AN H ADAPTIVE HIERARCHICAL PANEL METHOD FOR SOLVING A BIDIMENSIONAL POTENTIAL SUBSONIC FLOW

Raul Bernardo Vidal Pessolani
Universidade Federal Fluminense, Deto de Eng Mecânica
Rua Passos da Patria, 156, CEP 24220-000, Niterói, Rio de Janeiro
raul@vm.uff.br

Rodrigo Guimarães Moraes
Universidade Federal Fluminense, Deto de Eng Mecânica
Rua Passos da Patria, 156, CEP 24220-000, Niterói, Rio de Janeiro
rodrigo_sir@yahoo.com.br

Abstract. *An H-adaptive hierarchical panels method is proposed for investigating the 2-D incompressible inviscid flow around arbitrary bodies. The method is based on a combination of distribution of source and doublet applied over the boundary and using Dirichlet boundary condition to form the final system of equations. Initially, the singularities distributions are quadratic also for the discretization of curved boundaries. Then, the H strategy, that increase collocation points, is applied where the solution is not satisfactory. The hierarchical concept accumulates the computations of each iteration, reducing substantially computation time. A local error estimator used is based on the residue equation of each element. Finally, based on one example, it can be shown the applicability of the method.*

Keywords*: Panels method, Subsonic flow, adaptive procedures.*

1. Introduction

Using the numerical techniques applications, it's possible to solve Laplace equation, which governs potential flow through panels method. The panels method consists on the solution of equations, which provides the velocity field around a given geometry, using singles panels distributed on the boundary element. Those techniques allow the treatment more realistic of the given geometry through boundary condition on the surface of the body. The results are more economic, from the computational point of view, than others methods that solve the flow with elements distributed on the whole volume of fluid.

On the other hand, the adaptive program represents an area of great interest for researchers. Those techniques can be seen as a way for automates procedures, seeking to reduce errors. The user provides the adaptive program with the minimum necessary information to define geometry, loads and boundary conditions. Then, the program tries to generate the best mesh distribution that gives the most accurate solution within a certain tolerance. This procedure has the advantage of reducing the processing time and the dimension of the equation system and resulting in a great computational economy. At the same time, adaptive strategy helps the engineering in the problem analysis, avoiding occasional evaluation errors in the mesh distribution.

This paper is concerned with the development of an adaptive hierarchical panels method formulation, which applies the concepts of adaptive developed in other areas, specially the hierarchical concept (Pessolani 2002), on a panels method based on the distribution of quadratic sources and wavelets on the boundary.

2. The panels method

The fundamental formulation to develop a solution for the equation, which determines the potential in any flow, it's based on considering a physical body submerged in a potential flow. The possibility of the flow be an incompressible ($\nabla \cdot \mathbf{V} = 0$, where $\mathbf{V}$ is the vector velocity) and irrotational field it's justified by the fact that the thickness of the boundary layer is very small. Therefore, the viscous effects don't influence in the calculation of the pressure and lifting coefficients on the surface. So, the Laplace equation is given by Eq. (1):

$$\nabla^2 \Phi^* = 0 \; ou \; \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \tag{1}$$

The solution domain in the Panels Method must be determined to obtain an equation at each point of the mesh. The method uses analytic solutions of the Laplace's equations through a combination of elementary flows, that involves the distribution of sources, doublets and vortex in the segment which define the surface of the body (or panels), shown in the Fig. 1. With the influence of a uniform flow (or free current), it's possible to obtain the final solution.

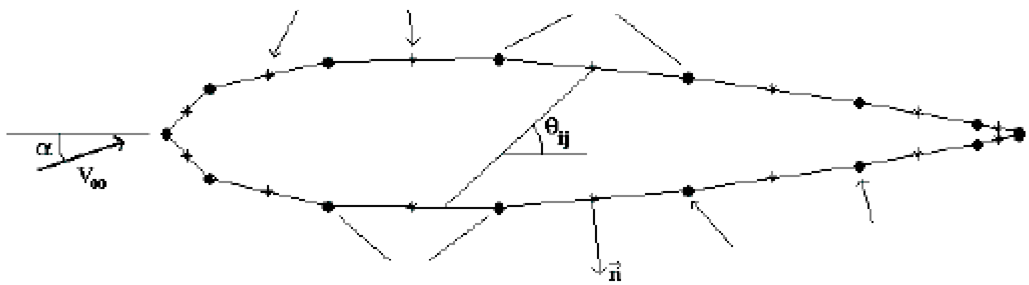

Figure 1. Panels on the body surface

Using the Green identity, it's possible to achieve the Potential general solution. This solution is given by the sum of a combined local distribution of doublets μ and sources σ, on each panel:

