

AN ANALYTICAL SOLUTION FOR THE SECOND ORDER MATRIX GITT TRANSFORMED DIFFERENTIAL EQUATION BY THE LAPLACE TRANSFORM TECHNIQUE

Heinen, Ismael R.

Universidade Federal do Rio Grande do Sul – PROMEC, Sarmento Leite, 425 – Porto Alegre – RS
heire@bol.com.br

Wortmann, Sérgio

Universidade Federal do Rio Grande do Sul – PPGMAP, Bento Gonçalves, 9500 – Porto Alegre – RS
wortmann@mat.ufrgs.br

Vilhena, Marco T.M.B.

Universidade Federal do Rio Grande do Sul – PPGMAP, Bento Gonçalves, 9500 – Porto Alegre – RS
vilhena@mat.ufrgs.br

Cotta, R.M.

Universidade Federal do Rio de Janeiro – COPPE - LTTC – Cidade Universitária - Rio de Janeiro – RJ
cotta@serv.com.ufrj.br

Abstract. *In this work, we report the recent advance in the GILTT technique, regarding the issue of solving analytically the linear second order ordinary GITT transformed equation. This kind of problems usually appears in the GITT solution of steady-state problems. The main idea consists in the order reduction of the transformed problem into a linear first order ordinary differential equation and solution of the resulting problem, analytically, by the Laplace transform technique. We specialize the application to the solution of the linear steady-state two-dimensional energy equation in parallel-plane channel. We also present numerical simulation and comparisons with available results in the literature.*

Keywords: *GITT, Laplace Transform Technique, Energy Equation, Order Reduction, GILTT.*

1. Introduction

Recently appeared in the literature the GILTT approach, which is a new version of the GITT method, in the sense it solves analytically the GITT transformed linear problems and also the nonlinear problems, in a semi-analytical manner. For the linear problems, the key feature of the GILTT technique consists in the solution of the first order linear matrix transformed differential equation by the Laplace transform and matrix diagonalization approaches (Wortmann, 2003), (Buske, 2004). On the other hand, the second order matrix transformed differential equation has been solved by the method of order reduction to a first order differential matrix equation (Heinen, 2005). At this point, we must emphasize that with the GILTT method, due the analytical character of the solution, it is possible to handle problems with a large number of eigenfunctions (up to 1500). Furthermore, we must recall that this methodology has only be applied to problems in fluid mechanics (Thum *et al.*, 1998), (Wortmann, 2003), and atmospheric pollution (Buske, 2004), (Wortmann *et al.*, 2003), whose transformed equations are first order linear matrix differential equation. For more detail see the work of (Wortmann *et al.*, 2005).

In this work, we step forward by solving a problem, whose transformed equation is a second order matrix differential equation. To this end, we specialize the application of the GILTT method in the solution of the linear steady-state, two-dimensional energy equation in a plane-parallel channel. To hit this objective, we outline the paper as follows: in section 2, we report the GILTT solution for the two-dimensional, steady-state, forced convection problem in a plane-parallel channel with a fully developed laminar velocity profile. In section 3, we present numerical simulations and comparisons with available results in the literature.

2. The GILTT Solution

In order to solve a second order transformed matrix differential equation problem by the GILTT method, let us consider the problem:

$$U(y) \frac{\partial \theta(x, y)}{\partial x} = \frac{1}{Pe} \left(\frac{\partial^2 \theta(x, y)}{\partial x^2} + \frac{\partial^2 \theta(x, y)}{\partial y^2} \right) \quad (1)$$

for $0 < x < \infty$ and $0 < y < 1$, subject to the boundary conditions:

$$\theta(0,y)=1 \quad (1a)$$

$$\left. \frac{\partial \theta(x,y)}{\partial x} \right|_{x=R} = 0 \quad (1b)$$

$$\left. \frac{\partial \theta(x,y)}{\partial y} \right|_{y=0} = 0 \quad (1c)$$

$$\theta(x,1)=0 \quad (1d)$$

Here, we assume that the speed U has the form:

