PANEL METHOD FORMULATION FOR OSCILLATING AIRFOILS IN SONIC FLOW

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Abstract. *The mathematical model for two-dimensional unsteady sonic flow, based on the classical diffusion equation with imaginary coefficient, is presented and discussed. The main purpose is to develop a rigorous formulation in order to bring into light the correspondence between the sonic, supersonic and subsonic panel method theory. Source and doublet integrals are obtained and Laplace transformation demonstrates that, in fact, the source integral is the solution of the doublet integral equation. It is shown that the doublet-only formulation reduces to a Volterra integral equation of the first kind and a numerical method is proposed in order to solve it. To the authors' knowledge this is the first reported solution to the unsteady sonic thin airfoil problem through the use of doublet singularities. Comparisons with the source-only formulation are shown for the problem of a flat plate in combined harmonic heaving and pitching motion.*

Keywords: *Unsteady aerodynamics, Sonic flow, Panel method, Linearized theory*

1. INTRODUCTION

In spite of the well-known fact that airfoil mathematical models must be essentially nonlinear in the transonic range, a linearized treatment appears to be meaningful whenever the motion presents a certain degree of unsteadiness and low amplitude as has been observed by Bisplinghoff et al (1955). The present work was undertaken basically due to the following considerations: 1) The observation by Garrick (1957) that the treatment of linearized nonstationary flow in the sonic range should not be disconsidered, for this provides a continuous bridge in analytical results from subsonic through transonic to supersonic speeds, and 2) the present work unifies numerical methods already presented for subsonic (Soviero, 1993) and supersonic (Soviero & Ribeiro, 1995) airfoils in oscillating compressible flow.

Satisfactory formulations for the problem of an airfoil in unsteady sonic flow induced by small vertical harmonic oscillations have been available since the 1950's. The work of Nelson & Berman (1953) for the problem is simple and results in a numerical quadrature involving an elementary function and the known downwash distribution along the airfoil chord. In fact, the arbitrary, unsteady motion of a thin airfoil in sonic or supersonic ranges can be represented by placing an appropriated sheet of sources along the airfoil chord. Alternatively, the problem can be analyzed by means of a Laplace transformation in the independent streamwise variable, as can be seen in Bisplinghoff et al (1955), Dowell et al (1973) and Landahl (1961).

In the present investigation the disturbance field corresponding to compression or expansion over an airfoil in sonic flow will be represented by the action of doublets, although sources alone may suffice (which leads, in fact, to the usual way to tackle the problem). The mathematical model for two-dimensional unsteady transonic flow based on the classical diffusion equation with imaginary coefficient (Morse & Feshbach, 1953) is presented and discussed. The aim is to develop a rigorous formulation that shows the correspondence between the sonic flow and the classical panel method theory. Through the application of Green's theorem to the diffusion equation an integral solution is obtained. It is shown that the source-only formulation is equivalent to that presented in Nelson & Berman (1953) and that the doublet-only formulation reduces to a Volterra integral equation of the first kind. The relationship between source and doublet formulation is derived by Laplace transformation, which demonstrates that the source-only formulation is a solution to the Volterra equation. Finally a numerical method is proposed in order to solve the Volterra equation. To the authors' knowledge this is the first reported solution that uses doublet-only singularities. The solution involves the concept of the finite part of an integral and its study is essential for the three-dimensional thin wing solution, as reported in Soviero & Pinto (2000).

2. MATHEMATICAL MODEL

In a reference frame that translates uniformly with the undisturbed flow velocity, U, close to the sound speed a , the complex velocity perturbation potential Φ due to the harmonic small-amplitude motion of a thin airfoil may be described mathematically by a linear differential equation, as given by Landahl (1961), in the frequency domain

$$\frac{\partial^2 \mathbf{\Phi}}{\partial Z^2} - \frac{2Ui\omega}{a^2}\frac{\partial \mathbf{\Phi}}{\partial X} + \frac{\omega^2}{a^2}\mathbf{\Phi} = 0 \qquad \mathbf{(1)}$$

The velocity U lies in the positive X direction and Φ is made nondimensional relative to U and a reference length L, in the present work the semi-chord of the airfoil.