$$\Phi^*(x,y,z) = \frac{1}{4\pi} \int_{S+w} \mu n \cdot \nabla\left(\frac{1}{r}\right) dS - \frac{1}{4\pi} \int_S \sigma \left(\frac{1}{r}\right) dS + \Phi_\infty \tag{2}$$

where "S" represents on the body surface, w the wake, r is the vector position and n the vector normal outside the panel.

The Dirichlet boundary condition gives that the internal potential of the physical body is constant:

$$\Phi_i^* = \left(\Phi + \Phi_\infty\right)_i = const \tag{3}$$

Putting a point P(x,y,z) inside the body and using the Dirichlet's boundary condition, the internal potential (three-dimensional) in terms of the singularity distribution on the surface is given by Eq. (4):

$$\Phi_i^*(x,y,z) = \frac{1}{4\pi} \int_{S+w} \mu \cdot \frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS - \frac{1}{4\pi} \int_S \sigma \left(\frac{1}{r}\right) dS + \Phi_\infty = cons \tag{4}$$

Similarly, if the singularity distribution were placed inside a two-dimensional body, the boundary condition proposed by Dirichlet becomes:

$$\Phi_i^*(x,z) = \frac{1}{2\pi} \int_S \left[\sigma \ln r - \mu \frac{\partial}{\partial n}(\ln r)\right] dS + \Phi_\infty = const \tag{5}$$

To construct a numerical solution, the body must be divided into N panels and the integration is performed at each panel such that:

$$\sum_{j=i}^{N} \frac{1}{2\pi} \int_S \sigma \ln r \, dS - \sum_{j=1}^{N} \frac{1}{2\pi} \int_S \mu \frac{\partial}{\partial n}(\ln r) dS + \Phi_\infty = cons \tag{6}$$

For constant strength singularity elements on each panel, the influence of panel j at a point P is given by:

$$C_j \equiv -\frac{1}{2\pi} \int_S \frac{\partial}{\partial n}(\ln r) dS|_j \tag{7}$$

When a doublet and source distribution on each panel, the influence of panel j at a point P is given by:

$$B_j \equiv \frac{1}{2\pi}\int_S (\ln r) dS|_j \tag{8}$$

The Eq. (6) becomes

$$\Phi_i^* = \sum_{j=1}^{N} B_j \sigma_j + \sum_{j=1}^{N} C_j \mu_j + \Phi_\infty = const \tag{9}$$

where μ_j it's the doublet to be calculated, and the source strength is given by:

$$\sigma_j = \mathbf{n}_j \cdot \mathbf{Q}_\infty \tag{10}$$

Let be $\phi_i^* = \phi_\infty$. The equation becomes

$$\sum_{j=1}^{N} B_j \sigma_j + \sum_{j=1}^{N} C_j \mu_j = 0 \tag{11}$$

The coefficients B and C can either be calculated using numerical or analytical integration. Katz e Plotkin (2002) deduce the B and C analytically expressions:

$$B = \frac{1}{4\pi}\left\{(x-x_1)\ln\left[(x-x_1)^2+z^2\right]-(x-x_2)\ln\left[(x-x_2)^2+z^2\right] -2(x_2-x_1)+2z\left(\tan^{-1}\frac{z}{x-x_2}-\tan^{-1}\frac{z}{x-x_1}\right)\right\} \tag{12}$$

$$C = \frac{1}{2\pi}\left[\tan^{-1}\frac{z}{x-x_2}-\tan^{-1}\frac{z}{x-x_1}\right] \tag{13}$$

where x1, x2 and z are given in terms of local coordinates (Fig.1).

