A METHOD FOR EIGENVALUE MAXIMIZATION IN STRUCTURAL OPTIMIZATION

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Abstract:
The maximization of the fundamental eigenvalue in structural optimization, such as the critical buckling load factor of finite element structures, suffers from extreme computational inefficiency. This is because optimization cycles (redesign) are guided by sensitivity analysis based on the current fundamental mode shape, which at the end of a redesign step, may not correspond to the new fundamental eigenvalue. This behavior is the cause of oscillation in the search of the optimal solution. This paper presents a new efficient strategy to improve dramatically the convergence characteristics of eigenvalue maximization in the context of the approximation concepts idea of structural optimization. The strategy is based on the solution of a set of simultaneous approximate optimization subproblems at a base design using the Rayleigh Quotient Approximation. The technique has been successfully applied to the maximization of the critical load of composite laminated plates including thermal residual stress stiffening effects, a type of problem where the convergence difficulties just described are usual. The technique was able not only to stabilize convergence but also to lead to correct optimal solutions. It is foreseen that this novel general technique can be extended to other problems of eigenvalue maximization in structural optimization.

Keywords: *Eigenvalue maximization method, approximation concepts, structural optimization, composite plates, finite element analysis, thermal residual stresses.*

1. Motivation

Solutions for the maximization of the fundamental eigenvalue in structural optimization, such as the critical buckling load factor in elastic stability problems or the fundamental natural frequency in vibration problems of structures modeled by finite elements, lack computational efficiency. The difficulty arises from the fact that an optimization cycle is guided by the sensitivity analysis of the current fundamental mode, which is not guaranteed to remain fundamental after the cycle. This means that material redistribution caused during an optimization cycle may cause a higher mode to become the new fundamental mode of the next optimization cycle. This behavior is the cause for oscillation in the convergence pattern to the optimal solution. In the reasoning it is assumed that material redistribution after a redesign cycle is small enough to maintain the mode shapes practically unaltered. Therefore, the change in the fundamental mode can be interpreted as due only to material redistribution.

The use of approximation concepts is the most efficient approach available to structural optimization, which reduces the computational effort to a minimum due to the smaller number of finite element analysis and sensitivity analysis needed for convergence when compared to direct approaches based solely on mathematical programming techniques. However, the approach is subject to the same oscillation pattern just mentioned, demanding for convergence the use of tight move limits for the design variables and consequently an increased number of complete finite element analyses and sensitivity analyses. The maximization is based on the generation of high quality explicit approximations of the eigenvalues in terms of the design variables, namely the RQA (Rayleigh Quotient Approximation). In the RQA the eigenvectors are supposed to be invariant to changes in the design.

Although the stabilization of convergence in the case of eigenvalue maximization is a challenge to be overcome in the realm of any approach in structural optimization this paper addresses such a new strategy in the context of the approximation concepts idea (Schmit, 1976). The idea will be developed over the optimization of laminated plates subject to inplane loads due to mechanical and thermal loading, a class of problems which is currently subject of research and which contain the necessary ingredients to illustrate the method. It is foreseen that this novel general technique can be extended to any problems of eigenvalue maximization in structural optimization.

2. Finite Element Eigenproblem

The eigenproblem used to illustrate the optimization method presented in this work corresponds to the linear buckling of a finite element composite laminated plate and is given by Eq. (1). The plate is subject to inplane loads due to direct mechanical action and also to inplane residual thermal stresses. Since the plate is symmetrical in the z direction, there is no coupling between membrane and bending behavior. The plate is modeled with lagrangean bi-cubic isoparametric 16-nodes element, with 5 degrees of freedom (u, v, w, q_x, q_y) per node. The eigenvalue l_m and eigenvector $\mathbf{f}_m$ are the solution of the following set of finite element equations:

$$\left(\mathbf{K}+\mathbf{K}_G^R - l_m \mathbf{K}_G^0\right)\mathbf{f}_m = \mathbf{0} \tag{1}$$

$\mathbf{K}$ is recognized as the global elastic stiffness matrix and $\mathbf{K}_G^R$ is the geometric stiffness matrix due to inplane thermal residual stresses (Almeida and Hansen, 1997). The matrix $\mathbf{K}_G^0$ is the geometric stiffness matrix associated to a direct mechanical loading; in the present example an inplane applied displacement at one of the plate's edges.

