2D ELASTODYNAMIC WITH BOUNDARY ELEMENT METHOD AND THE OPERATIONAL QUADRATURE METHOD

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Abstract. *The study of elastodynamic problems is important because, practically, all areas of engineering works with these type of analysis and the criteria involve, in many situations, conditions of security and/or confiability. The Boundary Element Method (BEM) is an important tool in the development of solutions for elastodynamic problems and the reserch are in continouos evolution. Some formulations has been studied using the time domain fundamental solution and the Laplace domain. Other formulations employs the much simpler elastostatic fundamental solution and an example is the dual reciprocity boundary element method. Recently, the Operational Quadrature Method (OQM) has been studied, principally in scalar wave problems and the results obtained verify the potentiality of the method. Motivated with the method, the authors are working with an extension to the simulation of crack problems (considered as a discontinuity) in juntion with the numerical Green's function. In the sequence was developed the formulation to 2D elastodynamic problem. In the Operational Quadrature Method the convolution integral is substituted by a quadrature formula, whose weights are computed using the fundamental solution in the Laplace domain, producing the direct solution to the problem in the time domain. Numerical examples are presented, confirming that the formulation is stable and accurate.*

Keywords*: Boundary Element Method, Operational Quadrature Method, Elastodynamic*

1. Introduction

Elastodynamic is one of the most important topics studied with the BEM (Brebbia, Telles and Wrobel, 1984). In general, the treatment of the 2D elastodynamic problems with the BEM can be analyzed into three approaches: one is a direct formulation in the time domain (Mansur, 1983) that is solved evaluating the classic BEM formulation in conjunction with conventional step-by-step time integration schemes. The second is the analysis in a transform domain using the Laplace transformation (Cruse, 1968). In this case, a numerical inverse transformation was required to bring the transformed solution back to the original time domain. The third is the dual reciprocity method where one of the advantages is that dynamic problems can be solved by a static fundamental solution (Partridge, Brebbia and Wrobel, 1992).

In this paper are presented extended applications of the time domain BEM (TD-BEM) solved with the Operational Quadrature Method (Lubich, 1988, 1988a, 1994). In this approximation the convolution integral is substituted by a quadrature formula, whose weights are computed using the Laplace transform of the fundamental solution and a linear multistep method. The final solution of the problem is obtained in the time domain.

2. The Time Domain formulation with the operational quadrature method.

Consider an elastic solid enclosed by a boundary surface, subjected to specified external dynamic loadings and in the absence of body force. The condition of dynamic equilibrium of a body is expressed by the equation:

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ji} = \rho \ddot{u}_i \tag{1}$$

where μ and λ are Lame's constant, ρ is the mass density and $\ddot{u}_i$ are acceleration components. To uniquely formulate the dynamic problem, boundary and initial conditions must be imposed which specify the state of displacements and velocities at time t_0. Following the usual procedure in boundary element formulation, the integral equation can be expressed as follows:

$$4\pi C_{ij}(\xi)u_j(\xi,t)=\int_0^{t^+}\int_\Gamma u_{ij}^*(x,t;\xi,\tau)\,p_j(x,\tau)\,d\Gamma(x)\,d\tau - \int_0^{t^+}\int_\Gamma p_{ij}^*(x,t;\xi,\tau)u_j(x,\tau)\,d\Gamma(x)\,d\tau \tag{2}$$

where u_{ij}^* and p_{ij}^* are boundary displacements and tractions, respectively and C_{ij} is the usual free coefficient dependent on the location of ξ (interior or boundary).