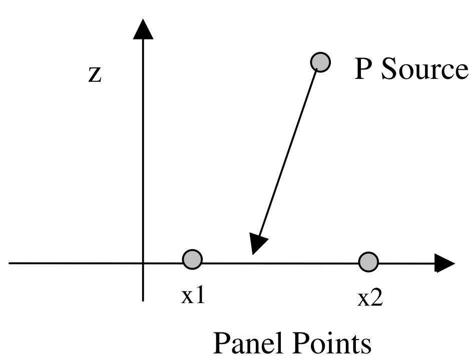

Figure 2. Local coordinates of each panel x_1, x_2, z

The wake influence is given by the Kutta condition, located at the trailing edge of the body. This can be seen below on the Figure 3:

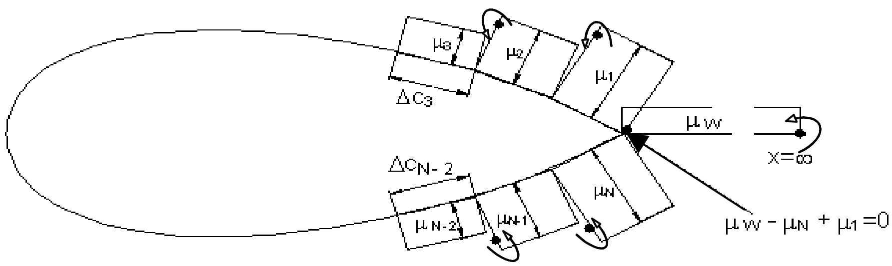

Figure 3. Constant-strength doublet panel elements near a body's trailing edge

Considering the Kutta's condition, the circulation at the trailing edge must be zero. The doublet in the panel of the wake added with the difference between the doublets of the first and last panels must be zero. The equation that represents this is given by:

$$\mu_w + (\mu_1 - \mu_N) = 0 \tag{14}$$

The elements c_{ii} of doublet matrix must be equal at 0.5, because they represent the performance strength of the doublet of the panel on itself. So, it's possible to reduce the order of the matrix C by relating the doublet in the wake panel with the doublet in the first and last panels. This relation is given by:

$$\begin{aligned} a_{ij} &= c_{ij}, \quad j \neq 1,N \\ a_{i1} &= c_{i1} - c_{iw}, \quad j=1 \\ a_{iN} &= c_{iN} + c_{iw}, \quad j = N \end{aligned} \tag{15}$$

where N is the last element matrix. The matrix system finally become:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1N} \\ b_{21} & b_{22} & \dots & b_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ b_{N1} & b_{N2} & \dots & b_{NN} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_N \end{pmatrix} = 0 \tag{16}$$

Once the strength of the doublets μ_j is known, the potential outside the surface can be calculated. Through of the boundary condition proposed by Dirichlet we arrive the following equation for the external potential, as shown in the start.

$$\Phi_u = \Phi_i - \mu \tag{17}$$

The local external tangential velocity component above each collocation point can be calculated by differentiating the velocity potential along the tangential direction, or be:

$$Q_t = \frac{\partial \Phi_u^*}{\partial l} \tag{18}$$

where l is the line along the surface.

The pressure coefficient can be calculated by using the following Eq. (19):

$$C_p = 1 - \frac{Q_t^2}{Q_\infty^2} \tag{19}$$

3. The h-adaptive hierarchical formulation

There are three types of adaptive formulations: The P adaptive technique increases the degree of the interpolation function in the critical areas, whereas H refines the mesh of elements, and HP is a combination of H and P techniques.

When the process is adaptive, several discretizations are necessary until the correct solution is found. Using conventional interpolation functions, like Lagrange's functions, any element subdivision or any increase of the number of collocations points requires that the system be reassembled with the calculation of all the coefficients in matrices "A" and "b" on the Fig. 4. For this reason, the classical method is not convenient to be applied to the adaptive procedure. An alternative approach uses hierarchical families of interpolation functions. Those functions are accumulative, that is, the higher order functions are generated without modifying lower order contributions. In the re-discretization, the new system is set up, increasing the previous system with the new collocation points

