ON THE PHYSICAL SIGNIFICANCE OF SOME DIMENSIONLESS NUMBERS USED IN HEAT TRANSFER AND FLUID FLOW

Clovis R. Maliska
Computational Fluid Dynamics Laboratory-SINMEC
Mechanical Engineering Department
Federal University of Santa Catarina
88040-900-Florianópolis-SC-Brazil
maliska@sinmec.ufsc.br

Abstract. *Dimensionless numbers are of key importance in parametric analysis of engineering problems. They are also extremely useful in understanding the similarity among problems belonging to the same broad class. However, in spite of its importance in phenomenological analysis, their physical interpretation is usually not given or is contradictory in the literature. Well-known dimensionless numbers, like Re and Ra, are frequently misinterpreted in textbooks widely used by engineering students. The main goal of this paper is to present a physical interpretation of the Reynolds, Peclet, Rayleigh and Boussinesq numbers based on the ratio of advective and diffusive fluxes of heat and momentum. With the help of scale analysis it is shown that when the dimensionless numbers are related to the ratio of advective and diffusive fluxes, the physical meaning is straightforward.*

1. INTRODUCTION

The use of dimensionless numbers in engineering and physics allows the important task of data reduction of similar problems. This means that a lot of experimental runs are avoided if data is correlated using appropriate dimensionless parameters. Recall, for example, the transient 1D heat conduction in a slab with a convection boundary condition. In this case, the parameters involved are the slab thickness (**L**), conductivity (**k**), specific heat ($\mathbf{c_p}$), density (ρ), heat convection coefficient (**h**), temperature (**T**) and a space coordinate (**x)**. Using dimensionless numbers the temperature dependence of six parameters reduces to a dependency of Biot, Fourier and x/L. Besides this fundamental application of the dimensionless numbers, they also serve as an important mechanism for understanding the physics of the phenomenon.

There are two widely used ways for obtaining the dimensionless numbers. The first one is the use of the well-known π-theorem (Langhaar, 1951), where it is, first, chosen the important variables of the physical process, including physical properties, geometry and flow variables, followed by the solution of a linear system for determining the exponents of the different variables which form the dimensionless numbers. This procedure requires foreknowledge, since if some important variable is forgotten, its influence in the dimensionless numbers is missed. And, missing an important parameter may result in the appearance of meaningless dimensionless numbers, which would correlate the physics of a non-existing phenomenon. The second approach for determining dimensionless numbers is through the use of the partial differential equations governing the physical phenomena. The key issue in this approach is the definition of the dimensionless dependent and independent variables. A good choice is required to end up in dimensionless numbers that properly correlate the physical data.

In both procedures the dimensionless numbers just come out of the algebraic manipulation, lacking a strong physical interpretation. A closer look at the areas of fluid mechanics and heat transfer reveals that in these fields important dimensionless parameters like Reynolds, Peclet and Rayleigh are frequently misinterpreted. Bejan (Bejan, 1994 and Bejan, 1995), using scale analysis, made strong contribution in clarifying several important aspects related to these numbers.

In this work it is presented a physical interpretation of the Reynolds, Peclet, Rayleigh and Boussinesq numbers using scale analysis in conjunction with the role played by the advection and diffusion of momentum and energy in fluid flows.

2. BOUNDARY LAYER CONCEPTS

Knowing the fundamental physical details of the growth of the momentum and thermal boundary layer of a flow, it considerably helps the understanding of the physical significance of some important dimensionless parameters used in heat transfer and fluid flow. Therefore, in this section it is introduced some basic boundary layer concepts.