3. Critical Buckling Load Parameter Optimization Strategies

The optimization problem is enunciated as the maximization of the fundamental eigenvalue, or the critical buckling load factor, as follows:

$$\text{Max} \qquad l_1(\mathbf{X}) \tag{2}$$

$$s.t. \qquad S w_e X_e - V_0 \text{ £ } 0 \qquad e = 1,\ldots, L_e \tag{3}$$

$$X_{k+1} - X_k \leq 0 \qquad k, k+1 \text{ Î } L_e \tag{4}$$

$$X_{il} \text{ £ } X_i \text{ £ } X_{iu} \qquad i = 1, \ldots, ndv \tag{5}$$

The design variables are the heights $\mathbf{X} = \{X_1, X_2, \ldots, X_n\}^T$ of the plate laminae, measured from the laminate midsurface. The plate volume (or its mass, if the laminae specific weights are the same) is limited to its initial value by the imposition of the constraint, Eq. (3), where w_e denotes the area in the x-y plane of a region of variable thickness and X_e is the height of the plate surface over this region. The next constraint in Eq. (4) means that an inner varying height cannot surpass the outer one in a group L_e of variables associated to a region of varying thickness. In Eq. (5) design variable limits, or side constraints, are imposed. As is usual in structural optimization applications, the constraints represented by the equilibrium equations, Eq. (1), have also to be satisfied; however, this task is left to the eigensolver of the finite element analysis software and not to the mathematical programming tool used in the optimization

Within the framework of the approximation concepts idea different strategies can be devised to solve the problem presented by Eq. (2)-(5). The investigation is carried out over the results of previous numeric experimentation and three strategies are constructed with a growing degree of complexity, culminating with an innovative solution (strategy #3).

3.1 An explicit eigenvalue approximation (RQA)

An eigenvalue can be expressed in terms of the Rayleigh quotient as

$$l_m = \frac{U_m}{T_m} = \frac{\mathbf{f}_m^T\left(\mathbf{K}+\mathbf{K}_G^R\right)\mathbf{f}_m}{\mathbf{f}_1^T \mathbf{K}_G^0 \mathbf{f}_1}, \tag{6}$$

where U_m and T_m are modal strain energies. An explicit Rayleigh Quotient Approximation (RQA) is created for the eigenvalue, from first order expansions of the modal energies U_m and T_m in terms of the design variables (Canfield, 1990):

$$\tilde{l}_m = \frac{\tilde{U}_{m0}}{\tilde{T}_{m0}} = \frac{U_{m0} + \sum_{l=1}^{ndv} U_{ml}\left(X_l - X_{0l}\right)F_l}{T_{m0} + \sum_{l=1}^{ndv} T_{ml}\left(X_l - X_{0l}\right)G_l} \tag{7}$$

where the coefficients of both first order expansions, $\tilde{U}_{m0}$ and $\tilde{T}_{m0}$, are given by

$$U_{m0} = \mathbf{f}_{0m}^T \left(\mathbf{K} + \mathbf{K}_G^R\right)\Big|_{\mathbf{X}_0} \mathbf{f}_{0m} \qquad U_{ml} = \mathbf{f}_{0m}^T \left.\frac{\partial\left(\mathbf{K} + \mathbf{K}_G^R\right)}{\partial X_l}\right|_{\mathbf{X}_0} \mathbf{f}_{0m}$$

$$T_{m0} = \mathbf{f}_{0m}^T \mathbf{K}_G^0\Big|_{\mathbf{X}_0} \mathbf{f}_{0m} \qquad T_{ml} = \mathbf{f}_{0m}^T \left.\frac{\partial \mathbf{K}_G^0}{\partial X_l}\right|_{\mathbf{X}_0} \mathbf{f}_{0m} \tag{8}$$

In Equation (8) the constant coefficients U_{m0}, U_{ml}, T_{m0} and T_{ml} are calculated at the base design $\mathbf{X}_0$, the point where the approximation is created. The approximation, Eq. (7) and the associated coefficients (Eq. (8)), is therefore created from the results of a complete finite element structural analysis and first order sensitivity analysis at the base design $\mathbf{X}_0$. In Eq. (7) the coefficient F_l and G_l assume the value "1 (one)" or "X_{ol}/X_l", depending if the direct (X_l) or reciprocal variable ($1/X_l$) is adopted (Vanderplaats, 1984) for the specific l-th term of the series. Analytical sensitivity is used for the matrices derivatives appearing in Eq. (8).

The explicit approximation, Eq. (7), can be used as a temporary substitute for the exact eigenvalue, avoiding the use of a complete structural finite element analysis every time an eigenvalue is needed. However the approximation is valid only in a neighborhood not very far from the base design $\mathbf{X}_0$, and therefore design variable move limits should be imposed (see Eq. (11)). The approximation is aimed to be updated as the optimization progresses towards the optimum, i.e., the coefficients in Eq. (8) are updated and new move limits are imposed.