Defining now a new complex potential as

$$\boldsymbol{\phi}(x,z) = \mathbf{\Phi}(X,Z)\, \mathbf{exp}\,(i\omega X/2aM) \qquad \mathbf{(2)}$$

where ω is the angular frequency of the motion, M the undisturbed flow Mach number, and using the transformation

$$x = X/L \qquad z = MZ/L \qquad \mathbf{(3)}$$

the governing equation of the problem, Eq. (1), can be rewritten as the classical diffusion equation with imaginary coefficient,

$$\frac{\partial^2 \phi}{\partial z^2} - 2ik\frac{\partial \phi}{\partial x} = 0 \tag{4}$$

where k = ω L / U is the reduced frequency. Equation (4) is parabolic and posseses source- and doublet-like elementary solutions.

In the transformed plane the linearized boundary condition on the airfoil surface is

$$w(x) = \frac{\partial \phi}{\partial z} = \frac{\exp(ikx/2)}{M}\left[\frac{\partial h}{\partial x} + ikh\right] \tag{5}$$

where $h\ (x)$ represents the airfoil surface nondimensional vertical displacement. After Eq. (4) is solved for ϕ, subjected to boundary condition (5), the complex pressure coefficient on the airfoil surface is obtained as

$$C_p(x) = -\frac{2}{UL}\exp(-ikx/2)\left[\frac{\partial \phi}{\partial x} + \frac{ik}{2}\phi\right] \tag{6}$$

Similar mathematical models for subsonic and supersonic flows leading respectively to the Helmholtz and to the hyperbolic Helmholtz equations are presented in Soviero (1993) and Soviero & Ribeiro (1995).

3. PANEL METHOD FORMULATION

Consider Eq. (4). The elementary source and doublet (in the z direction) solutions for this equation at point (x,z), are, respectively,

$$\phi_S\,(x,z) = \frac{1}{\sqrt{2ik\pi(x-x_0)}}\exp\left[-\frac{ik(z-z_0)^2}{2(x-x_0)}\right] \tag{7}$$

$$\phi_D\,(x,z) = \frac{ik(z-z_0)}{(x-x_0)\sqrt{2ik\pi(x-x_0)}}\exp\left[-\frac{ik(z-z_0)^2}{2(x-x_0)}\right] \tag{8}$$

for $x > x_0$ and zero otherwise, where (x_0 , z_0) is the position of the singularity. The doublet solution comes from the z_0 -derivative of the source solution as is usual in the classical panel method.

Application of Green's theorem to the diffusion equation is well known and is described in Morse & Feshbach (1953). For an airfoil on the x-axis (i.e., $z_0 = 0$) with leading edge at x = 0, it yields the following integral identity for the upper flow region (z > 0),

$$\phi(x,z) = \int_0^x \left(\frac{\partial\phi}{\partial z}\right)_{upper} \phi_S \, dx_0 - \int_0^x (\phi)_{upper} \, \phi_D \, dx_0 \qquad \textbf{(9)}$$

and similarly for the lower region. Notice the appearance of source and doublet integrals, like in the classical incompressible panel method. For a source-only formulation, only the first integral remains, providing a simple solution to the Eq. (4) on the airfoil surface, namely,

$$\phi(x) = \int_0^x \sigma(x_0) \, \frac{1}{2\sqrt{2ik\pi(x-x_0)}} \, dx_0 \qquad \textbf{(10)}$$

This is equivalent to the solution of the thin airfoil in small vertical harmonic motion presented in Bisplinghoff et al (1955) and Nelson & Berman (1953). One recognizes $\sigma(x_0)$ as the surface source density of the first fundamental solution, see Eq. (7). The above solution is completed with the application of the boundary condition (5). For a thin airfoil, the density of the source distribution over the chord is given by

$$\sigma(x) = 2\,w(x) = 2\,\frac{\partial\phi}{\partial z}(x) \qquad \textbf{(11)}$$

This result is analogous to the one obtained for the thin airfoil thickness problem in subsonic or supersonic steady flow. Observe, however, that the airfoil surface slope, w(x), and consequently the source density itself, may be different for upper- and lower-surface solutions.

For a doublet-only formulation, only the last integral in Eq. (9) remains. Differentiation in z and introduction of the boundary condition (5) on the left hand side lead to

$$\frac{\partial\phi}{\partial z}(x) = -\int_0^x \mu(x_0) \frac{ik}{2(x-x_0)\sqrt{2ik\pi(x-x_0)}} \, dx_0 \qquad \textbf{(12)}$$

which is a Volterra's integral equation of first kind. Again, one recognizes $\mu(x)$ as the doublet density of the second fundamental solution defined in Eq. (8), and obtained from

$$\mu(x) = 2\,\phi(x) \qquad \textbf{(13)}$$

where μ is placed on the chord and ϕ is taken on the airfoil surface. The integral equation, Eq.(12), has the same form as for the subsonic airfoil camber problem. In fact, the classical direct and inverse problems of airfoil theory are fully retrieved from Eqs. (10) and (12).