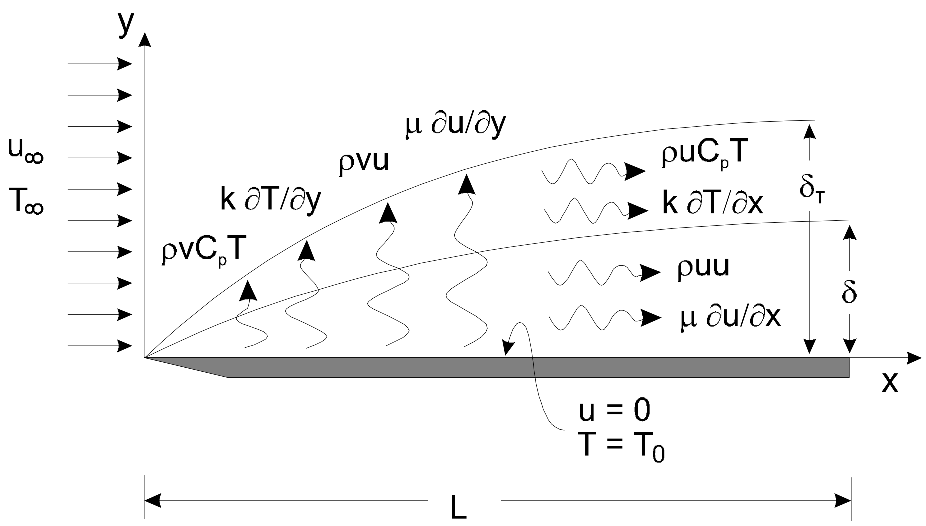

Fig. 1 – Transport fluxes in forced convection flows

Consider Fig 1, where it is shown the thickness of the momentum and thermal boundary layer for a steady-state flow over a flat plate with u_∞ constant. The x-momentum and energy equations, after scale analysis is performed (Bejan, 1995), read

$$\frac{\partial}{\partial x}(\rho\, u u)+\frac{\partial}{\partial y}(\rho\, v u)=\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{1}$$

$$\frac{\partial}{\partial x}\left(\rho\, u c_p T\right)+\frac{\partial}{\partial y}\left(\rho\, v c_p T\right)=\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial y}\right) \tag{2}$$

To begin interpreting the physics of a boundary layer flow it is important to recognize that the stress existing at the fluid/solid interface is, in fact, a diffusion flux of momentum per unit area in the y-direction. This momentum flux will alter the velocity profile in a similar manner as the heat flux at the wall will alter the temperature profile. The momentum flux by advection in the x-direction, by its turn, tends to maintain the uniform velocity profile given as boundary condition at the plate leading edge. Therefore, the momentum boundary layer thickness is determined by the relative strength of these momentum fluxes. A lower momentum flux by advection will allow the diffusion effects to further penetrate in the y-direction, thickening the boundary layer δ, while a higher velocity will decreased the thickness δ. Exactly the same happens with the penetration of the diffusion of heat in the y-direction and the tendency of the advection of energy in the x-direction in maintaining the uniform temperature profile prescribed as boundary condition at the plate leading edge.

Two important physical properties are, therefore, responsible for the growth of the boundary layers (thermal and momentum) and the relative thickness between them, the thermal diffusivity, α, and the momentum diffusivity,ν. The ratio of these two quantities is the well-known Prandtl number, given by

$$Pr=\frac{\nu}{\alpha} \tag{3}$$

Therefore, the larger the Prandtl number, the thicker will be the momentum boundary layer compared to the thermal boundary layer. The physical significance of the Prandtl number is, thus, very strong, since it is the only required dimensionless parameter that relates the thermal and momentum boundary layer thickness.

To complete the necessary information for analyzing the physical role of the Reynolds, Peclet, Rayleigh and Boussinesq numbers consider, again, Fig. 1 where it is shown schematically the momentum and heat fluxes by diffusion and advection in both x and y directions present in the flow. Define, for the forced flow over a flat plate, the following flux ratios to be used along this work. The ratio of momentum advection by momentum diffusion in the x-direction,

$$\frac{A_x^M}{D_x^M}\approx\frac{\rho\, u u}{\mu\,\dfrac{\partial u}{\partial x}} \tag{4}$$

where the subscript x and the superscript M refers to the x-direction and momentum, respectively, the ratio of momentum advection by momentum diffusion in the y-direction,

$$\frac{A_y^M}{D_y^M} \approx \frac{\rho\, v u}{\mu \dfrac{\partial u}{\partial y}} \tag{5}$$

and the ratio of momentum advection in the x-direction divided by momentum diffusion in the y-direction,

$$\frac{A_x^M}{D_y^M} \approx \frac{\rho\, u u}{\mu \dfrac{\partial u}{\partial y}} \tag{6}$$

In the above equations only part of the diffusion momentum flux is used, since the other part or is of the same or lower order of magnitude. Recall also that one is dealing only with the x-momentum equation, since the y-momentum equation is discarded in boundary layer flows. Therefore, when it is referred to a momentum flux by advection in the y-direction, it is meant the *x-momentum flux* by advection in the y-direction. The energy flux ratio by advection and diffusion is given by

$$\frac{A_x^H}{D_x^H} \approx \frac{\rho u c_p T}{k \dfrac{\partial T}{\partial x}} \tag{7}$$

