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INVERSE PROBLEMS OF PARAMETER AND FUNCTION ESTIMATION IN HEAT TRANSFER

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Abstract. *This paper addresses the solution of inverse heat transfer problems. Basic concepts and methods of solution are briefly described. The Levenberg-Marquardt method and the sequential estimation technique of parameter estimation, as well as the conjugate gradient method of function estimation, are applied to practical inverse problems.*

Keywords: *Inverse Problems, Ill-Posed Problems, Levenberg-Marquardt Method, Sequential Parameter Estimation Technique, Conjugate Gradient Method*

1. INTRODUCTION

Inverse heat transfer problems rely on temperature and/or heat flux measurements for the estimation of unknown quantities appearing in the analysis of physical problems in this field. As an example, inverse problems dealing with heat conduction have been generally associated with the estimation of an unknown boundary heat flux, by using temperature measurements taken below the boundary surface. Therefore, while in the classical direct heat conduction problem the cause (boundary heat flux) is given and the effect (temperature field in the body) is determined, the inverse problem involves the estimation of the cause from the knowledge of the effect.

Inverse problems are mathematically classified as *ill-posed*, whereas standard heat transfer problems are *well-posed* (Hadamard, 1923, Tikhonov and Arsenin, 1977, Beck and Arnold, 1977, Alifanov, 1994, Beck et al, 1985, Alifanov et al, 1995, Dulikravich and Martin, 1996, Sabatier, 1978, Murio, 1993, Trujilo and Busby, 1997, Hensel, 1991, Kurpisz and Nowak, 1995, Denisov, 1999, Yagola et al, 1999, Ramm et al, 2000, Ozisik and Orlande, 2000). The solution of a well-posed problem must satisfy the conditions of existence, uniqueness and stability with respect to the input data. The existence of a solution for an inverse heat transfer problem may be assured by physical reasoning. On the other hand, the uniqueness of the solution of inverse problems can be mathematically proved only for some special cases. Also, the inverse problem is very sensitive to random errors in the measured input data, thus requiring special techniques for its solution in order to satisfy the stability condition. For a long time it was thought that, if any of the conditions required for well-posedness were violated, the problem would be unsolvable or the results obtained from such a solution would have no practical importance. It was *Tikhonov's regularization procedure* (Tikhonov and Arsenin, 1977), *Alifanov's iterative regularization techniques* (Alifanov, 1994, Alifanov et al, 1995) and *Beck's sequential function specification approach* (Beck et al, 1985) that revitalized the interest in the solution of inverse heat transfer problems.

Inverse problems can be solved as a *parameter estimation* approach or as a *function estimation* approach. If some information is available on the functional form of the unknown quantity, the inverse problem can be reduced to the estimation of few unknown parameters. On the other hand, if no prior information is available on the functional form of the unknown, the inverse problem can be regarded as a function estimation approach in an infinite dimensional space of functions. Techniques for the solution of inverse problems, as parameter estimation and function estimation approaches, are presented below.

2. PARAMETER ESTIMATION

In parameter estimation problems, we consider that some information is available on the functional form of the unknown quantity, which can be, for example, the transient heat flux at the surface of a space vehicle reentering the atmosphere. Let us assume that the unknown function $g(t)$ can be represented in the following general linear form:

$$g(t) = \sum_{j=1}^{N_{par}} P_j C_j(t) \tag{1}$$

where P_j , j=1,...,N_{par} , are unknown constants and $C_j(t)$ are known trial functions. Therefore, the inverse problem of estimating the unknown function $g(t)$ is reduced to the problem of *estimating a finite number of parameters* P_j, where the number of parameters, N_{par}, is supposed to be chosen in advance. Another example of a parameter estimation problem is the identification of unknown constant thermophysical properties, such as thermal conductivity and volumetric heat capacity.