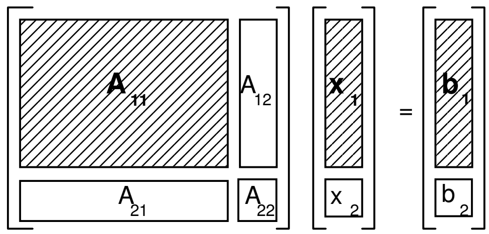

Figure 4. Resulting matrix

The H Adaptive hierarchical formulation consists in include intermediary collocation points in the element with interpolations functions of the same order of the previous, that can be the conventional Langrangeans. For a quadratic interpolation is defined a set of interpolations functions N_2, that is:

$$N_{2,kl} = \begin{cases} 0 & \text{if } \left|\eta - \eta_{2l}\right| \geq D_\eta / k \\ 1 - \dfrac{(\eta - \eta_{2l})^2}{\left(D_\eta / k\right)^2} & \text{if } \left|\eta - \eta_{2l}\right| \leq D_\eta / k. \end{cases} \tag{20}$$

where D_η is half the element to be refined in natural coordinates, k the number of requested divisions, η_{11} is the natural coordinate of the nodes used in the linear interpolation and η_{21} is the natural node coordinate used in the quadratic interpolation.

For each degree λ, $2^{\lambda-1}$ Bubbles functions are generated with the corresponding collocation points arranged symmetrically, equally spaced along the element. For example, the second hierarchical level would include collocation points according to the Fig 5.

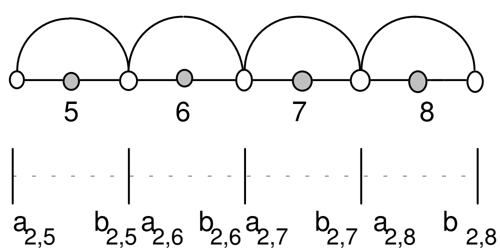

Figure 5. Second level of hierarchical discretization.

The approximate solution φ on the element is expressed in terms of the previous and the new values on the nodes:

$$\hat{\varphi} = \sum_{c=0}^{2} N_c(\xi)\hat{\varphi}_c + \sum_{c=3}^{4} N_c(\xi)\hat{\varphi}_c.. \tag{21}$$

Generically, let be the iteration λ, Λ the number of hierarchical levels and Π_λ the list of integers indicating which pair of hierarchical functions Π are being applied in the mesh (for each level λ there are $2^{\lambda-1}$ pairs). In iteration λ, it is:

$$N_{\lambda,n} = \begin{cases} N_{\lambda,2}(t_{\lambda,n}(\eta)) & \text{for } \eta \in [a,b]_{\lambda,n} \\ 0 & \text{otherwise} \end{cases}. \tag{22}$$

$$a_{\lambda,n} = -1 + (n-1)\frac{1}{2^{(\lambda-1)}} \tag{23}$$

$$b_{\lambda,n} = a_{\lambda,n} + \frac{1}{2^{(\lambda-1)}} \tag{24}$$

Also, $t_{\lambda,n}$ is the natural coordinate mapped in the interval $[a_{\lambda,n}, b_{\lambda,n}]$ and it is defined by:

$$t_{\lambda,n}(\eta) = 2^{\lambda}(\eta+1) - 2n + 1 \tag{25}$$

To accelerate the convergence rate it is quite convenient that the starting point of the adaptive scheme be a refinement that takes into account the points of singularity of the problem.

4. Example

An analysis of an uniform flow around a cylinder of 1m of radius is performed to see the applicability of the method.

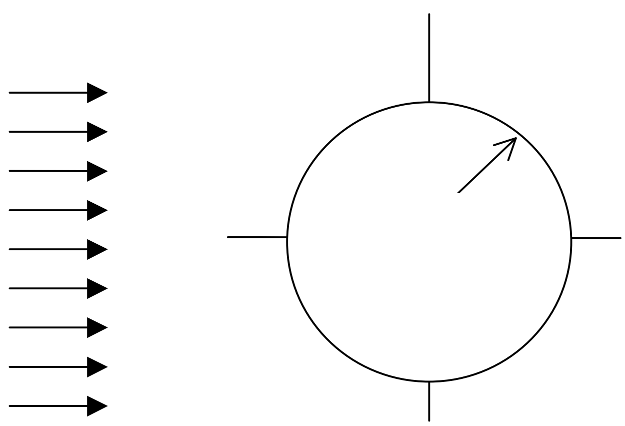

Figure 6. Uniform potential flux around a cylinder

The analytical tangential velocity and the coefficient of pression on a surface are (Anderson, 1994) is given by:

$$V_\theta = 1 - \text{sen}^2\, r \tag{26}$$

$$C_p = 1 - 4 * sin^2(\theta) \tag{27}$$

where r and θ are given in polar coordinates and represent the position of the point.

