of each other. Figure 4 shows the real part of the velocity amplitude associated with the complex conjugate eigenvalues σ_A and σ_E shown in Figure 2.

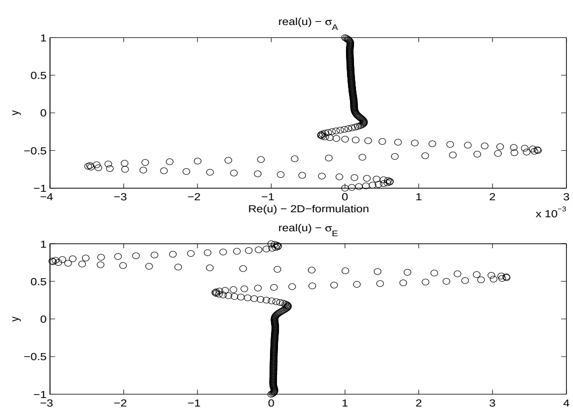


Figure 4. eigenvectors

The spectrum predicted by the two formulations presented in this work are compared in Figure 5; which shows the leading eigenvalues computed at $Re = 500$, $\alpha = 1.5$ and $\beta = 0$ with **formulations 2D** and **3D**.

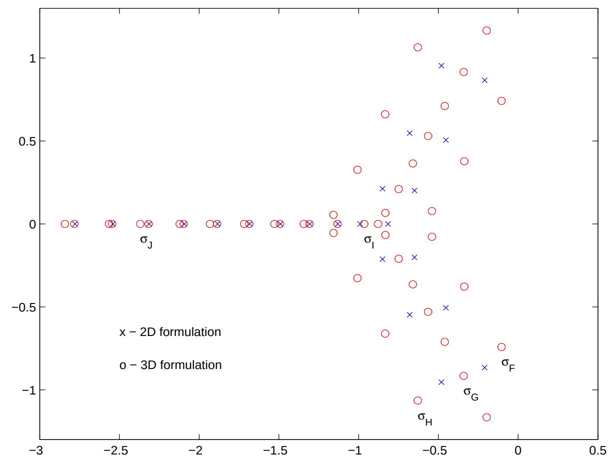


Figure 5. eigenvalues

Qualitatively the two formulations are equivalent although the spectrum is not the same. Note that in **3D formulation** there are three branches of eigenvalues near imaginary axis against two branches in the spectrum of **2D formulation**.

Observing the eigenvectors predicted by the **3D formulation**, the independence of w (velocity in z direction) with regard to u and v (velocity in x and y directions) is evident. The eigenvalues at the middle branch (σ_G) correspond to perturbations of the transversal velocity component (z direction), thus $u = v = 0$. The eigenvalues of other two branches are associated with perturbations on the plane xy, thus $w = 0$ for σ_F and σ_H in Figure 5.

As in **2D formulation** the oscillation frequency of the eigenvectors rises as the eigenvalues move away from the imaginary axis: compare σ_I's eigenvector with σ_J's eigenvector in Figure 6

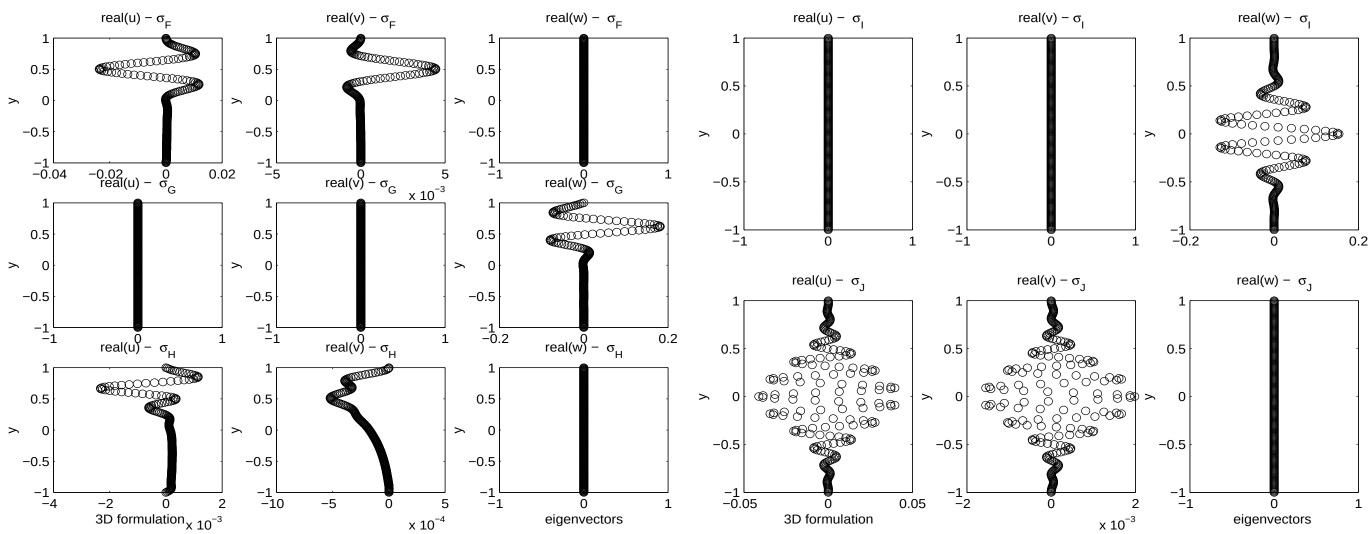


Figure 6. eigenvalues and eigenvectors

4. Concluding Remarks

In this work, a formulation for a linear stability analysis using primitive variables (velocity and pressure) in the system of governing equations is proposed. The differential equations that describe the evolution of the perturbation are discretized by Galerkin's finite element method. The results were compared with the literature, which uses a spectral method to discretize the Orr-Sommerfeld equation, with the stream function as the independent variable. With finite element method, more general geometries can be studied. The agreement with the literature was excellent and less computational effort is necessary using the finite element formulation.

An enlightening analysis involving eigenvectors was also shown. The eigenvectors related to the leading eigenvalues present a smooth oscillation and these eigenvalues have converged with the discretization. The oscillation frequency of the eigenvectors rise as the eigenvalue moves away from the region of interest for stability, the imaginary axis. When these frequency are larger then the mesh resolution; to capture all information it is necessary more elements, a richest mesh.

5. References

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