Like in Soviero & Ribeiro (1995), a remarkable result can be obtained when Eq. (12) is solved by Laplace transformation. After introducing Eq. (13) the integral equation can be rewritten as (Krasnov et al, 1977)

$$w(x) = -\phi(x) * \frac{ik}{x\sqrt{2ik\pi x}} \tag{14}$$

where the symbol * stands for the convolution product. Taking the Laplace transform of this equation, one obtains

$$w^*(s) = \frac{\sqrt{2ik}}{\sqrt{\pi}}\,\phi^*(s)\,\sqrt{s}\,\Gamma(1/2) = \sqrt{2ik}\,\phi^*(s)\,\sqrt{s} \tag{15}$$

where Γ is the Gamma function. An accessory boundary condition, $\phi(0) = 0$, is introduced in Eq.(15). Rearranging, one finds

$$\phi^*(s) = \frac{w^*(s)}{\sqrt{2ik}}\,\frac{1}{\sqrt{s}} \tag{16}$$

This relation, after an inverse Laplace transform, becomes

$$\phi(x) = \int_0^x \frac{w(x_0)}{\sqrt{2ik\pi(x-x_0)}}\,dx_0 \tag{17}$$

returning the same result as in Eqs. (10) and (11). This shows that the source integral is a solution to the doublet equation (12). In other words, Eq. (12) is the true boundary integral equation equivalent to the system formed by the diffusion equation, Eq. (4), and the boundary condition, Eq. (5).

4. NUMERICAL SOLUTION

In order to obtain a panel method solution for the doublet-only formulation - Eqs. (12) and (13) - the x axis between 0 and 2 (i.e., the airfoil chord) is uniformly divided into a number

N of small elements. As in classical vortex lattice method control points are chosen at the center of each panel. Doublet density μ, and consequently φ, is taken as piecewise constant and their values are discontinuous from one panel to the next as the solution proceeds along the airfoil chord.

The unitary induced velocity a_{kj} for a control point x_k, due to a panel extending from x_j to x_{j+1} and constant doublet density $\mu_j = 1$ is written as

$$a_{kj} = -\sqrt{\frac{ik}{8\pi}} \int_{x_j}^{x_{j+1}} \frac{dx_0}{(x_k - x_0)^{3/2}} \tag{18}$$

and might assume one of the following cases

$$a_{kj} = 0, \qquad x_c < x_j \tag{19a}$$

$$a_{kj} = \sqrt{\frac{ik}{2\pi}} \left[\frac{1}{\sqrt{x_k - x_{j+1}}} - \frac{1}{\sqrt{x_k - x_j}}\right], \qquad x_c > x_{j+1} \tag{19b}$$

$$a_{kj} = -\sqrt{\frac{ik}{2\pi}} \frac{1}{\sqrt{x_k - x_j}}, \qquad x_c = (x_j + x_{j+1})/2 \tag{19c}$$

In order to obtain a_{kj}, the finite-part integral concept (Heaslet & Lomax, 1957), originally introduced and generalized by Hadamard, must be employed when evaluating Eq. (18), since $x_j < x_k < x_{j+1}$. The discretized form of Eq. (12) on an arbitrary control point i, reads

$$w(x_i) + \sum_{j=1}^{i} a_{kj}\, \mu_j = 0, \qquad i = 1, N \tag{20}$$

This procedure is applied for each control point until the airfoil trailing edge (x = 2) is reached.

Observe that the above method is, as in the supersonic case (Soviero & Ribeiro, 1995) rigorously equivalent to the assembly of a linear matrix equation, given by

$$[A]\, \mu = B \tag{21}$$

where [A] is the influence coefficient matrix, μ is the vector containing the doublet density values at the panels and B contains the normal velocity component, w, at each control point. The matrix [A] is lower triangular as a consequence of the upwind dependency of sonic linearized phenomena.

5. RESULTS

The doublet-only method was used to compute the two basic unsteady flows applied to a flat plate: harmonic heaving and pitching motion. Due to the linearity of the mathematical model, these two solutions can be combined to produce more complex airfoil motions. The purpose of the computations were twofold: 1) to validate the present method through comparisons with previous solutions and, 2) to evaluate the accuracy of the panel method solution and its dependency upon discretization. For a complete study about convergence concerning heaving, pitching and combined motions the reader is referred to Pinto (1999).

