NUMERICAL AND EXPERIMENTAL INVESTIGATIONS OF THE NONLINEAR DYNAMICS AND CHAOS IN NON-SMOOTH SYSTEMS WITH DISCONTINUOUS SUPPORT

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Abstract. *Non-smooth systems are abundant in nature operating in different modes. In general, non-smooth characteristics are the source of difficulties for the modeling and simulation of natural systems. This contribution uses a smoothened switch model in order to analyze non-smooth systems. The procedure reveals to be effective to deal with this kind of system, presenting advantages for the numerical implementation. As an application of the general formulation, a single-degree of freedom oscillator with discontinuous support is analyzed. An experimental apparatus is developed in order to verify the capability of the mathematical formulation and the numerical procedure to describe the general behavior of the system. System dynamical behavior shows a rich response, presenting dynamical jumps, bifurcations and chaos. Besides, numerical simulation presents a close agreement with those obtained from experimental apparatus.*

Keywords: *Nonlinear dynamics, chaos, non-smooth systems, bifurcations, grazing.*

1. Introduction

Non-smooth nonlinearity is abundant in nature being usually related to either the friction phenomenon or the discontinuous characteristics as intermittent contacts of some system components. Non-smooth systems appear in many kinds of engineering systems and also in everyday life. Examples may be mentioned by the stick-slip oscillations of a violin string or grating brakes (Hinrichs et al., 1998). Some related phenomena as chatter and squeal causes serious problems in many industrial applications and, in general, these forms of vibrations are undesirable because of their detrimental effects on the operation and performance of mechanical systems (Andreaus & Casini, 2001).

The mathematical modeling and numerical simulations of non-smooth systems present many difficulties, which turns their description unusually complex. Moreover, the dynamical behavior of these systems is complex, presenting a rich response. Literature presents many reports dealing with non-smooth systems. In general, they concern the mathematical modeling, the proposition of proper numerical algorithms to treat these systems, and also experimental approaches employed in order to verify the obtained results. In all these works, it should be pointed out a complex dynamical response.

The idea that non-smooth systems can be considered as continuous in a finite number of continuous subspaces and also that the system parameters do not change in an abrupt manner, inspires some authors to try to describe non-smooth systems by a smoothed form. Wiercigroch (2000), Leine (2000), Leine et al. (2000), Leine & van Campen (2000, 2002a, 2002b) are some articles where interesting approaches are proposed in order to deal with mathematical discontinuity.

Since non-smooth systems present an unusually complex behavior and their description involves many mathematical and numerical difficulties, experimental studies are of great importance. Some of the cited references use experimental approaches to verify the proposed numerical methods. Other references discuss just the experimental point of view. Wiercigroch et al. (1998) and Wiercigroch & Sin (1998) present an experimental analysis of a base excited symmetrically bilinear oscillator. Virgin and co-workers also develops interesting analyses related to non-smooth systems (Todd & Virgin, 1996, 1997; Begley & Virgin, 1998; Slade et al., 1997; Piiroinen et al., 2004).

The objective of this research effort is the nonlinear dynamics analysis of a single-degree of freedom system with discontinuous support. Despite the deceiving simplicity of this problem, its nonlinear dynamics is very rich. Numerical and experimental approaches are employed. The mathematical model uses a smoothened switch model, proposed by Leine (2000) on the study of stick-slip vibrations. Basically, the switch model treats non-smooth systems defining different sets of ordinary differential equations. The smoothened system is built defining transition equations of motions that govern the dynamical response during the transition from one set to another. Therefore, the state space is split into subspaces that have its own smooth ordinary differential equation. The use of this approach smoothes the discontinuities and allows the use of classical numerical procedures for each subspace. Numerical investigations of the single-degree of freedom system with a discontinuous support shows efficiency and allows one to analyze many aspects related to the non-smooth system dynamics.

On the other hand, experimental approach considers an experimental apparatus developed in order to verify the numerical results. Basically, the apparatus is composed by an oscillator constructed by a car, free to move over a rail, connected to an excitation system. The discontinuous support is constructed considering a spring and a gap related to the car position. This apparatus is instrumented making possible to obtain all state variables of the system. This approach makes possible the comparison of the experimental results with those obtained by numerical simulations. In general, it is possible to say that numerical and experimental results are in close agreements.