$$U(y) = \frac{3}{2}(1-y^2). \quad (1e)$$

Focusing our attention to the issue of getting the transformed equation, in what follows, we apply the standard procedure of the GITT method, identical to the GILTT, to obtain the equation:

$$\frac{d^2 \bar{\Theta}(x)}{dx^2} + F \frac{d \bar{\Theta}(x)}{dx} + G \bar{\Theta}(x) = 0 \quad (2)$$

subject to the boundary conditions

$$\bar{\Theta}(0) = Q \quad (2a)$$

$$\left. \frac{d \bar{\Theta}(x)}{dx} \right|_{x=R} = 0 \quad (2b)$$

where F and G are $(M \times M)$ matrices with constant entries and $\theta(x)$ is the unknown column vector with M components of the truncated GITT series expansion. For the complete description of the mentioned matrices and vector, see the work of Heinen (2005).

According the work of Wortmann *et al.* (2005) we recast, the problem (2) using the reduction order procedure and we obtain:

$$\frac{d Z(x)}{dx} + H \cdot Z(x) = 0 \quad (3)$$

now subject to the initial condition:

$$Z(0) = \begin{bmatrix} Q \\ 0 \end{bmatrix} \quad (3a)$$

Here, $Z(X)$ is the unknown vector with $2M$ components, H is a $(2M \times 2M)$ block matrix and Q is a known vector. Again these matrix and vector are defined by (Heinen, 2005). At this point, appears the difference between the GITT and GILTT approaches. Indeed, the GILTT method solves Eqs. (3) analytically by the Laplace transform technique, meanwhile the GITT solves it numerically. In what follows, we briefly report the GILTT solution. Therefore, applying the Laplace transform technique and performing the diagonalization of the matrix H , we read:

$$\bar{Z} = X(s \ I + D)^{-1} X^{-1} Z(0) \quad (4)$$

Where $\bar{Z}$ is the Laplace transform of Z respect to x variable, I is the identity matrix, X the eigenfunction matrix and D the diagonal matrix of eigenvalues. Using standard results of the Laplace transform technique to perform the inversion, we promptly obtain

$$Z(x) = X B X^{-1} Z(0) \quad (5)$$

Finally the solution for problem (4), has the form

$$\theta(x, y) = \sum_{i=1}^M \frac{1}{N_i^{1/2}} \overline{t_i(x)} \psi_i(y) \quad (6)$$

where $\overline{t_i(x)}$, $i = 1 : M$ are the M first components of the vector $Z(x)$.

3. Numerical Results and Conclusions

In order to check the aptness of the GILTT method, under computational point of view, to solve second order linear matrix differential equation, in the sequel, we present numerical results for the two-dimensional, steady-state convection problem in a plane-parallel channel with a fully developed laminar flow, at different positions and the Péclet number take the values 1, 1440 and 2000. To illustrate the numerical convergence of the results attained with the increasing of the eigenfunction number, in table 1, we present numerical simulations for the bulk temperature, for M ranging from 5 to 60.

Tabela 1. Numerical convergence of the bulk temperature.

Pe=1440			Number	of	eigenvalues		
x	5	10	20	30	40	50	60
0,04608	0,999574	0,999470	0,999114	0,998979	0,998962	0,998961	0,998961
0,46080	0,997240	0,995510	0,994677	0,994663	0,994662	0,994661	0,994661
4,60800	0,978066	0,975243	0,975088	0,975078	0,975077	0,975077	0,975077
46,0800	0,886775	0,886138	0,886060	0,886055	0,886055	0,886055	0,886055
460,800	0,498347	0,498084	0,498053	0,498051	0,498051	0,498051	0,498051
4608,00	0,002185	0,002184	0,002184	0,002184	0,002184	0,002184	0,002184

To validate the results obtained, in Tab. 2, we display the numerical results achieved for the bulk temperature and comparisons with the ones got by Shah (1978) and Heinen (2005) neglecting the longitudinal diffusion, for the Péclet numbers: 1 and 2000. It is important to mention that we adopt the notation θ_B for bulk temperature, θ_s for the Shah results and θ_H for the Heinen results neglecting the longitudinal diffusing.

