FORCED AND NATURAL CONVECTION IN ANNULAR CONCENTRIC CHANNELS AND CAVITIES BY INTEGRAL TRANSFORMS

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Abstract. *A unified formulation for the study of forced and natural convection in annular concentric channels and cavities is presented. The Navier-Stokes and energy equations are solved using the Generalized Integral Transform Technique (GITT), which transforms the partial differential equations into a coupled ordinary differential system. The resulting system is solved numerically using well-known subroutines of scientific mathematical libraries. Numerical results for the three different physical situations are analyzed, related to forced convection in annular channels, natural convection in horizontal and vertical cavities, for various values of the governing parameters in each situation. Sets of benchmark results are generated in each case, to illustrate the proposed methodology, as well as to allow for critical comparisons with the available literature results.*

Keywords: *Forced and Natural Convection, Navier-Stokes Equations, Annular Ducts, Integral Transforms*

1. INTRODUCTION

The aim of this paper is to demonstrate the suitability of the Generalized Integral Transform Technique (GITT) (Cotta, 1993), as a tool in obtaining engineering or benchmark results in problems of forced or natural convection inside annular concentric ducts and cavities. This study is of permanent interest in light of the numerous engineering applications connected with heat transfer inside such geometric configurations.

The present work considers three different physical situations: forced convection in concentric ducts, natural convection in vertical and horizontal concentric annular cavities. A brief literature review for each case is here presented. In the case of forced convection in annular concentric ducts the literature results are quite limited, despite the subject importance in engineering applications. Using a boundary layer model, Shumway and McEliot (1971) studied this forced convection problem using the finite difference method. They analyzed three different types of boundary conditions with various aspect ratios. Coney and El-

Shaarawi (1975), also using finite differences, solved this problem considering boundary conditions of first and second kinds. Solving the Navier-Stokes and energy equations for laminar flow with constant properties, Fuller and Samuels (1970) studied forced convection in concentric annular ducts with a finite differences approach. They presented results for Reynolds numbers less than 500 and aspect ratio of 0.5. It is always worth mentioning that Shah and London (1978) made an important compilation of forced convection for internal laminar flows.

De Vahl Davis and Thomas (1969) and Thomas and De Vahl Davis (1970) studied natural convection in vertical concentric annular cavities. The coupled Navier-Stokes and energy equations were solved using finite differences. They investigated the influence of Rayleigh number and provided results for annular and rectangular cavities. El-Shaarawi and Sarhan (1980), using a boundary layer model, studied this problem through finite differences. Prasad and Kulacki (1985) experimentally developed an analysis of natural convection phenomena in a vertical annular cavity, for different heights of the cavity. They also used different fluids in their experiments. Aung et al. (1991) and Tsou and Gau (1992) considered temperature dependent fluid properties and solved the problem via finite differences. Rogers and Yao (1993) performed an instability analysis in vertical concentric annular cavities.

The problem of natural convection in horizontal concentric annular channels was studied by Kuehn and Goldstein (1976), who made an experimental and theoretical study of this same problem. They presented results for the temperature distribution and the for local heat transfer coefficients. Tsui and Trambley (1984) considered both permanent and transient regimen using the ADI scheme. Rao et al. (1985) investigated transient two-dimensional and permanent three-dimensional situations. Mahony et al. (1986), using finite differences, solved the variable properties model.

This contribution revisits each of these three problems, and through integral transformation and its inherent automatic error control capability, provides sets of reference results for validation purposes, here employed in critical comparisons against some of the above cited previous works. The present analysis is a natural extension in the development of this hybrid numerical analytical approach for heat and fluid flow problems, and some of the previous more representative contributions, related to the present work, are listed as follow: Pérez Guerrero and Cotta (1992, 1996), Pereira et al. (1998, 1999), Leal et al. (1999), Pérez Guerrero et al. (2000) and Pereira (2000).