2. Discontinuous Support Model

This contribution analyzes the dynamical response of a single-degree of freedom system with discontinuous support, shown in Figure 1, using this approach. The oscillator is composed by a mass *m*, two linear springs with stiffness *k* and linear damping with coefficient *c*. Moreover, the support is massless, having a linear spring with stiffness k_s and a linear damping with coefficient c_s. The displacement of the mass is denoted by *x*, relative to the equilibrium position, while the displacement of the support is denoted by *y*. A gap *g* defines the distance between the mass and the support. Therefore, the system has two possible modes, represented by a situation where the mass presents contact with the support and other situation when there is no contact.

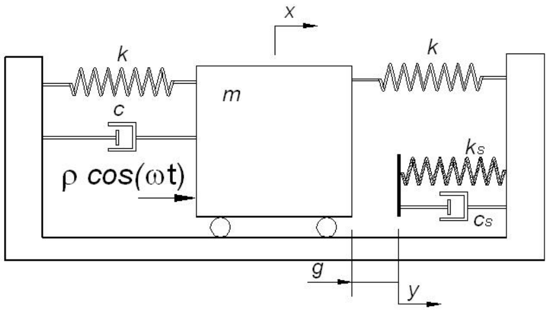

Figure 1 – Non-smooth system with discontinuous support.

The governing equations of the system may be defined by two different equations. One, for situations without contact and the other with contact:

$$\begin{cases} m\ddot{x} + 2kx + c\dot{x} = f_0 \cos(\omega t)\,, \text{ without contact.} \\ m\ddot{x} + 2kx + k_s(x-g) + (c+c_s)\dot{x} = f_0 \cos(\omega t)\,, \text{ with contact.} \end{cases} \tag{1}$$

In order to use the Filippov theory, it is necessary to perform a correct description of the transitions. The contact between the mass and the support occurs if the displacement becomes equal to the gap *g*. On the other hand, the mass loses contact with the support when the contact force vanishes, *i.e.*, if $f_s = -[k_s(x-g) + c_s\dot{x}] = 0$. Therefore, there are two subspaces with two different transition situations. The subspaces are defined by two indicator functions:

$$\begin{aligned} h_\alpha(x,\dot{x}) &= x - g \\ h_\beta(x,\dot{x}) &= -k_s(x-g) - c_s\dot{x} \end{aligned} \tag{2}$$

Assuming a transition through a narrow band with thickness η defined around the discontinuity, system can be written as follows (Leine, 2000; Divenyi *et al.*, 2005),

$$\dot{u} = f(u,t) = \begin{cases} f_-, u \in \Gamma_- \\ f_\alpha, u \in \Sigma_\alpha \\ f_\beta, u \in \Sigma_\beta \\ f_+(u,t), u \in \Gamma_+ \end{cases} \tag{3}$$

where,

$$f_-(u,t) = \begin{Bmatrix} \dot{x} \\ -\dfrac{2k}{m}x - \dfrac{c}{m}\dot{x} + \dfrac{f_0}{m}\cos(\omega t) \end{Bmatrix} \tag{4a}$$

$$f_+(u,t) = \begin{Bmatrix} \dot{x} \\ -\dfrac{2k}{m}x - \dfrac{k_s}{m}(x-g) - \dfrac{c+c_s}{m}\dot{x} + \dfrac{f_0}{m}\cos(\omega t) \end{Bmatrix} \tag{4b}$$

and the transitions associated with hyper-surfaces Σ_α and Σ_β, given by:

$$f_\alpha = \begin{Bmatrix} \dot{x} \\ -\dfrac{2k}{m}x - \dfrac{c+c_s}{m}\dot{x} + \dfrac{f_0}{m}\cos(\omega t) - \dfrac{k_s}{m}(x-g+\eta) - \dfrac{c_s}{m}\dot{x}_a\left(1 - \dfrac{x-g+\eta}{2\eta}\right) \end{Bmatrix} \tag{5a}$$

$$f_\beta = \begin{Bmatrix} \dot{x} \\ -\dfrac{2k}{m}x - \dfrac{c}{m}\dot{x} + \dfrac{f_0}{m}\cos(\omega t) - \dfrac{k_s}{m}(x-g+\eta) - \dfrac{c_s}{m}\dot{x} \end{Bmatrix} \tag{5b}$$