Following the OQM procedure, the convolution integrals presented in Eq. (2) can be approximated by

$$\int_0^{t^+}\int_\Gamma u_{ij}^*(x,t;\xi,\tau)\,p_j(x,\tau)\,d\Gamma(x)\,d\tau=\sum_{k=0}^{n}{}^{n-k}g_{ij}^e(x,\xi,\Delta t)\;{}^k p_j^e(x)\quad n=0,1,\ldots,N \tag{3}$$

and

$$\int_0^{t^+}\int_\Gamma p_{ij}^*(x,t;\xi,\tau)u_j(x,\tau)\;d\,\Gamma(x)\,d\,\tau=\sum_{k=0}^{n}{}^{n-k}h_{ij}^e(x,\xi,\Delta t)\;{}^k u_j^e(x)\qquad n=0,1,\ldots,N \tag{4}$$

The weights ***g*** and ***h*** in Eqs. (3) and (4), respectively, are computed with the expressions,

$${}^n g_{ij}^e(x,\xi,\Delta t)=\frac{\rho^{-n}}{L}\sum_{l=0}^{L-1}\int_{\Gamma^e}\hat{u}_{ij}^*\left(x,\xi,\frac{\gamma\left(\rho e^{i l 2\pi/L}\right)}{\Delta t}\right)\Phi^e(x)\,d\,\Gamma(x)\,e^{-i n l 2\pi/L} \tag{5}$$

and

$${}^n h_{ij}^e(x,\xi,\Delta t)=\frac{\rho^{-n}}{L}\sum_{l=0}^{L-1}\int_{\Gamma^e}\hat{p}_{ij}^*\left(x,\xi,\frac{\gamma\left(\rho e^{i l 2\pi/L}\right)}{\Delta t}\right)\Phi^e(x)\,d\,\Gamma(x)\,e^{-i n l 2\pi/L} \tag{6}$$

where $\Phi^e(x)$ represents the interpolation function utilized in the boundary discretization (in fact, quadratic interpolation were employed).

Equation (2) is rewritten in a discretized form

$$4\pi C_{ij}(\xi)u_j(\xi,t_n)=\sum_{e=1}^{E}\sum_{k=0}^{n}{}^{n-k}g_{ij}^e(x,\xi,\Delta t)\;{}^k p_j^e(x)-\sum_{e=1}^{E}\sum_{k=0}^{n}{}^{n-k}h_{ij}^e(x,\xi,\Delta t)\;{}^k u_j^e(x) \tag{7}$$

Equation (7) can now be written for all boundary nodes in terms of global matrices to give the complete system of equations

$$C u^n=\sum_{k=0}^{n}G^{n-k}\,p^k-\sum_{k=0}^{n}H^{n-k}\,u^k \tag{8}$$

Here, ***C*** is a quasi diagonal matrix that is formed by the coefficients $C_{ij}(\xi)$; *n* e *k* correspond to the variables of the time discretization $t_n=n\,\Delta t$ e $t_k=k\,\Delta t$, respectively.

The system of equations is solved step-by-step. Thus, for the first time step

$$(C+H^0)u^1=G^0\,p^1+(G^1\,p^0-H^1\,u^0) \tag{9}$$

The columns of the matrices ***H*** and ***G*** in Eq. (9) must be reordered considering the boundary conditions, obtaining the following expression

$$A^0 y^1 = f^1 + f^0 \tag{10}$$

where $\boldsymbol{f}^1$ is formed by the boundary contributions at $t = \Delta t$ and

$$f^0 = G^1 p^0 - H^1 u^0 \tag{11}$$

For an interval $t_n = n\Delta t$ the expression can be written as

$$A^0 y^n = f^n + \sum_{k=0}^{n-1} f^k \tag{12}$$

and

$$f^k = G^{n-k} p^k - H^{n-k} u^k \tag{13}$$

3. Fundamental Solution

The fundamental solution for a elastodynamic 2D problem in the Laplace domain (Barra, 1996) is written as:

$$\hat{u}^*_{ij}(x,\xi,s) = \frac{1}{\rho c_s^2}\left[\varphi(r)\delta_{ij} - \chi(r) r,_i r,_j\right] \tag{14}$$