2. ANALYSIS

We consider in all three cases studied here, the physical situation of a Newtonian fluid in laminar regime with constant thermophysical properties. The models considered involve the full Navier-Stokes and energy equations for two-dimensional steady incompressible flow, assuming the validity of the Boussinesq approximation for the free convection cases. The streamfunction-only formulation is preferred over the primitive variables one, as in previous contributions with the GITT for the solution of flow problems (Pérez Guerrero and Cotta, 1992; 1996 and Pereira et al., 1998) due to the enhanced convergence characteristics achievable and demonstrated.

2.1. Forced convection in concentric annular channels

The dimensionless streamfunction and energy equations are given by:

$$E^4\psi = \frac{Re}{2(1-\kappa)}\left[\frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial(E^2\psi)}{\partial r} - \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial(E^2\psi)}{\partial z} - \frac{2}{r^2}\frac{\partial\psi}{\partial z}E^2\psi\right] \tag{1}$$

$$\nabla^2\Theta = \frac{Pe}{2(1-\kappa)}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial\Theta}{\partial r} - \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial\Theta}{\partial z}\right) \tag{2}$$

where E^4 and ∇^2 are the biharmonic and Laplacian operators in cilyndrical coordinates, respectively, as described in Pereira et al. (1999) and Pereira (2000). The dimensional boundary conditions considered are described in Fig. 1, where the inner wall of the channel is isothermic and the outer wall is insulated.

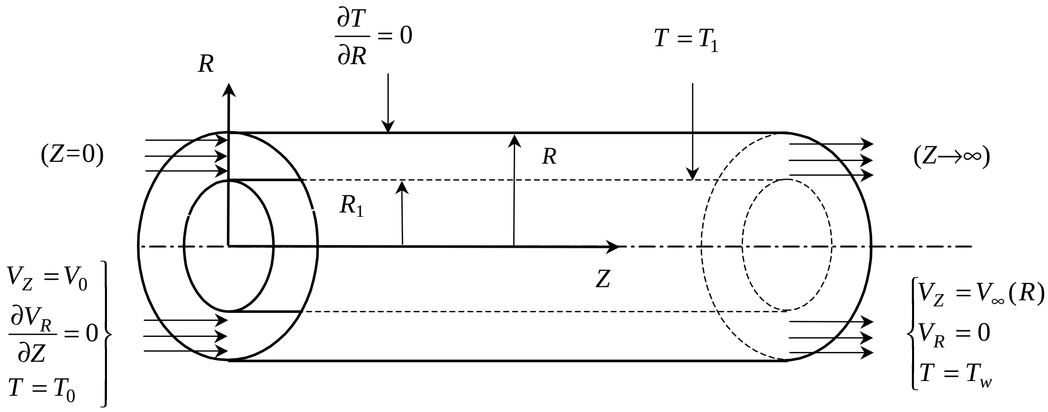

Figure 1. Forced convection in concentric annular channel

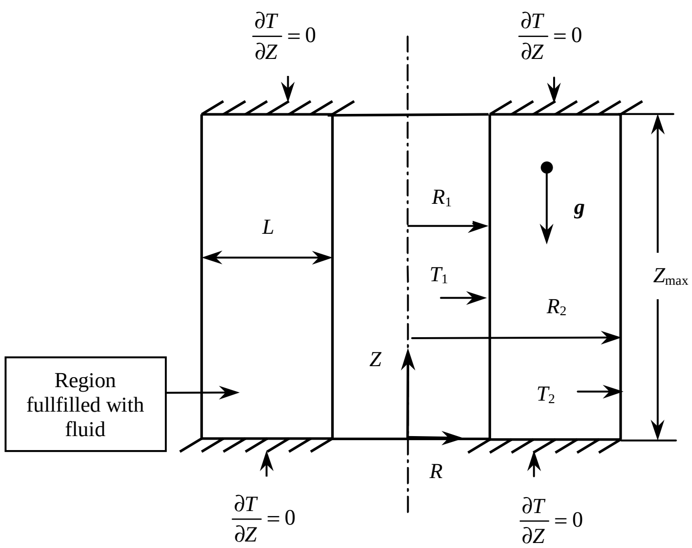

Figure 2. Natural convection in vertical concentric annular cavity

2.2. Natural convection in vertical concentric annular cavities

The dimensionless streamfunction and energy equations are given by:

$$\mathrm{E}^4\psi = \frac{1}{Pr}\left[\frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial(\mathrm{E}^2\psi)}{\partial r} - \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial(\mathrm{E}^2\psi)}{\partial z} - \frac{2}{r^2}\frac{\partial\psi}{\partial z}\mathrm{E}^2\psi\right] + Ra_L\frac{\partial\Theta}{\partial r} \tag{3}$$

$$\nabla^2\Theta = \frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial\Theta}{\partial r} - \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial\Theta}{\partial z} \tag{4}$$

with boundary conditions as indicated in Fig. 2 above.

2.3. Natural convection in horizontal concentric annular channels

We consider two concentric infinite cylinders with prescribed uniform temperatures, with the inner wall temperature being greater than that of the outer one. A two-dimensional situation is assumed without axial flow induction. The symmetry of the problem is taken into account, and the coupled streamfunction and energy equations are written in dimensionless form as:

$$\nabla^4\psi = \frac{1}{\mathrm{Pr}}\left[\frac{1}{r}\frac{\partial\psi}{\partial\theta}\frac{\partial(\nabla^2\psi)}{\partial r} - \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial(\nabla^2\psi)}{\partial\theta}\right] + Ra_L\left(\sin\theta\frac{\partial\Theta}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial\Theta}{\partial\theta}\right) \tag{5}$$

$$\nabla^2\Theta = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\frac{\partial\Theta}{\partial r} - \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial\Theta}{\partial\theta} \tag{6}$$

The boundary conditions that complete the formulation of the problem are as indicated in Fig.3.

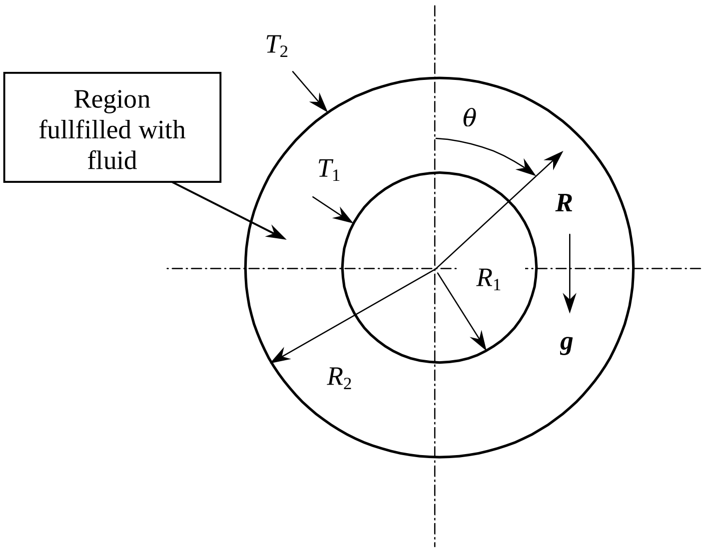

Figure 3. Natural convection in horizontal concentric annular cavity

3. INTEGRAL TRANSFORM SOLUTION

The nonlinear elliptic partial differential systems described in section 2 were solved using the GITT approach, following the procedure established in Cotta (1993). The first step is the selection and solution of the auxiliary eigenvalue problems, for each of the potentials, streamfunction and temperature, yielding eigenvalues, eigenfunctions and the related orthogonality properties. Next, one defines the inverse-integral transform pairs for streamfunction and temperature. The partial differential system is then transformed using the definitions of the transforms and inverse formulae. The result is an ordinary differential system, where infinite summations are truncated to a sufficiently large order for computational purposes. The truncated ordinary differential system is solved using subroutines with automatic error control, as the subroutine BVPFD of IMSL (1989). The integral ODE system coefficients, which appear from the integral transformation procedure, are numerically evaluated, since the internal products that form the integrands involve Bessel functions, and most of them do not allow for analytical integration. Details of the solution and computational algorithms are found in Pereira (2000).