The pressure distribution is shown on Fig. 7.

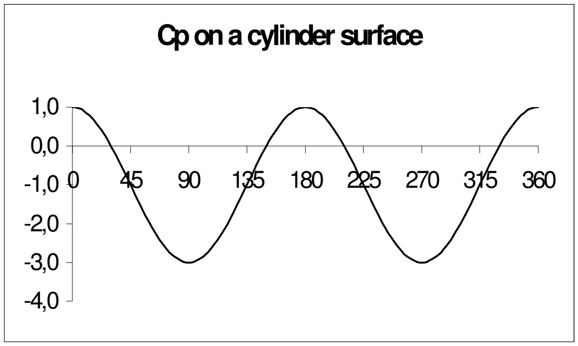

Figure 7. Distribution of the pressure on a cylinder surface

Since the results are symmetrical about the x and y axis, only a quarter of a cylinder is shown. The result with H strategy is shown on the Fig. 8 with the initial and final meshes while that the Fig. 9 shown the error

The initial mesh distribution for H technique was 8 collocation points distributed over the quarter of circle. The H refinement has 3 iterations with 24 collocation points equally spaced.. It can be demonstrated that it depends on how the initial mesh approximates the geometry of the boundary.

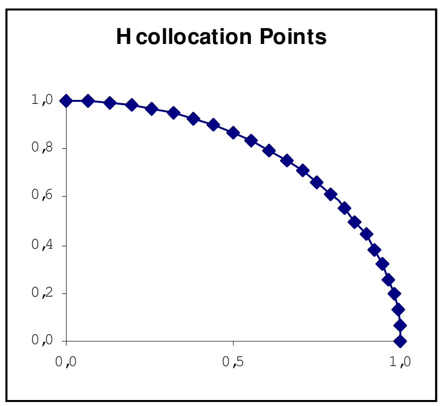

Figure 8. H discretization

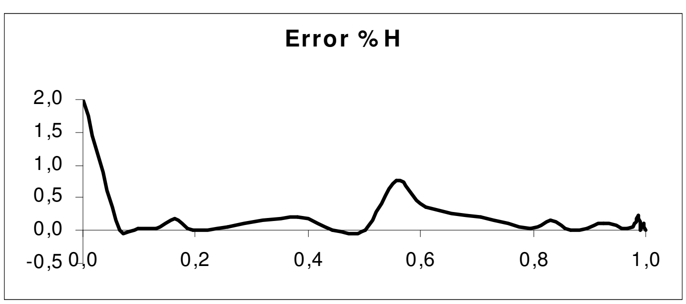

Figure 9. Error Cp over the quarter cylinder surface

5. Conclusions

An H adaptive panels method formulation was proposed for the solution of potential flow around arbitrary bodies in two dimensions. This program can automates process, and also can be executed in personal computers of small capacity, since it requests less memory and computational effort than the conventional analysis.

The example showed the applicability and the precision that can obtained with few elements. Although, the method has to be tested for more airfoils and extended for 3D problems.

6. References

Anderson, 1994. *Theory of Aerodynamics*. 2nd Edition, Springer Verlag.

Katz, J., & Plotkin, A., 2002. *Low Speed Aerodynamics*. Cambridge University Press, 2nd Edition

Zienkiewicz O.C., Zhu, J.Z., & Gong, N.G., 1989. Effective and Practical H-P Version Adaptive Analysis Procedures goes the FEM, *International Journal for Numerical Methods in Engineering*, vol 28, 879-891.

Parreira P., & Dong Y.F., 1992. Adaptive Hierarchical Boundary Elements. *Advances in Engineering. Software*, pp. 249 to 259.

Pessolani, R.B.V., 2002. An Hp-Adaptive Hierarchical Formulation for the Boundary Element Method Applied to Elasticity in Two Dimensions, *Journal of the Brazilian Society Mechanical Sciences*, Vol XXIV, march.