2.1 Estimation Techniques

For the solution of parameter estimation problems we consider here the use of minimization techniques. An objective function is then defined, involving the difference between measured and estimated variables, like temperature, for example. In order to appropriately choose the objective function, some hypotheses regarding the measurement errors are required. Let us assume valid the following statistical hypotheses (Beck and Arnold, 1977): the errors in the measured variables are additive, uncorrelated, normally distributed, with zero mean and known constant standard-deviation; only the measured variables appearing in the objective function contain errors; and there is no prior information regarding the values and uncertainties of the unknown parameters. In this case, the least squares norm becomes a minimum variance estimator (Beck and Arnold, 1977). The least squares norm can be written in matrix form as:

$$S_{OLS}(\mathbf{P}) = [\mathbf{Y} - \mathbf{T}(\mathbf{P})]^T [\mathbf{Y} - \mathbf{T}(\mathbf{P})] \tag{2}$$

where **P** is the vector of unknown parameters and

$$[\mathbf{Y}-\mathbf{T}(\mathbf{P})]^T = [\vec{Y}_1-\vec{T}_1(\mathbf{P}), \vec{Y}_2-\vec{T}_2(\mathbf{P}), \cdots, \vec{Y}_I-\vec{T}_I(\mathbf{P})] \tag{3.a}$$

The row vector $[\vec{Y}_i-\vec{T}_i(\mathbf{P})]$ contains the difference between measured (Y) and estimated (T) variables for each of the M sensors at time t_i, i = 1, …, I, that is,

$$[\vec{Y}_i-\vec{T}_i(\mathbf{P})] = [Y_{i1}-T_{i1}(\mathbf{P}), Y_{i2}-T_{i2}(\mathbf{P}), \cdots, Y_{iM}-T_{iM}(\mathbf{P})] \qquad \text{for } i=1,\ldots,I \tag{3.b}$$

The iterative procedure of the *Levenberg-Marquardt Method* for the minimization of the *ordinary least squares norm* (2) is given by (Beck and Arnold, 1977, Ozisik and Orlande, 2000**)**:

$$\mathbf{P}^{k+1} = \mathbf{P}^k + (\mathbf{J}^T \mathbf{J} + l^k \mathbf{\Omega}^k)^{-1} \mathbf{J}^T [\mathbf{Y} - \mathbf{T}(\mathbf{P}^k)] \tag{4}$$

where k denotes the number of iterations, **J** is the sensitivity matrix, $\mathbf{\Omega}$ is a diagonal matrix and l is a scalar named damping parameter (Beck and Arnold, 1977, Ozisik and Orlande, 2000**)**. The purpose of the matrix term $l^k \mathbf{\Omega}^k$ in equation (4) is to damp oscillations and instabilities due to the ill-conditioned character of the problem.

If we relax the statistical hypotheses described above, by considering that some information regarding the unknown parameters is available, we can use the *maximum a posteriori objective function* in the minimization procedure (Beck and Arnold, 1977**)**. Such an objective function is defined as:

$$S_{MAP}(\mathbf{P}) = [\mathbf{Y} - \mathbf{T}(\mathbf{P})]^T \mathbf{W} [\mathbf{Y} - \mathbf{T}(\mathbf{P})] + (\boldsymbol{\mu} - \mathbf{P})^T \mathbf{V}^{-1} (\boldsymbol{\mu} - \mathbf{P}) \tag{5}$$

where **P** is assumed to be a random vector with known mean $\boldsymbol{\mu}$ and known covariance matrix **V**. Therefore, the mean $\boldsymbol{\mu}$ and the covariance matrix **V** provide *a priori* information regarding the parameter vector **P** to be estimated. Such information can be available from previous experiments with the same experimental apparatus or even from the literature data. By assuming valid the other statistical hypotheses described above regarding the experimental errors, the weighting matrix **W** is a diagonal matrix with the inverse of the covariance of the measurements on its diagonal (Beck and Arnold, 1977**)**.

The iterative procedure for the minimization of the maximum a posteriori objective function (5) is given by (Beck and Arnold, 1977**)**:

$$\mathbf{P}^{k+1} = \mathbf{P}^k + [\mathbf{J}^T \mathbf{W} \mathbf{J} + \mathbf{V}^{-1}]^{-1} \{\mathbf{J}^T \mathbf{W} [\mathbf{Y} - \mathbf{T}(\mathbf{P}^k)] + \mathbf{V}^{-1} (\boldsymbol{\mu} - \mathbf{P}^k)\} \tag{6}$$

The iterative procedure given by equation (6) can be written in a more convenient form for computational purposes, which avoids matrix inversions, by employing the *sequential estimation technique* (Beck and Arnold, 1977**)**. In such a case, the measurements in the estimation procedure are sequentially used, so that estimates are obtained based on the current measurement and on the estimates for the parameters obtained with the measurements previously used in the analysis.