To interpret the Rayleigh and Boussinesq numbers a natural convection flow in a vertical flat plate is used. Considering Fig. 2, the following advection and diffusion fluxes of momentum and energy, respectively, can de defined:

$$\frac{A_y^M}{D_y^M} \approx \frac{\rho\, v v}{\mu \dfrac{\partial v}{\partial y}} \tag{8}$$

$$\frac{A_y^H}{D_y^H} \approx \frac{\rho\, v c_p T}{k \dfrac{\partial T}{\partial y}} \tag{9}$$

Recall that in the above equations the v velocity is not known. Its replacement by the known input variables of the natural convection flow problem makes the Rayleigh and Boussinesq numbers to appear. Other fluxes relations could be also defined for the natural convection flow, but only the two defined above are of interested in this paper.

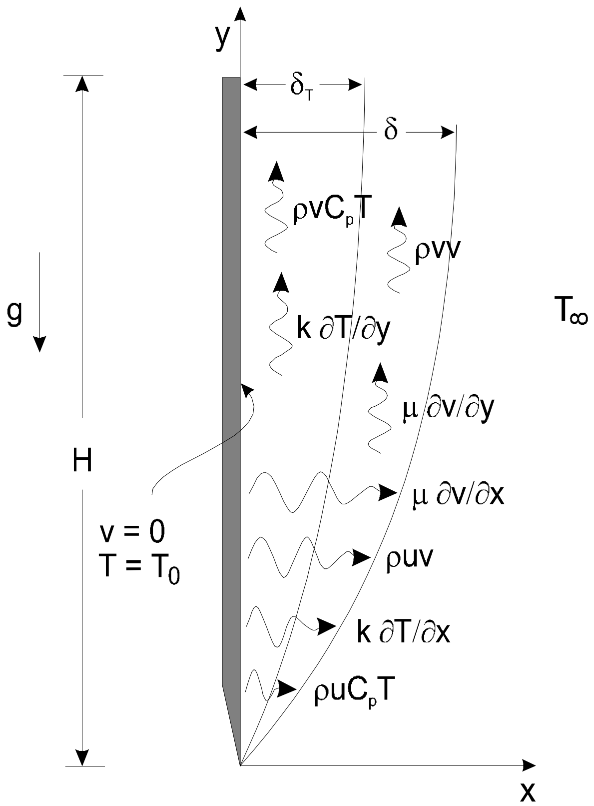

Fig. 2 – Transport fluxes in natural convection flows

PHYSICAL INTERPRETATION OF Re, Pe, Ra and Bo.

The Reynolds number. The Reynolds number is, certainly, the most important dimensionless number in fluid mechanics, since it is an input parameter for all forced flows and a criterion used for classifying the laminar and turbulent regimes. Before addressing comments on how the Reynolds number is normally interpreted in most textbooks, let's define it based on the physical process with governs the fluid flows, namely the advection and diffusion fluxes, leading to a clear understanding of its role in fluid mechanics. Using scale analysis (Bejan, 1995), recalling that $u \sim u_\infty$ and $x \sim L$, Eq.(4) results

$$\frac{A_x^M}{D_x^M} \approx \frac{\rho\, u_\infty L}{\mu} = Re_L^u \tag{10}$$

demonstrating that this momentum fluxes ratio is, precisely, the Reynolds number, where the characteristic length is a dimension measured along the flow direction. Therefore, the Reynolds number is a measure of the relative importance between the momentum flux by advection and by diffusion in the same direction. This is in fully accordance with the physical reasoning used when the diffusion term, in a specified direction, is neglected in the momentum equation when the Reynolds number is large in that direction. It is important to keep in mind that velocity and length are taken in the same direction. This suggests that one can define the Reynolds number based on other advection/diffusion flux ratios. Applying scale analysis to Eq.(5), which relates the fluxes in the y-direction, one finds

$$\frac{A_y^M}{D_y^M} \approx \frac{\rho\, v\delta}{\mu} = Re_\delta^v \tag{11}$$

which can be defined as the Reynolds number for the y-direction where, again, the characteristic dimension and the velocity entering the definition is in the same direction, the y-direction in this case. It is clear that this number is small for a boundary layer flow, indicating that the diffusion of momentum in the y-direction is more important than the advection of momentum in that direction, as is physically known.