The subspaces and transitions hyper-surfaces are now defined by:

$$\Gamma_- = \{u \in R^2 \;/\; h_\alpha(u) < -\eta \text{ or } h_\beta(u) > \eta\} \tag{6a}$$

$$\Gamma_+ = \{u \in R^2 \;/\; h_\alpha(u) > \eta \text{ and } h_\beta(u) < -\eta\} \tag{6b}$$

$$\Sigma_\alpha = \{u \in R^2 \;/\; -\eta < h_\alpha(u) < +\eta \text{ and } h_\beta(u) < -h_\alpha(u)\} \tag{7a}$$

$$\Sigma_\beta = \{u \in R^2 \;/\; h_\alpha(u) > -h_\beta(u) \text{ and } -\eta < h_\beta(u) < +\eta\} \tag{7b}$$

Figure 2 shows either the subspaces or the hyper-surfaces related to the single-degree of freedom dynamics, considering both non-smooth and smoothened systems.

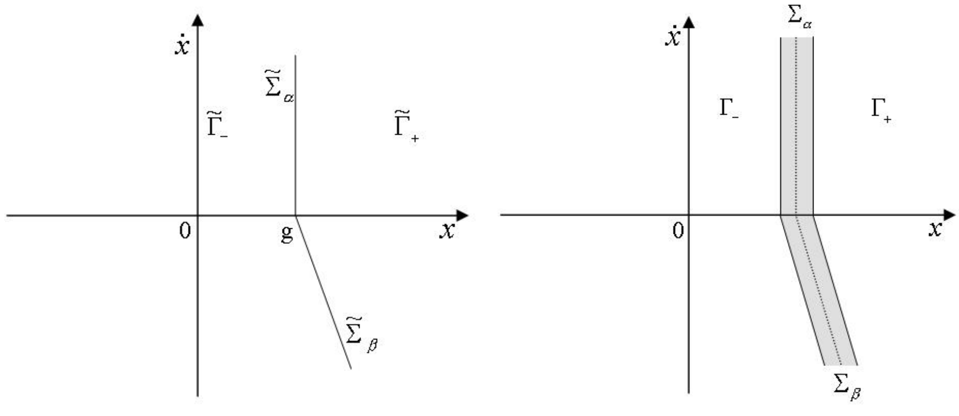

Figure 2 – Subspaces related to the system dynamics.

This approach allows one to deal with non-smooth systems employing a smoothened system. The thickness parameter of the narrow band η need to be chose appropriately for each physical problem (Leine, 2000). This switch model is introduced as an appropriate procedure to perform mathematical modeling allowing an efficient numerical procedure to describe non-smooth systems.

3. Experimental Apparatus

In order to perform an experimental analysis of non-smooth systems, it is developed an experimental apparatus of the one-degree of freedom oscillator discussed in the previous section (Figure 3). Basically, the apparatus is composed by an oscillator constructed by a car (4), free to move over a rail (2), connected to an excitation system composed by springs (3), strings and a DC motor (1) (PASCO ME-8750 with 0-0.3 Hz, 0-12 V e 0-0.3 A). Moreover, the apparatus has a magnetic damping device (5). The discontinuous support (6) is constructed considering a spring with a gap related to the car position. The movement is measured with the aid of a rotary sensor (7), PASCO CI-6538, which has a precision of ±0.25 degrees, maximum velocity of 30 rev/s and maximum sampling frequency of 1000 Hz.