It is considered for the sequential estimation technique that one single measurement is added to the estimation procedure at a time. Even if transient measurements of multiple sensors are used, they can also be arranged in such a manner, that is,

$$\mathbf{Y}^T = [Y_1, Y_2, \cdots, Y_n, \cdots Y_N] \tag{7}$$

where $N = I\,M$ and $n = m + (i-1)M$, for i=1,…,I and m=1,…,M.

The *computational algorithm for the sequential estimation technique* consists of the following basic steps (Beck and Arnold, 1977**)**:

Step 1. Initialize the iterative procedure by setting the iteration index k to 0 and making $\mathbf{P}^0 = \boldsymbol{\mu}$.

Step 2. Compute the estimate for the vector of unknown parameters sequentially, for n=0,…,(N-1), by using

$$\mathbf{A} = \mathbf{V}_n \mathbf{J}_{n+1}^T \tag{8.a}$$

$$\Delta = \mathbf{J}_{n+1} \mathbf{A} + W_{n+1}^{-1} \tag{8.b}$$

$$\mathbf{K} = \mathbf{A} \Delta^{-1} \tag{8.c}$$

$$E_{n+1} = Y_{n+1} - T_{n+1}(\mathbf{P}^k) \tag{8.d}$$

$$\mathbf{P}_{n+1}^{k+1} = \mathbf{P}_n^{k+1} + \mathbf{K}[E_{n+1} - \mathbf{J}_{n+1}(\mathbf{P}_n^{k+1} - \mathbf{P}^k)] \tag{8.e}$$

$$\mathbf{V}_{n+1} = \mathbf{V}_n - \mathbf{K}\,\mathbf{J}_{n+1} \mathbf{V}_n \tag{8.f}$$

$$\text{where} \quad \mathbf{V}_0 = \mathbf{V} \qquad \mathbf{P}_0^k = \boldsymbol{\mu} \qquad \mathbf{J}_n = \left[\frac{\partial T_n}{\partial P_1}, \cdots, \frac{\partial T_n}{\partial P_{N_{par}}} \right] \qquad W_{n+1} = s^{-2} \tag{8.g-j}$$

Step 3. Check convergence of the values estimated sequentially with all N measurements, that is,

$$\left\| \mathbf{P}_N^{k+1} - \mathbf{P}_N^k \right\| < e \tag{9.a}$$

If the criterion given by equation (9.a) is not satisfied, increment k, make

$$\mathbf{P}^k = \mathbf{P}_N^k \tag{9.b}$$

and return to step 2.

In equation (8.j), s denotes the standard deviation of the measurement errors, which is assumed constant and known. Note that the above algorithm does not contain matrix inversions because Δ is a scalar. Also, it was derived for a case where previous estimates were available for the vector of parameters and its covariance matrix. However, it can also be used for cases where no previous estimations are available, or if available, they have large uncertainty. For such cases, take $\boldsymbol{\mu}$ as any vector, say, with null components. Also, take $\mathbf{V}$ as a diagonal matrix with large values on the diagonal as compared to the square of the expected values for the parameters.

2.2 Statistical Analysis

By performing a statistical analysis it is possible to assess the accuracy of $\hat{P}_j$, which are the values estimated for the unknown parameters P_j, j=1,...,N_{par}. By assuming valid the statistical hypotheses about the measurement errors described above, the *covariance matrix*, of the estimated parameters $\hat{P}_j$, corresponding to the *ordinary least squares norm*, is given by (Beck and Arnold, 1977**)**:

$$\text{cov}(\hat{\mathbf{P}}) = (\mathbf{J}^T \mathbf{J})^{-1} s^2 \tag{10.a}$$

The *covariance matrix* of the estimated parameters $\hat{P}_j$, corresponding to the *maximum a posteriori objective function*, is given by (Beck and Arnold, 1977**)**:

$$\text{cov}(\hat{\mathbf{P}}) = [\mathbf{J}^T \mathbf{W} \mathbf{J} + \mathbf{V}^{-1}]^{-1} \tag{10.b}$$