The Reynolds number defined according Eq(10) and Eq.(11) quantifies the ratio between momentum fluxes by advection and diffusion in a specified direction. It is not an indication about the condition of the flow, if laminar or turbulent. For this purpose, the Reynolds number must take into account the role played by the penetration of momentum by diffusion normal to the principal flow direction and the momentum advection in the principal flow direction.

Recent theories (Bejan, 1981) suggest that a fluid stream will become turbulent when it passes from a viscous (stable) to an inviscid (unstable) condition. It can be shown that the wavelength of the meandering of a fluid stream is related to the thickness of this stream normal to the flow. Therefore, it is easy to understand that the larger the momentum advected in the x-direction, the larger will be the chance of the flow to become turbulent, since the time scale gets smaller as the flow velocity increases. In the other hand, the smaller the time scale to propagate the momentum flux by diffusion across the fluid stream, the bigger is the chance of the flow to remain viscous or, in other words, stable. This suggests that a ratio between the momentum flux by advection in the x-direction (tendency to become unstable) and the momentum flux by diffusion in the y-direction (tendency to be stable) is the appropriate parameter to indicate the conditions (laminar or turbulent) of the flow. This ratio is given by

$$\frac{A_x^M}{D_y^M} \approx \frac{\rho\, u\, u}{\mu \dfrac{\partial u}{\partial y}} \tag{12}$$

Applying scale analysis (Bejan, 1995), it results

$$Re_\delta^u = \frac{\rho\, u\delta}{\mu} \tag{13}$$

where it can be realized that now the characteristic length is the thickness of the fluid stream and the velocity is along the fluid stream. If the flow is inside a duct, the length δ will be replaced by the duct diameter, recovering the well-known Reynolds number, $\rho uD/\mu$, which indicates the condition (laminar or turbulent) of the flow inside a duct. Bejan, 1995, shows that this parameter, as defined by Eq. (13), with the proper thickness of the fluid stream, is of the order 10^2 for general flows. In this work we are not concerned with the determination of the magnitude of the Reynolds number that defines the transition from laminar to turbulent, but just to show, again, that the Reynolds number, being a relation between momentum fluxes by advection and diffusion, carries a strong physical meaning.

The above analysis clearly shows that the Reynolds number can be defined in two different ways, each one with a different physical interpretation. If the ratio of momentum fluxes is taken in the same direction, the Reynolds number tells about the boundary layer characteristics of the flow. If defined considering the diffusion transversal to the flow direction, it tells us about the flow regime, if laminar or turbulent. This interpretation also puts clear the importance played by the characteristic length in the Reynolds number.

Finally, few comments about the usual interpretation for the Reynolds number encountered in the majority of the textbooks in fluid mechanics, is worthwhile. The statement that the Reynolds number is the ratio between inertia and viscous forces does not encounter physical support. In fact, for inertia forces to exist it is required to have momentum variation. For example, in a fully developed flow inside a duct the momentum flow by advection is constant and, as a consequence, the inertia forces are zero. Therefore, the usual interpretation of the Reynolds number is not correct. Bejan, (Bejan, 1995), quotes that, apparently, the only significance of the Reynolds number is related to its square root, which is the ratio between the length of the plate and the boundary layer thickness (L/δ). This is in agreement with the interpretation of the Reynolds number as been a parameter telling about the boundary layer characteristics of the flow. In fact, $(L/\delta)^2$ is the ratio of advection and diffusion of momentum in the principal flow direction. It is our feeling that seeing the Reynolds number as a momentum flux relation helps in understanding its role in fluid flows.

The Peclet number. The Peclet number can be interpreted as the Reynolds number counterpart for thermal energy transfer. Therefore, it applies the previous analysis done for the Reynolds number, replacing the flux of momentum by the flux of thermal energy in Eqs. (4), (5) and (6).