Table 2. Comparison between the bulk temperatures to the two energy equations.

Pe = 1					
x	θ_B	θ_s	θ_H	$ \theta_B - \theta_s $	$ \theta_B - \theta_H $
0,02304	0,97356	0,99926	0,90814	0,02570	0,06542
0,18432	0,81175	0,99705	0,64415	0,18530	0,16760
3,45600	0,02589	0,97937	0,00134	0,95348	0,02455
11,5200	0,00000	0,95425	0,00000	0,95425	0,00000
23,0400	0,00000	0,92774	0,00000	0,92774	0,00000
92,1600	0,00000	0,82065	0,00000	0,82065	0,00000
184,320	0,00000	0,71860	0,00000	0,71860	0,00000
921,600	0,00000	0,27241	0,00000	0,27241	0,00000
1843,20	0,00000	0,08152	0,00000	0,08152	0,00000
4608,00	0,00000	0,00218	0,00000	0,00218	0,00000
Pe=2000					
0,02304	0,99950	0,99926	0,99940	0,00024	0,00010
0,18432	0,99770	0,99705	0,99763	0,00065	0,00007
3,45600	0,98343	0,97937	0,98340	0,00406	0,00003
11,5200	0,96319	0,95425	0,96317	0,00894	0,00002
23,0400	0,94181	0,92774	0,94180	0,01407	0,00001
92,1600	0,85530	0,82065	0,85529	0,03465	0,00001
184,320	0,77259	0,71860	0,77258	0,05399	0,00001
921,600	0,38190	0,27241	0,38189	0,27242	0,00001
1843,20	0,16020	0,08152	0,16020	0,07868	0,00000
4608,00	0,01182	0,00218	0,01182	0,00964	0,00000

Finally, in Fig. 1, in order to show the temperature behavior, we plot the temperature distribution for $Pe = 1$ and 2000.

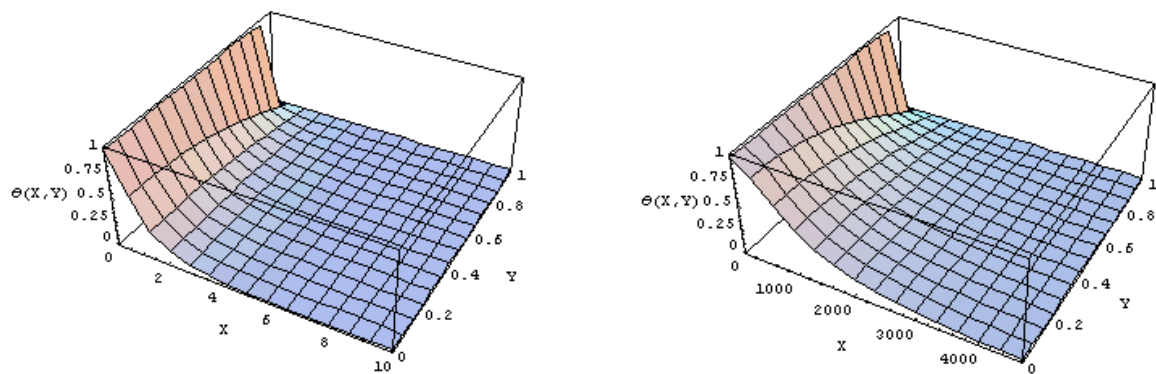


Figure 1. Temperature distribution: a) $Pe = 1$, b) $Pe = 2000$.

Given a closer look to the results reported in Tab. 1 and Tab. 2, we promptly realize the numerical convergence of the GILTT solution, in the sense that the prescribed accuracy of 6 significant digits was obtained with $M = 60$. Furthermore, as expected the relative difference between the solution presented decrease when the Péclet number varies from 1 to 2000. Concluding, the above analysis shows the robustness of the GILTT method to solve problem involving second order linear matrix differential equation

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