The *standard deviations* for the estimated parameters can thus be obtained from the diagonal elements of $\text{cov}(\hat{\mathbf{P}})$ as

$$s_{\hat{P}_j} \equiv \sqrt{\text{cov}(\hat{P}_j, \hat{P}_j)} \qquad \text{for } j=1,...,N_{par} \tag{11}$$

Confidence intervals for the estimated parameters at the 99% confidence level can be obtained as

$$\hat{P}_j - 2.576\, s_{\hat{P}_j} \le P_j \le \hat{P}_j + 2.576\, s_{\hat{P}_j} \qquad \text{for } j=1,...,N_{par} \tag{12}$$

2.3 Design of Optimum Experiments

Optimum experiments can be designed by minimizing the hypervolume of the confidence region of the estimated parameters, in order to ensure minimum variance for the estimates. This can be accomplished by maximizing the determinant of *Fischer's Information Matrix*, which is given by $\mathbf{F} \equiv \mathbf{J}^{\mathrm{T}} \mathbf{J}$ (Beck and Arnold, 1977**)**, in the so-called *D-optimum* approach. If the restriction of a large but fixed number of transient measurements of M sensors is considered, optimum experiments can be designed by examining an alternative form of the matrix **F**, the elements of which are given by

$$[\mathbf{F}_I^*]_{r,s} = \frac{1}{M t_f} \sum_{m=1}^{M} \int_{t=0}^{t_f} \left(P_r \frac{\partial Y_m}{\partial P_r} \right) \left(P_s \frac{\partial Y_m}{\partial P_s} \right) dt \qquad \text{for } r,s = 1,...,N_{par} \tag{13}$$

where t_f is the duration of the experiment.

3. FUNCTION ESTIMATION

In this section we present a powerful iterative minimization scheme called the *Conjugate Gradient Method of Minimization with Adjoint Problem*, for solving inverse heat transfer problems of *function estimation*. In this approach, no *a priori* information on the functional form of the unknown function is considered available (Alifanov, 1994, Alifanov et al, 1995, Ozisik and Orlande, 2000), except for the functional space that it belongs to. To illustrate this technique, we consider the estimation of an unknown function $g(t)$, by using the transient readings $Y(t)$ of a single sensor located at x_{meas}. We assume that the unknown function belongs to the Hilbert space of square-integrable functions in the time domain (Alifanov, 1994, Alifanov et al, 1995, Ozisik and Orlande, 2000), denoted as $L_2(0,t_f)$, where t_f is the duration of the experiment.

In order to solve the present function estimation problem, the functional $S[g(t)]$ defined as

$$S[g(t)] = \int_{t=0}^{t_f} \{Y(t) - T[x_{meas}, t; g(t)]\}^2 dt \tag{14}$$

is minimized under the constraint specified by the corresponding direct problem. This is achieved with an iterative procedure by proper selection of the direction of descent and of the step size in going from iteration k to $k + 1$. The iterative procedure of the conjugate gradient method (Alifanov, 1994, Alifanov et al, 1995, Ozisik and Orlande, 2000) for the estimation of the function $g(t)$ is given by:

$$g^{k+1}(t) = g^k(t) - \beta^k d^k(t) \tag{15.a}$$

where β^k is the *search step size* and $d^k(t)$ is the *direction of descent*, defined as

$$d^k(t) = \nabla S[g^k(t)] + \gamma^k d^{k-1}(t) \tag{15.b}$$

The *conjugation coefficient* γ^k can be computed with different expressions, including:

$$\gamma^k = \frac{\int_{t=0}^{t_f} \nabla S[g^k(t)]\{\nabla S[g^k(t)] - \nabla S[g^{k-1}(t)]\}\, dt}{\int_{t=0}^{t_f} \{\nabla S[g^{k-1}(t)]\}^2 dt} \quad \text{for } k = 1,2,\ldots \text{ with } \gamma^0 = 0 \quad \text{for } k = 0 \tag{15.c}$$

Expressions for the search step size, β^k, and for the gradient of the functional, $\nabla S[g^k(t)]$, can be obtained with the solution of two auxiliary problems, namely, the sensitivity problem and the adjoint problem (Alifanov, 1994, Alifanov et al, 1995, Ozisik and Orlande, 2000).