3.2 Basic Strategy (Strategy #1)

The *basic strategy* for the optimization problem defined in Eq. (1)-(5) consists of the solution of a sequence of approximate subproblems, each one with the following statement:

Max $\tilde{l}_1(\mathbf{X})$ (9)

s.t. $S w_e X_e - V_0$ £ 0 $\quad e = 1,\ldots, L_e$ (10)

$X_{k+1} - X_k$ £ 0 $\quad k, k+1$ Î L_e (11)

X_{il} £ X_i £ X_{iu} $\quad i = 1, \ldots, ndv$ (12)

The subproblem of Eq. (9)-(12) is approximate in the sense that the fundamental eigenvalue being maximized is the RQA defined by Eq. (7); however, the solution will converge to the exact one. Move limits are individually applied to each design variable with the purpose of preserving the accuracy of the RQA and to allow a smother convergence to the final optimum. The move limits are adjusted after each complete optimization cycle, according to a strategy based on the comparison of the exact objective function value and its approximation at the end of the current subproblem. Individual increment is activated when an upper or lower bound is reached for a design variable in two consecutive optimization cycles (Thomas et all, 1992). A complete optimization cycle is defined as the solution of a subproblem defined by Eq. (9)-(12). The final optimum will be approached in the converging solutions of the sequence of approximate subproblems, individually solved by DOT (VMA Engineering, 1989). The number of optimization cycles (subproblems) needed to converge to the exact solution depends on the quality of the approximations for the eigenvalue. For a lower quality tighter move limits are needed and consequently a higher number of complete optimization cycles.

The strategy just defined has the advantage inherent to the approximation concepts approach in structural optimization (Schmit, 1976); namely, to reduce the number of complete finite element analyses and sensitivity analyses. However it is subjected to the same drawbacks of any other method that tries to solve the problem stated in Eq. (2) to (5), that is, to engage is an oscillatory pattern to the optimum. The reason for this behavior is that material redistribution caused during an optimization cycle may cause a higher mode to become the new fundamental mode of the next optimization cycle. Therefore, the higher modes play an important role in the problem, though they are not included in the problem statement.

The described strategy is applied to the presented examples, where we will be able to appreciate its advantages and drawbacks.

3.3 Active Mode Constraint Strategy (Strategy #2)

The oscillatory characteristics associated to the *basic strategy* (#1) can be overcome by the introduction in the problem defined by Eq. (9)-(12) of new constraints imposing that the eigenvalues corresponding to higher modes are bigger than the fundamental eigenvalue, resulting in the following new statement of the approximate subproblem:

$$\text{Max} \quad \tilde{\lambda}_1(\mathbf{X}) \tag{13}$$

$$s.t. \quad \Sigma w_e X_e - V_0 \le 0 \qquad e = 1,\dots, L_e \tag{14}$$

$$\tilde{g}_k(\mathbf{X}) = 1 - \frac{\tilde{\lambda}_k}{\tilde{\lambda}_1} \le 0 \qquad k = 1,\dots, p \tag{15}$$

$$X_{k+1} - X_k \le 0 \qquad k, k+1 \in L_e \tag{16}$$

$$X_{il} \le X_i \le X_{iu} \qquad i = 1, \dots, ndv \tag{17}$$

The new constraints given by Eq. (14) apply to "*p*" active modes, which are defined as those closer to the fundamental mode within a given tolerance. This format of the approximate problem should stabilize the convergence, however, it can be foreseen that this strategy will have the tendency to force the current fundamental mode shape to prevail until to the end of the optimization. Consequently, it can be expected that this strategy won't lead strictly to the optimum, but eventually close enough.

Naturally appears the question of how to idealize an optimization process that has computational efficiency, in the sense that the number structural optimization cycles or finite element analysis/sensitivity is small and at the same time leading to the true optimum.

3.4 Critical Mode Identification Strategy – CMIS (Strategy #3)

Not only the current fundamental mode but also any of the active modes are potentially critical in the sense that it may eventually turn to be the fundamental mode at the end of an optimization cycle. The active modes are those closer to the first mode within a given tolerance. So, a strategy is needed to identify the critical mode, which would in fact be equivalent to identify the problem objective function among the set of active modes (Andrade, 2002). At the beginning one of the active modes is used as the approximate objective function (RQA), while lower limit constraints are imposed on each of the remaining eigenvalues approximations (RQA) of modes in the active set. The process happens in such a way that each one of the active modes becomes the objective function whereas appropriate modal constraints are imposed on the others. At the end only one among the solutions in the set will be chosen as the optimal solution to the optimization cycle. This approach is herein called *critical mode identification strategy* – CMIS (Andrade, 2002).