For a flat plate in harmonic heave, the vertical positions of all its points vary according to

$$h(x,t) = h(x)\exp(i\omega t) = h_0 \exp(i\omega t) \tag{22}$$

For harmonic pitch around point x_p, the angle-of-attack varies according to

$$\alpha = \alpha_0 \exp(i\omega t) \tag{23}$$

which means that the plate surface location is given by

$$h(x,t) = -\alpha_0 (x - x_p) \exp(i\omega t) \tag{24}$$

The lift and moment coefficients for the airfoil are given by

$$C_l(t) = [C_{lr} + iC_{li}]\exp(i\omega t) = \int_0^2 (C_{p,lower} - C_{p,upper})\frac{dx}{L} \tag{25a}$$

$$C_m(t) = [C_{mr} + iC_{mi}]\exp(i\omega t) = \int_0^2 (C_{p,lower} - C_{p,upper})(x - x_{ref})\frac{dx}{L^2} \tag{25b}$$

In the present work the reference point, x_{ref}, for pitching moment is taken as the leading-edge of the flat plate.

Figure 1 shows real and imaginary pressure coefficient differences, ($C_{p,lower}$ - $C_{p,upper}$), along the profile chord in combined heaving and pitching about the leading edge. Results obtained with both the doublet-only and source-only formulations are shown.

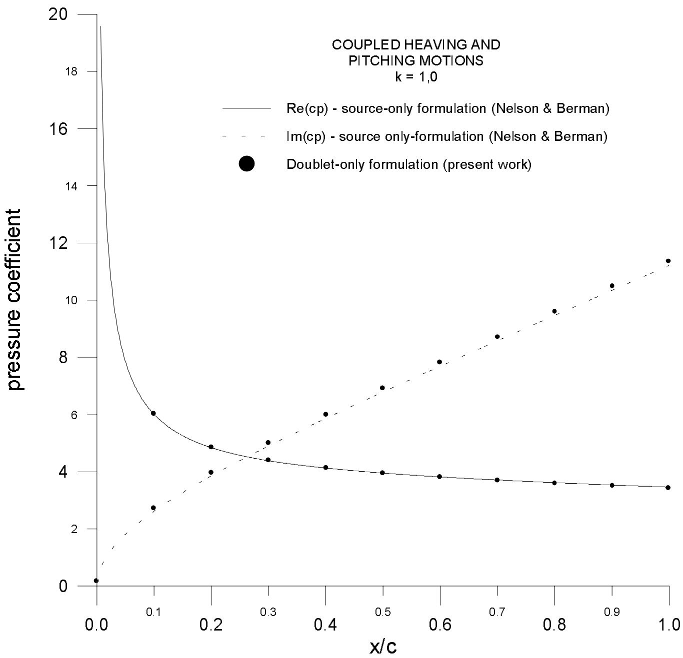

Figure 1 - Pressure coefficient jump

It must be pointed out the singular behavior of the real part coefficient at the leading edge, which is typical of the subsonic flow, and the non-zero pressure coefficient jump at the trailing edge, which is typical of the supersonic one. The pattern represented by Fig. 1 is common to all calculations made for reduced frequencies ranging from 0^+ to 3.5 as can be observed in Pinto (1999).

Figures 2 and 3 show real and imaginary parts of lift and moment coefficients computed by the doublet-only method for a flat plate in combined pitching and heaving motion. The wide range of frequencies shown, not all of then realistic confirms the consistency of the proposed numerical scheme .Also plotted in Figures 2 and 3 are values computed by Nelson & Berman obtained from numerical integration of Eq. (10) and boundary condition, Eq.(6), that is, the source- only formulation. As can be observed, the overall agreement is fairly good. Some of the computations required about 2000 panels but even those cases did not exceed 4 seconds of CPU time on a personal computer (Pentium, 133 MHz).