Referring again to Fig. 1, if the energy flux by advection in the x-direction is divided by the energy flux by diffusion in the same direction, one obtains

$$\frac{A_x^H}{D_x^H} \approx \frac{\rho u c_p T}{k \frac{\partial T}{\partial x}} \tag{14}$$

Applying scale analysis (Bejan, 1995), recalling that for Pr<<1 the order of the velocity which prevails inside the thermal boundary layer is u_∞, one can write

$$\frac{u_\infty L}{\alpha} = Pe_L^u = Re_L^u \, Pr \tag{15}$$

which is the well known Peclet number. This parameter also tells about the boundary layer characteristics of the flow, now related to the fluxes of energy. It should be observed that the Prandtl number, multiplying the Reynolds number, is a factor that amplifies or diminishes the thermal boundary layer behavior compared to the momentum boundary layer behavior of the flow.

Similarly to what was done for the momentum fluxes, Eq(11), (12) and (13) applies, replacing the Reynolds number by Peclet number, what means to replace the scalar momentum by the scalar energy. This means that the Peclet number can also be defined using different flux ratios, giving raise to dimensionless numbers with different interpretations. For Pr>>1 u scales as $u_\infty.\delta_T/\delta$, since only part of the free stream u_∞ velocity acts inside the thermal boundary layer, giving for the flux relation

$$\frac{A_x^H}{D_x^H} \approx Re_L Pr \, Pr^{-\frac{1}{3}} \tag{16}$$

showing now that the Peclet number is the ratio of the transport fluxes times $Pr^{1/3}$.

The Rayleigh and Boussinesq numbers

The dimensionless numbers so far considered were obtained relating the momentum and energy fluxes in a forced convection flow. Similar analysis can be done for natural convection flows, where the thermal boundary layer is coupled with the momentum boundary layer. Again, it is possible, using scale analysis, to demonstrate that the governing equations for a 2D incompressible flow for determining the u, v, p and T variables reduces to a system of equations comprising the mass, v-momentum and energy conservation equations (Bejan, 1995), as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{17}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 v}{\partial x^2} + g\,\beta\,(T - T_\infty) \tag{18}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial x^2} \tag{19}$$

Pressure is related to the hydrostatic flow outside the boundary layer and buoyancy is represented in terms of temperature using the equation of state for ideal gases. Boussinesq approximation is employed, whereby the density variation is taken into account only in the buoyancy term. Recall that the cross momentum equation, direction x in this case, is again discarded. This means that the natural convection flow in a vertical plate can be also considered as a boundary layer flow.

Fig. 2 depicts the thermal and momentum boundary layers for a fluid with Prandtl number greater than one and the energy fluxes by advection and diffusion involved in the flow. For natural convection flows the velocity field is a result of the physical process in which a fluid is submitted to a temperature difference. Therefore, the velocity can't take part, directly, into an input dimensionless number for this type of flow, since, as already pointed out, the scale of the v velocity is not known. Applying scale analysis in the region (H-$\delta_{T\}}$, for the energy equation, following (Bejan, 1995) one obtains the relation between v and δ_T, given by

$$v \approx \frac{\alpha H}{\delta_T{}^2} \tag{20}$$

This equation clearly shows the coupling between the y-momentum and energy equations and is valid for any Prandtl number. To find the other relations for permitting that the scale of v and δ_T be determined, the momentum equations need to be used. Performing a scale analysis of the momentum equation in the same (H-δ_T) region, one obtains

$$\left(\frac{H}{\delta_T}\right)^4 Ra_H^{-1} Pr^{-1} \quad , \quad \left(\frac{H}{\delta_T}\right)^4 Ra_H^{-1} \quad , \quad 1 \tag{21}$$

For high Prandtl number fluids the δ_T scales as

$$\delta_T \approx H\, Ra_H^{-\frac{1}{4}} \tag{22}$$

and for low Prandtl number fluids, δ_T scales as

$$\delta_T \approx H\, Bo_H^{-\frac{1}{4}} \tag{23}$$

where Bo is the Boussinesq number given by Ra.Pr. Knowing the scales of v and δ_T it is now possible to use the flux ratios in order to finding out the dominant parameters of natural convection flows.