4. RESULTS

In all cases studied here, a careful convergence control of the streamfunction and temperature was undertaken, automatically varying the truncation order of the ordinary differential system.

Table 1 shows the convergence history of the bulk mean temperature of the fluid, at different axial positions for the forced convection case in concentric annular channels, with a radii ratio between the concentric cylinders of κ=0.25, and Reynolds number Re=2000, taking air as the fluid (Pr=0.7). The first column of Table 1 shows the truncation orders for the streamfunction (NF) and temperature (NT) expansions, here taken equal for simplicity in the analysis. Only around 11 terms were required in the eigenfunction expansions for achieving fully converged results at the axial positions far downstream, while up to 29 terms were needed for full convergence to four significant digits at those positions considered closer to the channel inlet.

Table 1. Convergence of fluid bulk temperature, $\Theta_b(\kappa=0.25,\ Re=2000$ e $Pr=0.72)$ – Forced convection in annular channels

	z^+				
NF=NT	**2.5×10^{-4}**	**1.0×10^{-3}**	**2.5×10^{-3}**	**2.5×10^{-2}**	**2.5×10^{-1}**
5	9.930E-01	9.810E-01	9.632E-01	8.211E-01	2.169E-01
11	9.914E-01	9.783E-01	9.617E-01	8.205E-01	2.168E-01
17	9.903E-01	9.780E-01	9.615E-01	8.204E-01	2.168E-01
23	9.901E-01	9.779E-01	9.614E-01	8.204E-01	2.168E-01
29	9.900E-01	9.778E-01	9.614E-01	8.204E-01	2.168E-01
35	9.900E-01	9.778E-01	9.614E-01	8.204E-01	2.168E-01
Ref.#	**9.899E-01**	**9.779E-01**	**9.614E-01**	**8.207E-01**	**2.177E-02**
Ref.##	**9.902E-01**	**9.783E-01**	**9.621E-01**	**8.243E-01**	**-**

Shumway e McEligot (1971) – Boundary layer model

Coney e El-Shaarawi (1975) – Boundary layer model (Pr=0.7)

A critical comparison is performed against the boundary layer model results of Coney and El-Shaarawi (1975), as shown in Fig. 4 below, for the local Nusselt numbers along the channel length. Two radii ratios are considered, and a Nusselt number enhancement effect is observed as the radii ratio is decreased.