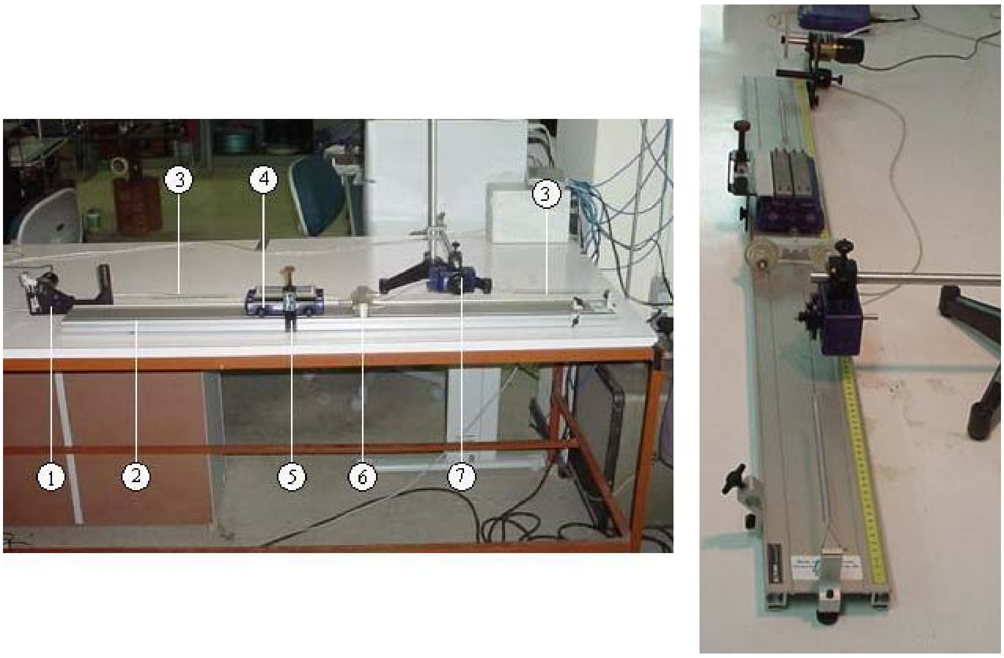

Figure 3 − Experimental apparatus.

This device is represented by parameters related to the oscillator and also to the support, as shown in the modeling presented in the previous section. Therefore, besides the mass, m, there are the stiffness of the main spring, k, and of the support, k_s. Moreover, there are parameters related to the dissipation of the oscillator, c, and of the support, c_s. Another important characteristic of the system is the gap, g, the excitation frequency, ω, and the amplitude, ρ. The following section discusses the identification of these parameters.

3.1. Parameters Identification

The mass of the system can be measured with a weight scale, and it is identified as m = 0.838kg. Since the springs of the oscillator and of the support have different characteristics, their stiffness are identified by different procedures. At first the stiffness is identified analyzing the slope of a force-displacement curve, plotted with the aid of two sensors: the rotary sensor shown in Figure 3 and a force transducer (PS-2104), which has a range of ±50N, with 1% of accuracy and resolution of 0.03N. This analysis yields k = 8.47 N/m, while the support has k_s = 1210 N/m.

The magnetic device responsible for the system dissipation may be modeled by a linear viscous damping in the considered range. The determination of the oscillator dissipation parameter, c or $\xi = c/2m\omega_0$, is done analyzing the frequency response of the system. Basically, the idea is to fit the maximum amplitude obtained by numerical simulations with those obtained by the experimental measures, adjusting the dissipation parameter. Therefore, it results in: ξ = 0.115 or $c = 2\xi m\omega_0$ = 0.87 Ns/m.

The dissipation parameter related to the support is identified assuming the logarithmic decrement procedure, which is defined verifying the ratio between any two consecutive displacement amplitudes. Therefore, analyzing an impulsive response of the system, it is possible to employ classical expressions in order to define c_s = 0.60 Ns/m, or ξ_s = 0.0075.

Now, the excitation characteristic is focused on. Initially, the frequency excitation is analyzed, establishing a relation between this frequency and the DC motor voltage. Moreover, the forcing amplitude is evaluated defining a relation between this force and the motor position, which is related to the length of rotating crank, a. Figure 8 presents the voltage-frequency curve and also the forcing amplitude-motor position curve. The relation of the force with the motor position is done with the aid of a force transducer (PS-2104).

Table 1 summarizes the system parameters experimentally identified and used in all simulations.

Table 1 − System parameters experimentally identified.

k (N/m)	k_s (N/m)	c (N s / m)	c_s (N s / m)	m (kg)	ω_0 (rad/s)
8.47	1210	0.87	0.60 N.s/m	0.838	4.60

4. Numerical and Experimental Results

This section considers the analysis of the system with discontinuous support considering both numerical and experimental approaches. Parameters experimentally identified in the previous section are used in all simulations. The proposed mathematical formulation is integrated with the aid of the Runge-Kutta-Fehlberg method, and numerical parameters need to be proper chosen. Divenyi *et al.* (2005) show the importance of the choice of the thickness of the narrow band η, which is related to the fact that system response passes in all subspaces Γ_- and Γ_+, and the hyper-surfaces Σ_α and Σ_β. On this basis, it is assumed $\eta = 10^{-5}$ for all simulations.