We note that the iterative procedure of the conjugate gradient method is not capable of providing by itself regularized solutions for inverse problems. However, the use of the conjugate gradient method may result on stable solutions if the *Discrepancy Principle* (Alifanov, 1994, Alifanov et al, 1995, Ozisik and Orlande, 2000) is used to specify the tolerance for the stopping criterion of the iterative procedure.

4. RESULTS

We now examine the application of the techniques described above to inverse problems of parameter and function estimation in heat transfer. The physical problem under picture in this paper consists of a one-dimensional slab of a solid material with thickness L, initially at the temperature T_i, which is lower than the temperature of ablation T_{ab}. The surface of the slab at $x = L$ is heated, while the other surface at $x = 0$ is kept insulated. As the slab is heated, eventually the temperature of the heated surface reaches the temperature of ablation, the material is removed because of physical and chemical phenomena and a moving boundary problem is established (Rey Silva and Orlande, 2002, Oliveira and Orlande, 2002).

The mathematical formulation of this problem in dimensionless form, for the pre-ablation and ablation periods, is given respectively by:

Pre-ablation Period $(0 < t < t_0)$:

$$C\frac{\partial q(X,t)}{\partial t} = k\frac{\partial^2 q(X,t)}{\partial X^2} \quad \text{in} \quad 0 < X < 1, \quad \text{for} \quad 0 < t \le t_0 \tag{16.a}$$

$$\frac{\partial q(X,t)}{\partial X} = 0 \quad \text{at} \quad X = 0, \quad \text{for} \quad 0 < t \le t_0 \tag{16.b}$$

$$k\frac{\partial q(X,t)}{\partial X} = Q(t) \quad \text{at} \quad X = 1, \quad \text{for} \quad 0 < t \le t_0 \tag{16.c}$$

$$q(X,t) = 0 \quad \text{in} \quad 0 < X < 1, \quad \text{for} \quad t = 0 \tag{16.d}$$

Ablation Period $(t > t_0)$:

$$C\frac{\partial q(X,t)}{\partial t} = k\frac{\partial^2 q(X,t)}{\partial X^2} \quad \text{in} \quad 0 < X < B(t), \quad \text{for} \quad t > t_0 \tag{17.a}$$

$$\frac{\partial q(X,t)}{\partial X} = 0 \quad \text{at} \quad X = 0, \quad \text{for} \quad t > t_0 \tag{17.b}$$

$$q(X,t) = 1 \quad \text{at} \quad X = B(t), \quad \text{for} \quad t > t_0 \tag{17.c}$$

$$q(X,t) = q_0(X) \quad \text{in} \quad 0 < X < B(t), \quad \text{for} \quad t = t_0 \tag{17.d}$$

where the energy balance at the surface is given in the form:

$$n\frac{d\,B(t)}{d\,t} = -Q(t) + k\frac{\partial q(B(t),t)}{\partial X}, \quad \text{with} \quad B(t_0) = 1, \quad \text{for} \quad t > t_0 \tag{18}$$

For the non-dimensionalization of the problem, the following dimensionless groups were defined:

$$X = \frac{x}{L}, \quad t = \frac{k_R}{C_R L^2}t, \quad B(t) = \frac{b(t)}{L}, \quad q = \frac{(T - T_i)}{(T_{ab} - T_i)}, \quad Q(t) = \frac{q(t)L}{k_R(T_{ab} - T_i)},$$
$$k = \frac{k^*}{k_R}, \quad C = \frac{C^*}{C_R}, \quad n = \frac{H^* r^*}{c_{pR}(T_{ab} - T_i) r_R} \tag{19.a-h}$$