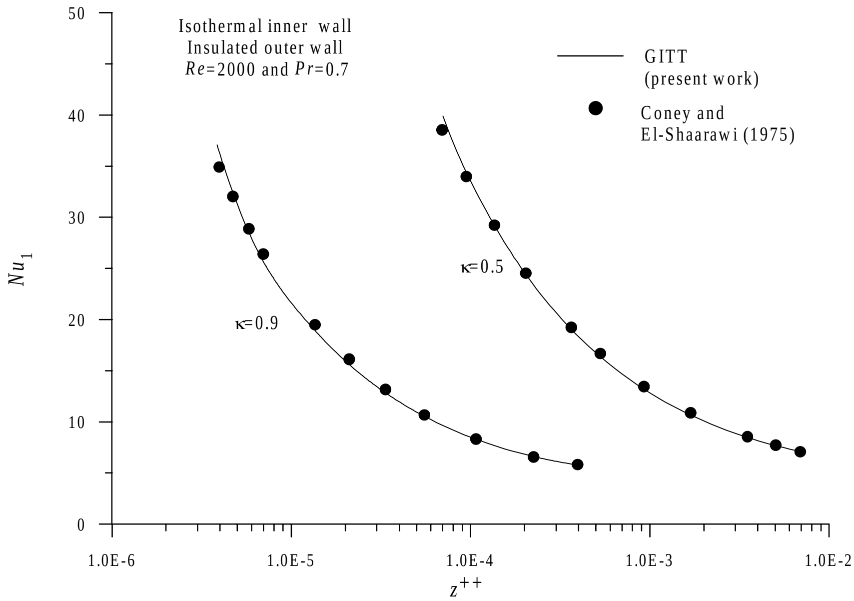

Figure 4. Local Nusselt number distributions for *Re*=2000 and *Pr*=0.7 – Forced convection

Table 2 illustrates the convergence of the steamfunction and temperature fields for natural convection at different positions within the vertical concentric annular cavity for $Ra_L=10^6$.

Table 2. Convergence of streamfunction and temperature fields (H=1, ϖ=2, $Ra_L=10^6$, Pr=0.7) - Natural convection in vertical annular cavity

Ψ(*r*,*z*)		***z* = 0.5**	
NF=NT	***r* =1.1**	**1.5**	**1.9**
10	1.805E+01	2.048E+01	1.867E+01
18	1.773E+01	2.029E+01	1.865E+01
30	1.780E+01	2.035E+01	1.868E+01
40	1.781E+01	2.035E+01	1.868E+01
Θ(*r*,*z*)		***z* = 0.5**	
NF=NT	***r* =1.1**	**1.5**	**1.9**
10	3.442E-01	3.331E-01	3.534E-01
18	3.414E-01	3.371E-01	3.573E-01
30	3.437E-01	3.386E-01	3.583E-01
36	3.436E-01	3.387E-01	3.585E-01
40	3.436E-01	3.387E-01	3.585E-01

The structure of the streamlines and isotherms for $Ra_L=10^6$ are presented in Fig.5, where the finer boundary layer structure is observed at the cavity walls, and two secondary vortices are noticeable. A critical comparison with previous work is presented in Fig.6, for the local Nusselt number at the inner wall as a function of the Rayleigh number, with perceptible differences between the two simulations, which could however be due to different Nusselt number definitions used and analyzed by Pereira (2000) and Kumar and Kalam (1991).