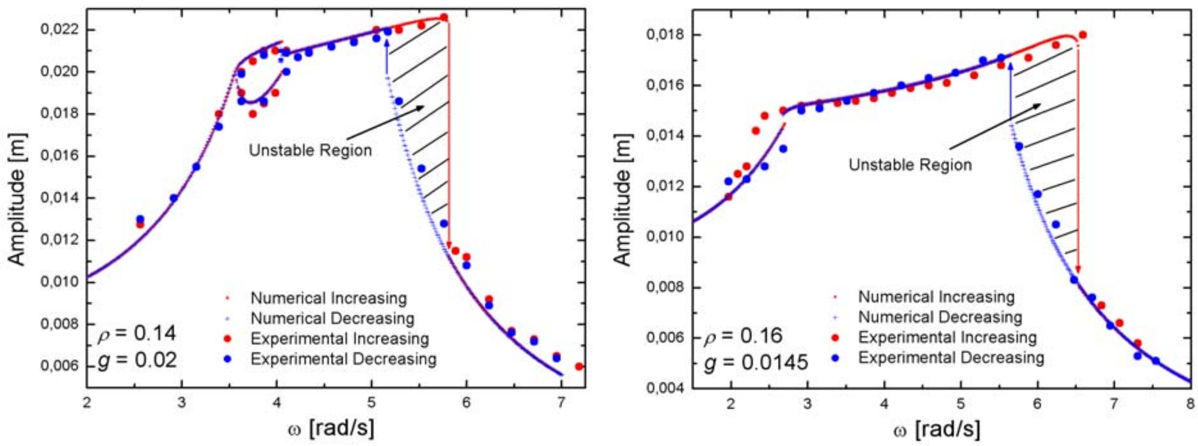

Figure 4 − Frequency response. (a) ρ = 0.14 N and g = 0.02 m; (b) ρ = 0.16N and g = 0.0145m.

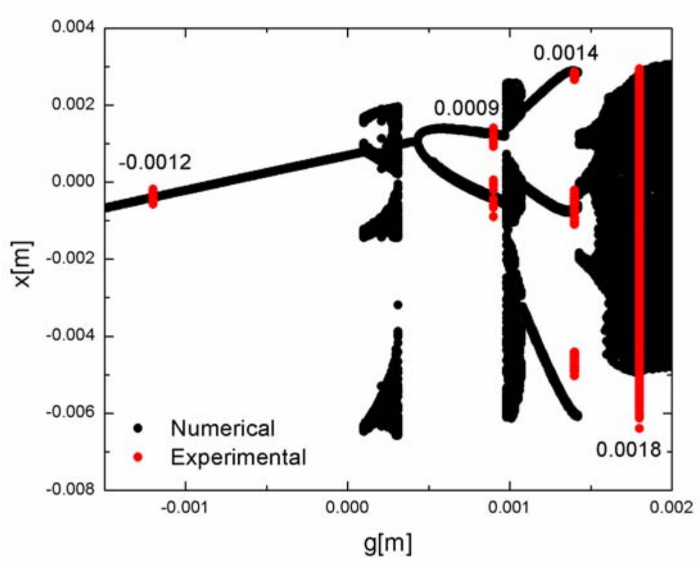

Figure 5 − Bifurcation diagram for ω = 11.15 rad/s and ρ = 0.33 N, varying gap.