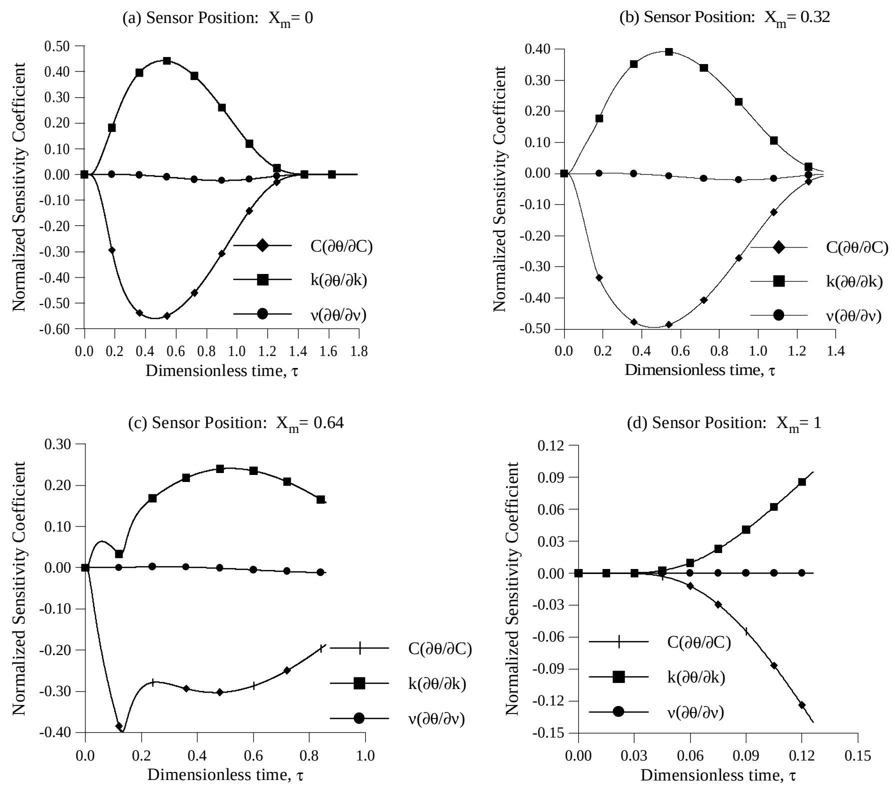
where k_R, C_R and r_R are reference values for thermal conductivity, volumetric heat capacity and density, respectively, L is the initial thickness of the slab, $b(t)$ denotes the position of the ablating front, $q(t)$ is the applied heat flux and H^* is the heat of ablation. In equation (17.d), $q_0(X)$ denotes the temperature distribution inside the slab at the moment that ablation begins, i.e., t_0.

The *direct problem*, associated with the formulation of the physical problem described above, involves the determination of the transient temperature field $q(X,t)$ in the slab, from the knowledge of initial and boundary conditions, as well as of the physical properties appearing in the formulation. Two different *inverse problems* are considered here: (i) the simultaneous estimation of the dimensionless thermal conductivity, volumetric heat capacity and heat of ablation, that is, k, C and n, respectively, by assuming that the other quantities appearing in the formulation are exactly known for the inverse analysis; and (ii) the estimation of the surface heat flux $Q(t)$, by assuming that the other quantities appearing in the formulation are exactly known for the inverse analysis. For the test-cases examined below, teflon is considered as the ablating material, with thermophysical properties given by (Rey Silva and Orlande, 2002, Oliveira and Orlande, 2002): $k^* = 0.22$ *W/mK*, $r^* = 1922$ kg/m^3, $C^* = 2.414\text{x}10^6$ J/m^3K, $H^* = 2.326\text{x}10^6$ J/kg and $T_{ab} = 560$ ^oC. The initial temperature was assumed as $T_i = 25$ ^oC. The reference values k_R, C_R and r_R were taken equal to those of teflon, so that $k = C = 1$ and $n = 3.46$.

4.1 Estimation of Thermophysical Properties of Ablating Materials (Rey Silva and Orlande, 2002)

Figures 1.a-d present the normalized sensitivity coefficients with respect to C, k and n at the measurement positions $X_m = 0$, 0.32, 0.64 and 1, respectively, for a constant heat flux $Q = 2.5$. For this case, the ablation of the surface starts at $t = 0.13$, when the sensitivity coefficients at $X_m = 1$ vanish. We can notice in these figures that the normalized sensitivity coefficient with respect to n, despite being much smaller than those with respect to C and k, attain values different from zero after ablation begins, at the positions $X_m = 0$, 0.32 and 0.64. Figures 1.a-d show that the normalized sensitivity coefficients with respect to C and k are linearly dependent, except at the measurement position X_m=0.64. The analysis of the sensitivity coefficients reveals that measurements taken during the pre-ablation period do not contribute with useful information for the estimation of the heat of ablation, because its sensitivity coefficient is null. Besides that, the estimation of n is quite difficult even with measurements taken after ablation begins, because the magnitude of the sensitivity coefficient is rather small. In order to overcome such difficulties, the following approach is used to estimate $\mathbf{P} = [C, k, n]$: (*i*) By utilizing the Levenberg-Marquardt method with measurements taken during the pre-ablation period, estimates for the volumetric heat capacity, C, and for the thermal conductivity, k, are obtained; (*ii*) The heat of ablation, n, is then estimated, by taking into account the values just estimated for C and k with their respective uncertainties, by using the sequential parameter estimation technique with measurements taken during the ablation period.