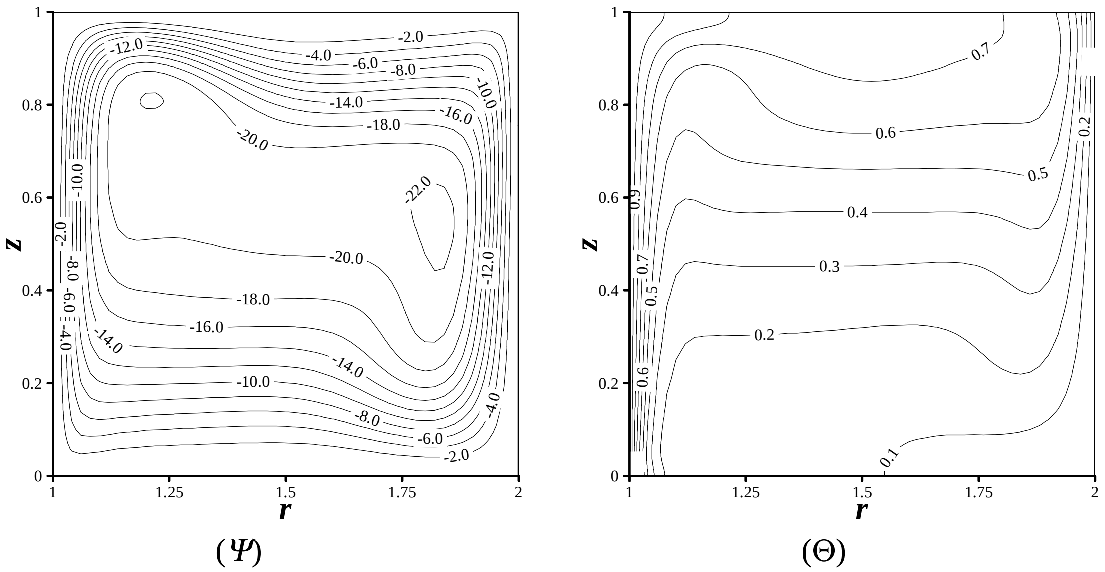

Figure 5. Isolines of streamfunction and temperature (Pr =0.7, H=1, ϖ =2.0, $Ra_L=10^6$)
- Natural convection in vertical annular cavity

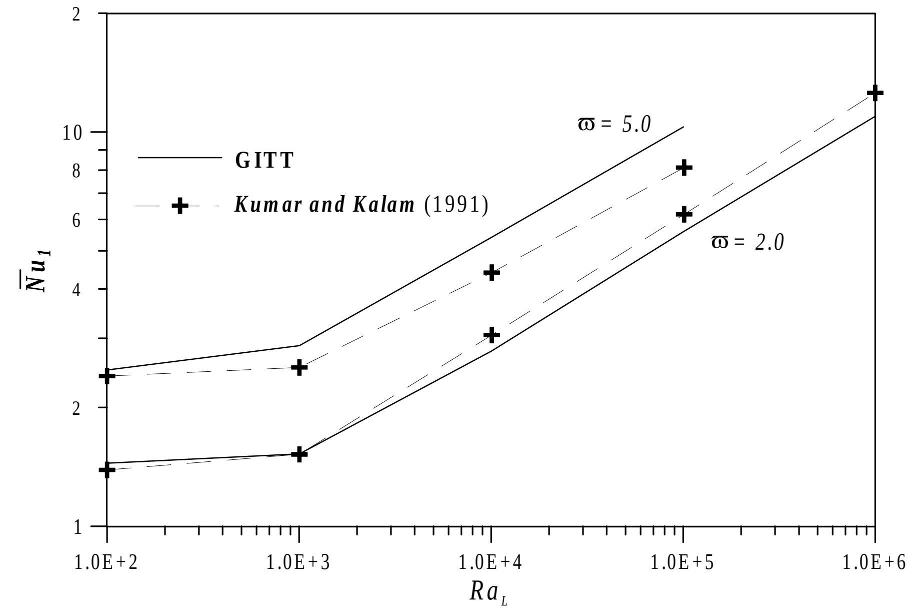

Figure 6. Local Nusselt numbers at the inner wall against Ra_L (Pr =0.7, H=1)
- Natural convection in vertical annular cavity

Table 3 below illustrates the excellent convergence behavior of the streamfunction and temperature fields for natural convection between horizontal infinite cylinders, at selected radial and angular positions within the cavity. A comparison of the local Nusselt numbers obtained through GITT against the experimental results of Kuehn and Goldstein (1976) is shown in Fig.7, where the agreement is found to be very good. A typical flow structure is illustrated through the isolines of streamfunction and temperature in Fig.8, for Ra_L=5x10^4.

Table 3. Convergence of Streamfunction and Temperature fields (ϖ=2.6, Ra_L=5x10^4, Pr=0.7) - Natural convection in horizontal annular cavity

$\Psi(r,z)$		$\theta = 90^o$	
NF=NT	***r* =0.725**	**1.125**	**1.525**
8	6.784E+00	2.157E+01	4.674E+00
16	7.233E-01	4.678E-01	3.339E-01
24	7.233E-01	4.677E-01	3.340E-01
28	7.233E-01	4.677E-01	3.340E-01
30	7.233E-01	4.677E-01	3.340E-01
$\Theta(r,z)$		$\theta = 90^o$	
NF=NT	***r* =0.725**	**1.125**	**1.525**
8	5.284E-01	3.158E-01	1.527E-01
16	5.287E-01	3.160E-01	1.528E-01
24	5.287E-01	3.161E-01	1.528E-01
28	5.287E-01	3.161E-01	1.528E-01
30	5.287E-01	3.161E-01	1.528E-01