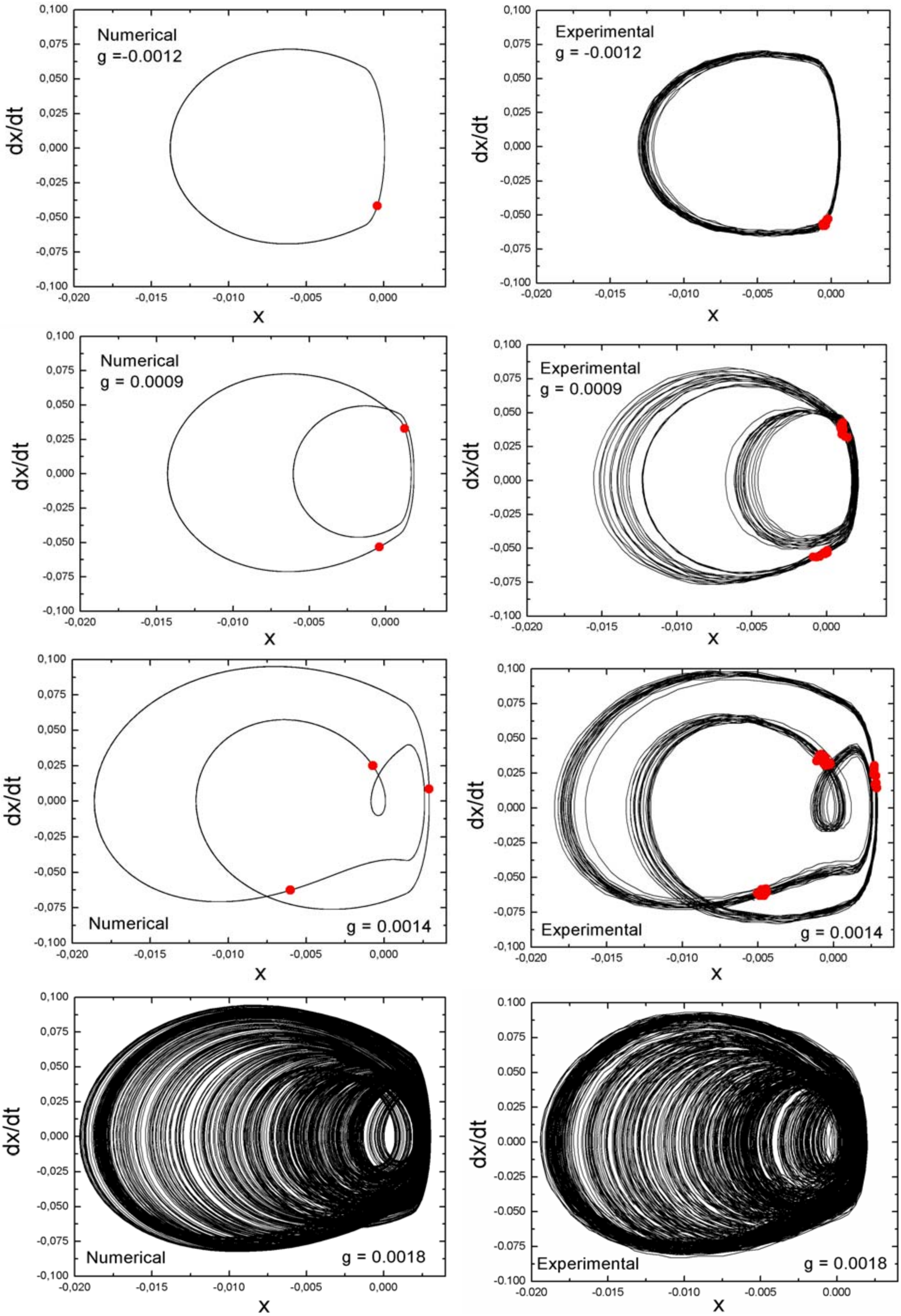

Figure 6 − State space comparing numerical and experimental results, ω = 11.15rad/s and ρ = 0.33N. (a) g = −0.0012 m; (b) g = 0.0009 m; (c) g = 0.0014 m; (d) g = 0.0018 m.

The forthcoming analysis considers a comparison between numerical and experimental results. At first, frequency domain analysis is in focus considering the resonance curve of the system with discontinuous support assuming $\rho = 0.14$ N and $g = 0.02$ m (Figure 4a). The resonance curve has a typical characteristic of nonlinear systems, presenting dynamical jumps due to different trajectories by increasing and decreasing frequency parameter. Therefore, in order to analyze these jumps, two situations are considered: the first considers frequency parameter increase while the second analyzes the decrease. It should be pointed out that numerical and experimental results are in close agreements. By observing the frequency increase, notice that there is a small period-2 region near $\omega = 3.63$ rad/s, and after that, a dynamical jump occurs near $\omega = 5.88$ rad/s. On the other hand, decreasing the frequency, the dynamical jump occurs in a different position, $\omega = 5.29$ rad/s, and after that the period-2 response appears again. Notice also that there is an unstable region defined inside the jumps associated with frequency increase and decrease. Figure 4b presents the frequency response considering different parameters: $\rho = 0.16$N and $g = 0.0145$m. These results have the general behavior presented by the previous one. Nevertheless, notice that, for this set of parameters, the period-2 response does not exist anymore.