Figures 2.a,b present the determinant of the matrix $\mathbf{F}_I^*$, the elements of which are given by equation (13), for the pre-ablation and ablation periods, respectively, for different number of sensors. For the pre-ablation period, the unknown parameters were considered as C and k, while for the ablation period the three parameters C, k, and n were regarded as unknown. For the case of a single sensor ($M = 1$), its position was taken as $X_1 = 0$; for $M = 2$, the two sensors were located at $X_1 = 0$ and $X_2 = 1$; for $M = 3$, the sensors were located at $X_1 = 0$, $X_2 = 0.49$ and $X_3 = 1$; and for $M = 4$, the sensors were located at $X_1 = 0$, $X_2 = 0.32$, $X_3 = 0.64$ and $X_4 = 1$. We can notice in figure 2.a that the determinant of $\mathbf{F}_I^*$ increases when the number of sensors increases. Figure 2.a shows that $\det(\mathbf{F}_I^*)$ is practically null when a single sensor is used in the analysis, as a result of the linear dependence of the sensitivity coefficients for C and k shown in figure 1.a. Also, figure 2.a shows that measurements shall be taken until ablation begins for the estimation of C and k during the pre-ablation period, when $\det(\mathbf{F}_I^*)$ is maximum. Differently from the pre-ablation period, the curves for $\det(\mathbf{F}_I^*)$ for different number of sensors for the ablation period (see figure 2.b) are practically identical. Therefore, the use of multiple sensors is not necessary to improve the accuracy of the parameters estimated during the ablation period. Figure 2.b shows that there exists an optimum duration of the experiment (around t = 1) for the estimation of the parameters C, k, and n during the ablation period, when $\det(\mathbf{F}_{\mathrm{I}}^{*})$ is maximum. The different number of sensors examined in this paper does not affect such an optimum duration of the experiment. As expected from the analysis of the sensitivity coefficients shown in Figs. 1.a-d, a comparison of Figs. 2.a,b reveals that the maximum values for $\det(\mathbf{F}_{\mathrm{I}}^{*})$ during the pre-ablation period are much larger than those during the ablation period. This is a result of the small values of the sensitivity coefficients for n in the ablation period.



Figures 1. Normalized sensitivity coefficients for Q = 2.5 at: (a) X_m=0, (b) X_m=0.32, (c) X_m=0.64 and (d) X_m=1.

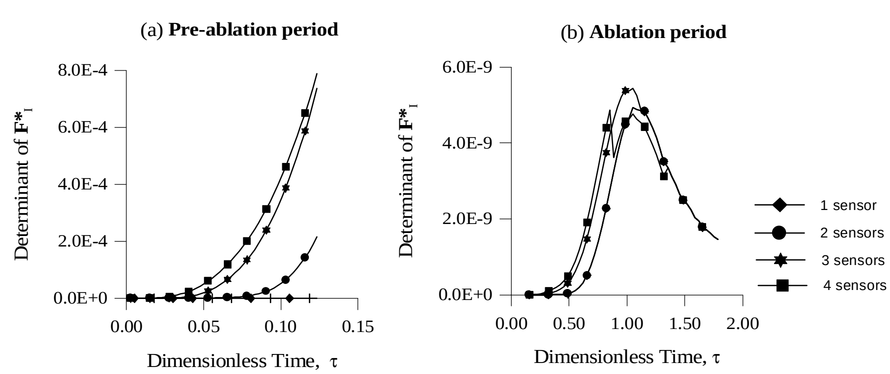


Figure 2. Determinant of de Information Matrix for (a) Pre-ablation Period and (b) Ablation Period

Table 1 presents the results obtained with simulated measurements of two sensors, containing random errors of standard-deviation σ = 0.005. For a 99% confidence level, such standard-deviation can result on errors of the order of 7°C. The results presented in Table 1 were averaged over 100 runs of the program, in order to reduce any bias introduced by the random number generator used to calculate the simulated measurements. Table 1 shows that very accurate estimates were obtained for the unknown parameters, even with large experimental errors.