To start, it is considered the ratio of momentum fluxes by advection and diffusion, given by

$$\frac{A_y^M}{D_y^M} \approx \frac{\rho\, v v}{\mu \dfrac{\partial v}{\partial y}} \tag{24}$$

For high Prandtl number fluids the flux ratio is given by

$$\frac{A_y^M}{D_y^M} \approx \frac{H^2}{\delta_T^2 P_r} \approx \frac{\sqrt{Ra_H}}{Pr} \tag{25}$$

and for low Prandtl number fluids by

$$\frac{A_y^M}{D_y^M} \approx \frac{\sqrt{Bo}}{Pr} \tag{26}$$

Following the same procedure the ratio between the advection and diffusion of energy fluxes can be written as

$$\frac{A_y^H}{D_y^H} \approx \frac{\rho\, v c_p T}{k \dfrac{\partial T}{\partial y}} \approx \frac{H^2}{\delta_T^2} \tag{27}$$

Using the v scale and the expression for the thermal boundary layer, one finds, for high Prandtl number fluids,

$$\frac{A_y^H}{D_y^H} \approx \sqrt{Ra_H} \tag{28}$$

and for low Prandtl number fluids

$$\frac{A_y^H}{D_y^H} \approx \sqrt{Bo_H} \tag{29}$$

Therefore, the square root of **Ra** and **Bo** is in fact the parameter that contains strong information about the physics of heat transfer in natural convection flows. For example, when Ra= 10^6 it means that the energy transported by advection is of order of 10^3 greater than the energy transported by diffusion in the same direction. This, of course, is extremely useful for understanding the character of the flow, giving to the **Ra** and **Bo** numbers a rich physical meaning.

One could also start from the previous knowledge that the Reynolds and Peclet numbers represents the ratio of the advection and diffusion fluxes in a flow, irrespective if the flow is forced or natural. Therefore, starting from the definition of Reynolds and Peclet numbers given by Eq. (10) and (15) and their counterparts for Pr>>1, and using the scale for velocity according to Eq. (20) one can find the relation between Reynolds and Peclet with Ra, or Bo. Recall again that in natural convection flows Re and Pe are not input parameters, but calculated after the flow is determined. Using the previous defined

Reynolds and Peclet numbers given by vH/ν and vH/α, respectively, and using Eq.(20) and the scale for δ_T, one finds, for Pr>>1,

$$\frac{A_y^M}{D_y^M} \approx \frac{H^2}{\delta_T^2 Pr} \approx \frac{\sqrt{Ra_H}}{Pr} \approx Re_H^v \qquad (30)$$

and

$$\frac{A_y^H}{D_y^H} \approx \sqrt{Ra_H} \approx Pe_H^v \qquad (31)$$

The above equations show that, again, the ratio of advection and diffusion of momentum and energy gives rise to the Reynolds and Peclet numbers when the proper velocity scale is used for calculating them in natural convection flows, as expected. Since in these flows these parameters are not known, the Rayleigh number is used as input parameter which correlates to the Reynolds and Peclet numbers according to Eqs.(30) and (31).

For Pr<1 fluids, Ra_H should be substituted by Bo_H in Eqs.(30) and (31). Therefore, if the Reynolds number is interpreted as the ratio between the advection and diffusion fluxes of momentum, its counterpart in natural convection is the square root of Ra_H/Pr (or square root of Bo_H/Pr if Pr<1.) For the Peclet number the counterpart is the square root of Ra_H (or square root of Bo_H if the Pr<1). Table 1 below summarizes the relation between the fluxes ratio and the dimensionless numbers.

3. CONCLUSIONS

This paper presented a physical interpretation of some of the most important dimensionless numbers used in fluid mechanics and heat transfer, like Reynolds and Rayleigh numbers. The definition based on the relative importance of the transport mechanisms of advection/diffusion and on scale analysis allows a clear interpretation of Reynolds, Peclet and Rayleigh numbers. The dimensionless numbers defined in this manner makes easy the interpretation of basic physical phenomena as well as to better understand some physical assumptions, as made in boundary layer flows, for example. It also helps to understand the role of the characteristic length appearing in each dimensionless number.

Table 1 – Summary of the relation between fluxes ratios and the dimensionless numbers

	FORCED FLOW		NATURAL FLOW	
	$\frac{A_x^M}{D_x^M}$	$\frac{A_x^H}{D_x^H}$	$\frac{A_y^M}{D_y^M}$	$\frac{A_y^H}{D_y^H}$
$Pr << 1$	Re	Pe	$\sqrt{Bo_H}\ Pr^{-1}$	$\sqrt{Bo_H}$
$Pr >> 1$	Re	$PePr^{-\frac{1}{3}}$	$\sqrt{Ra_H}\ Pr^{-1}$	$\sqrt{Ra_H}$

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