Now, the gap influence is focused on. In order to develop this analysis, it is assumed that $\omega = 11.15$ rad/s and $\rho = 0.33$. The gap varies in the range $-0.0012 \le g \le 0.0018$ m, and bifurcation diagram is used to present results (Figure 5). Again, it should be pointed out the agreement between numerical and experimental results that shows periodic and chaotic responses. Figure 6 presents the state space for different values of the gap, g (-0.0012, 0.0009, 0.0014 and 0.0018) showing, respectively, period-1, period-2, period-3 and chaotic responses.

Strange attractors related to the chaotic response of the previous simulations ($g = 0.0018$m) are presented in Figure 7, considering different positions of the Poincaré section. This Figure presents numerical simulations (left side) and also results obtained by the experimental apparatus (right side). Once again, notice the agreement between them. Notice that there is a limit in the right part of the attractor which is physically related to the constraint imposed by the support.

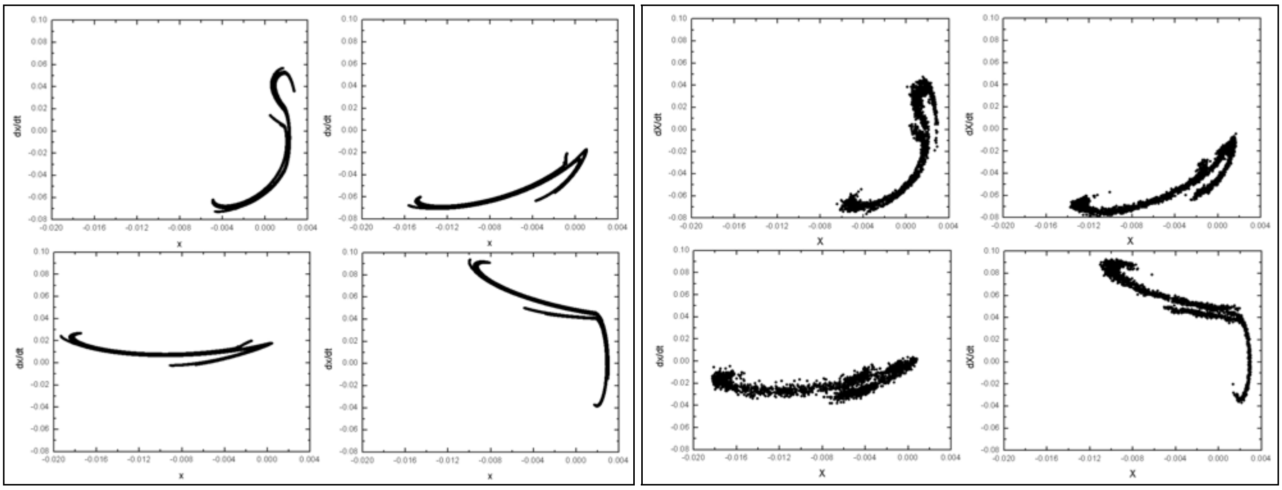

Figure 7 − Strange attractors related to numerical and experimental results.
$\rho = 0.33$ N, $\omega = 11.15$ rad/s and $g = 0.0018$ m.

5. Conclusions

This contribution presents the analysis of a non-smooth system with discontinuous support, considering both numerical and experimental approaches. A smoothened switch model is employed splitting the phase space into subspaces, defining finite regions to describe transitions among them. This procedure is useful for numerical simulations representing an effective form to integrate non-smooth equations. Moreover, an experimental apparatus is constructed in order to verify the numerical results. Numerical and experimental investigations are carried out allowing the analysis of different aspects related to the system dynamics. In general, numerical and experimental results are in close agreement. Both approaches show a very rich dynamics, presenting dynamical jumps, bifurcations and chaos. Finally, the authors believe that the proposed procedure may be useful for the analysis of other non-smooth systems.

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