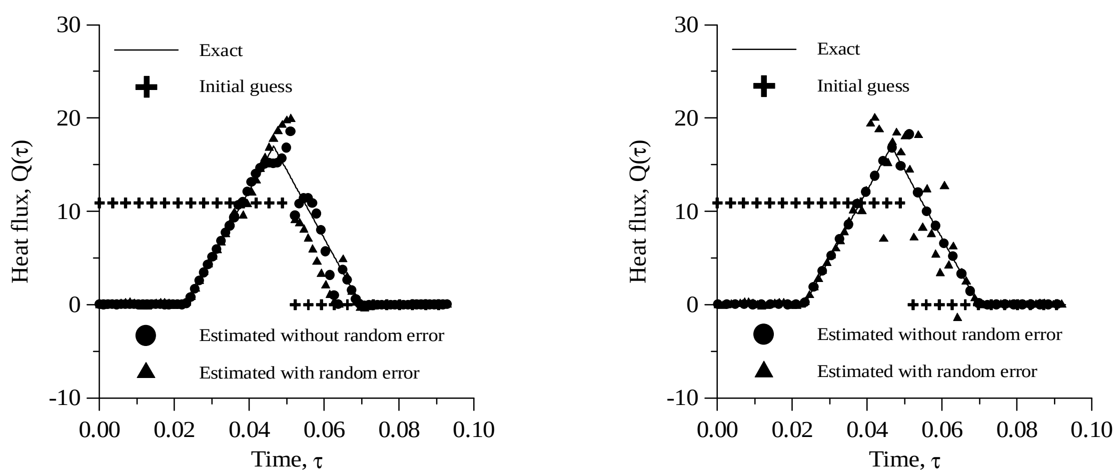
Table 1. Estimated parameters and confidence intervals

Parameter	Exact Parameter	Initial Guess	Estimated Parameter	Confidence Interval
C	1.000	10^{-4}	1.001	(1.000, 1.002)
k	1.000	10^{-4}	1.001	(1.000, 1.002)
n	3.46	0.5	3.46	(3.45, 3.47)

4.2 Estimation of the Heat Flux at the Surface of Ablating Materials (Oliveira and Orlande, 2002)

The inverse problem now under picture is concerned with the estimation of the surface heat flux $Q(t)$ by using the transient measurements of temperature sensors located inside the slab. Furthermore, we assume available for the inverse analysis the transient measurements of a sensor capable of measuring the position of the ablating surface, during the time interval $0 \leq t \leq t_f$.

Accurate estimates were obtained for the unknown function for a peak-flux of 100 *kW/m*^{2}, by using only temperature measurements in the inverse analysis, as illustrated in figure 3.a. The temperature sensor was located at X = 0.9. Figure 3.a shows that stable results were obtained for measurements containing random errors, but the peak-flux was overestimated, even when errorless measurements were used in the inverse analysis. Figure 3.b presents the estimated functions obtained by using temperature measurements during the no-ablation periods, and surface position measurements during the ablation period. The temperature sensor was located at X = 0.9. We note that the heat flux was exactly recovered when errorless measurements were used in the inverse analysis, which was not the case when only temperature measurements were assumed available (see figure 3.a). A comparison of figures 3.a and 3.b reveals an increase in the oscillations of the solution after ablation began, when surface position measurements containing random errors were used in the inverse analysis.



(a) (b)

Figure 3. Solution of the inverse problem for a peak-flux of 100 *kW/m*^{2} : (a) obtained with only temperature measurements and (b) obtained with temperature and surface position measurements

5. CONCLUSIONS

The objective of this paper was to discuss some fundamental aspects of inverse problems and to give practical examples of its applications in heat transfer. In the talk, the Levenberg-Marquardt method of parameter estimation, the sequential parameter estimation technique and the conjugate gradient method of function estimation will be described in detail and further examples will be given.

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