and the fundamental traction is,

$$\begin{aligned}\hat{p}^*_{ij}(x,\xi,s) = \Bigg\{ &\left[\frac{d\varphi(r)}{dr} - \frac{\chi(r)}{r}\right]\left(\delta_{ij}\frac{\partial r}{\partial n} + r,_j n_i\right) - 2\frac{\chi(r)}{r}\left(n_j r,_i - 2 r,_i r,_j \frac{\partial r}{\partial n}\right) \\ &- 2\frac{d\chi(r)}{dr} r,_i r,_j \frac{\partial r}{\partial n} + \left(\frac{c_p^2}{c_s^2} - 2\right)\left(\frac{d\varphi(r)}{dr} - \frac{d\chi(r)}{dr} - \frac{\chi(r)}{r}\right) r,_i n,_j \Bigg\}\end{aligned} \tag{15}$$

where the functions $\chi(r)$ and $\varphi(r)$ are defined as follow

$$\chi(r) = \kappa_2\left(\frac{sr}{c_s}\right) - \frac{c_s^2}{c_p^2}\kappa_2\left(\frac{sr}{c_p}\right) \tag{16}$$

and

$$\varphi(r) = \kappa_0\left(\frac{sr}{c_s}\right) + \left(\frac{sr}{c_s}\right)^{-1}\left[\kappa_1\left(\frac{sr}{c_s}\right) - \frac{c_s}{c_p}\kappa_1\left(\frac{sr}{c_p}\right)\right] \tag{17}$$

where *r* is the distance between ξ and *x*; c_p is the P-wave velocity and c_s is the S-wave velocity; κ_j is the modified Bessel function of the second kind and δ_{ij} is the Kronecker delta.

4. Numerical Examples

Example 1. The first example concerns a rectangular plate subjected to an tensile traction of a time-dependent Heaviside type as depicted in Fig. 1. The material properties are: Poisson's ratio $\nu = 0$, Young's modulus $E = 1N/m^2$, and mass density $\rho = 1.0\,kg/m^3$. The dimension of the rectangular plate are $h = 12m$ and $b = 6m$. The boundary element discretization consists of 72 quadratic elements, one internal point and dual nodes in the four square of the plate. The total time is 100 s and the β parameter considered is 0.83 ($\beta = c\Delta t/l$); where $c = 1m/s^2$ is the wave velocity, *l* is the element length.

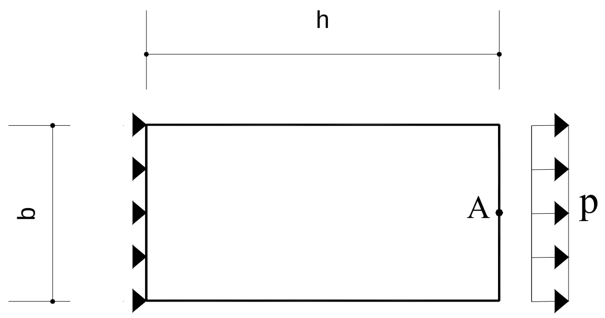

Figure 1. Rectangular plate under a Heaviside type forcing function.

The time histories of the displacement at point A, obtained with the OQM BEM formulation, are compared with the exact solution as is shown in Fig. 2.

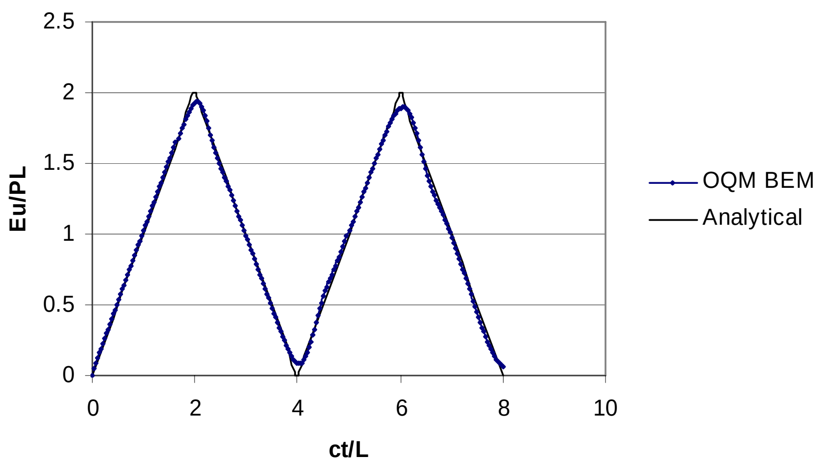

Figure 2. Results at point A (h , b/2).