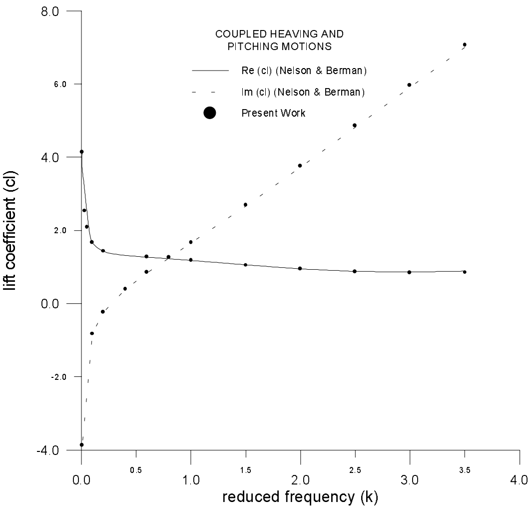

Figure 2 - Lift coefficient

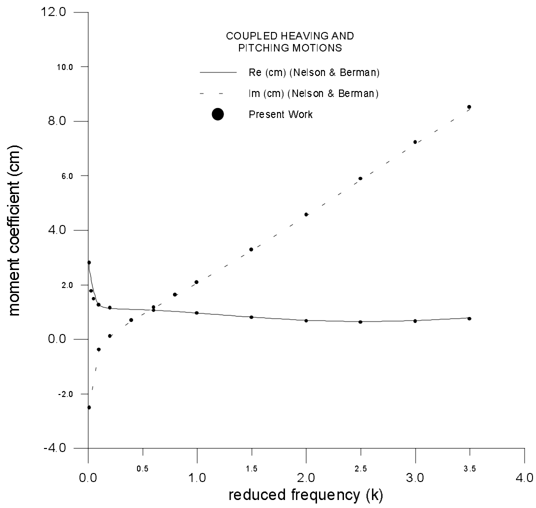

Figure 3 - Pitching moment coefficient

6. CONCLUSION

This work presents a rigorous formulation for the application of panel methods to unsteady sonic flow. It is shown that in analogy with subsonic and supersonic flows there is also a doublet solution to the problem. The lack of interest for the using of doublet is due to the simplicity of the problem, what allows a source-only approach and produces a solution by straightforward quadrature. Whereas in the subsonic case, sources and doublets are necessary to solve the thickness and camber problems, respectively, in the sonic regime either one of those singularities can be used. It has been shown that the source and doublet integrals are both solutions to the diffusion equation with imaginary coefficient applied to airfoil at sonic linearized flow. The numerical implementation of both formulations showed that the doublet-only method is as accurate and efficient as the source method.

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REFERENCES

Bisplinghoff, R. L., Ashley, H. and Halfman, R. L., 1955, Aeroelasticity, Addison-Wesley, Reading, Ma.

Dowell, E. H. et al., 1978, A Modern Course in Aeroelasticity, Sijthoff & Nordhoff, Alphen aan den Rijn, The Netherlands.

Garrick, I. E., 1957, Nosteady wing characteristics, in High Speed Aerodynamics and Jet Propulsion, Vol. 7, Sec. F, eds A. F. Donovan and H. R. Lawrence, Princeton University Press, Princeton, NJ.

Heaslet, M. A. & Lomax, H., 1957, Supersonic and transonic small perturbation theory, in High Speed Aerodynamics and Jet Propulsion, Vol. 6, Sec. D, eds A. F. Donovan and H. R. Lawrence, Princeton University Press, Princeton, NJ.

Krasnov, M., Kissélev, A. and Makarenko, G., 1977, Équations Integrales, Editions Mir, Moscou.

Landahl, M., 1961, Unsteady Transonic Flow, Pergamon Press, Oxford.

Morse, P. M. & Feshbach, H., 1953, Methods of Theoretical Physics, Part I, McGraw-Hill Book Company, New York, NY.

Nelson, H. C. & Berman, J. H., 1953, Calculations of the forces and moments for an oscillating wing-aileron combination in two-dimensional potential flow at sonic speed, NACA R 1128.

Pinto, F. H. L., 1999, Generalized vortex lattice method for sonic linear and unsteady flow regime, M.Sc. thesis, Instituto Tecnológico de Aeronáutica, São José dos Campos, São Paulo, Brazil, (in Portuguese).

Soviero, P. A. O., 1993, Generalized vortex lattice method for oscillating thin airfoil in subsonic flow, AIAA Journal, Vol. 31, No. 12, pp. 2380-2382.

Soviero, P. A. O. & Ribeiro, R. S., 1995, Panel method formulation for oscillating airfoils in supersonic flow, AIAA Journal, Vol. 33, No. 9, pp. 1659-1666.

Soviero, P. A. O. & Pinto, F. H. L., 2000, Unsteady lifting surface theory in sonic flow: the problem revisited. AIAA Journal, Vol. 38, No. 5, pp. 931-933.