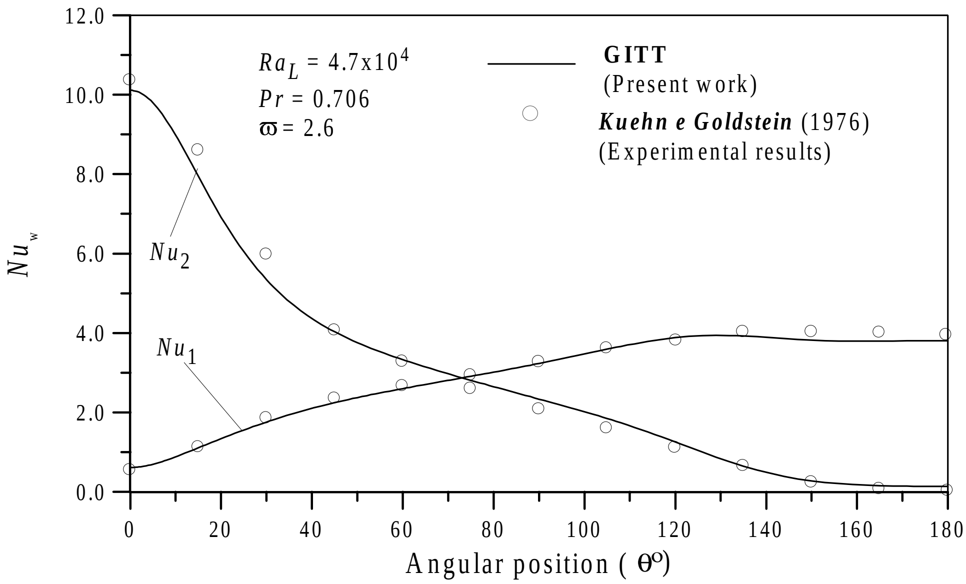

Figure 7. Comparison of local Nusselt numbers from GITT against experimental results for Ra_L=4.7x104 and Pr=0.706: - Natural convection in horizontal annular cavity

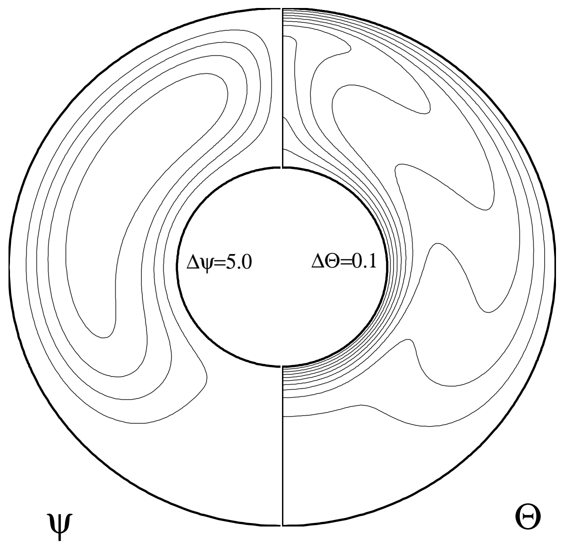

Figure 8. Isolines for streamfunction and temperature (Ra_L=5x10^4, Pr=0.7 e ϖ=2.6) - Natural convection in horizontal annular cavity

5. CONCLUSIONS

The present work illustrates the applicability of the integral transform approach in the analysis of steady forced or natural convection within cylindrical channels or cavities. This approach can be either employed for benchmarking purposes, yielding sets of reference results with controlled accuracy, or alternatively as an engineering simulation tool, with lower truncation orders and exceptional computational performance. The extension of the present analysis towards transient and variable properties formulations should now follow, as recently accomplished for rectangular geometries.

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