Example 2. This example concerns a rectangular plate subjected to an end tensile traction of a triangular time variation, as depicted in Fig. 3. The material properties are: Poisson's ratio $\nu = 0$, Young's modulus $E = 1N/m^2$, and mass density $\rho = 1.0\,kg/m^3$. The dimension of the rectangular plate are $h = 20m$ and $b = 1m$. The boundary element discretization consists of 44 quadratic elements, one internal point and dual nodes in the four square of the plate. The total time is 150 s and the β parameter considered is 0.60 and the wave velocity is $c = 1m/s^2$.

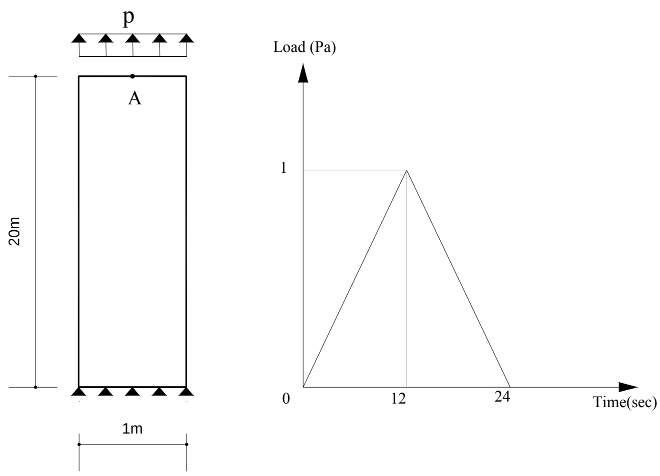

Figure 3. A long and rectangular plate under a triangular tensile load.

The OQM displacements of the point A are compared with the exact solution, and the results are showed in Fig. 4.

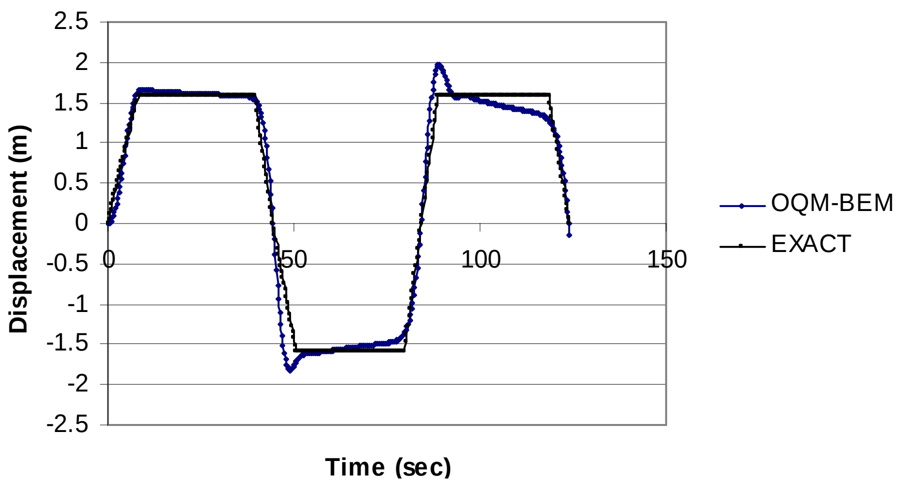

Figure 4. Time variations of vertical displacement of point A of a rectangular plate under a triangular tensile load.

5. Conclusions

A time domain boundary element formulation solved with the operational quadrature method are presented in this paper to analyze the transient problem of two dimensional elastic solids. The numerical results indicate that the solution technique have good accuracy and stability.

This work was a consequence of the initial research of the BEM and the OQM in potential problems; with the objective of solve crack problems applying the Numerical Green's Function (Vera-Tudela and Telles, 2002). The good results obtained are a good reason to extend the study to elastodynamic 2D crack problems and 3D problems.

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7. References

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