In fact, the CMIS is equivalent to the solution of a set of subproblems solved by the previous strategy (#2), with the difference that instead of the first mode each of the active modes become the objective function (RQA). The subproblem corresponding to the CMIS is stated as follows:

$$\underset{a}{\text{Max}}\,\underset{\mathbf{X}_a}{\text{Max}}\ \tilde{\lambda}_a \qquad a = 1,\dots,p \tag{18}$$

$$s.t. \quad \Sigma w_e X_e - V_0 \le 0 \qquad e = 1,\dots, L_e \tag{19}$$

$$\tilde{g}(\mathbf{X}_a) = 1 - \frac{\tilde{\lambda}_k}{\tilde{\lambda}_a} \le 0 \qquad k = 1,\dots, p;\ k \ne a \tag{20}$$

$$X_{k+1} - X_k \le 0 \qquad k, k+1 \in L_e \tag{21}$$

$$X_{il} \le X_i \le X_{iu} \qquad i = 1, \dots, ndv \tag{22}$$

In Equation (18) "*p*" is the number of active modes. The Equations (18) to (22) represent a subproblem which is composed of a set of $a = 1,\dots, p$ lower level optimization problems. In the *a-th* lower level problem the *a-th* eigenvalue (RQA) becomes the objective function while the others participate as constraints (Eq. (20)). After having solved the "*p*" lower level optimization problems the optimal solution will be chosen as the one ($\mathbf{X}_a$) with the highest objective function. The subproblem solution will be the optimum design variable vector $\mathbf{X}_a$ that possesses the maximum objective function $\tilde{l}_a$ among the "*p*" optimal solutions.

The strategy can be very easily implemented, and because it is based on the eigenvalues explicit approximations (RQA) it is also computationally inexpensive. The technique has been successfully applied to the optimization of composite plates (Andrade, 2002) and two examples will be presented here to attest its effectiveness.

4. Numerical Examples

These numerical examples apply to a plate configuration, whose mechanical and physical properties (related to T300/5208 carbon-epoxi), finite element discretization, mechanical and thermal loads and boundary conditions detailed in Andrade, Almeida and Hernandes (2001a).

4.1 Buckling Load Parameter Optimization of a Square Orthotropic Plate

A laminated composite plate will be optimized for maximization of the critical buckling load factor. The plate is under the action of direct inplane mechanical loading and residual thermal stresses arising from the curing process. The mechanical inplane loading which is a imposed compressive displacement at one edge tends to buckle the plate. Because inplane thermal stresses can generate stiffening effects the purpose of the optimization is to take advantage of this field introducing tensions which stiffen the plate and, therefore, increase its buckling load. The thermal stresses can created by designing a laminate with patches of distinct properties, such as the direction of the fibers, thickness and stacking sequence.

Before going into the details of the design variables definition it is helpful to present the finite element model of the studied composite plate. It represents a simply supported orthotropic composite plate as shown in Fig.(1). The plate is square, such that $a = b = 360$ *mm*. It is subject to a prescribed uniform displacement, d, applied to its top edge, in the y axis direction. The material properties assumed for a 0.15*mm* thick carbon/epoxy T300/5208 layer are (Andrade, 2002):

- Elastic longitudinal modulus, E_1 = 154500 MPa;
- Elastic transversal modulus, E_2 = 11130 MPa;
- Poisson's ratio (inplane), n_{12} = 0.304;
- Shear modulus (inplane), G_{12} = 6980 MPa;
- Shear modulus (transverse), G_{13} = 6980 MPa;
- Shear modulus (transverse), G_{23} = 3360 MPa;
- Specific mass, r = 1.56 x 10^3 kg/m^3;
- Longitudinal thermal expansion coefficient, a_1 = - 0.17 x 10^{-6} /$^\circ$C;
- Transverse thermal expansion coefficient, a_2 = 23.1 x 10^{-6} /$^\circ$C.

Now we can define the design variables, which are the heights of eight longitudinal parallel stiffeners along the direction of the applied displacement. Fig.(2) shows the plate cross section where the design variables, X_1 to X_8, are represented in the lower half of the plate. It is important to remark that the plate is symmetrical with respect to the z axis. These variables have lower limits equal to half of the base plate thickness. The base plate has a uniform constant thickness and is formed by the inner lamina; the base plate is not allowed to change during optimization.

The lamination angles in Fig.(2) are measured from the x axis. Therefore, the 90^o plies have the fiber direction aligned with the loading as the prescribed pre-buckling displacements are applied along the y direction. The 0^o plies have the fiber orientation orthogonal to the prebuckling load direction.

Variables X_1 and X_2 control the heights of two identical parallel stiffeners defined by elements 4, 12, 20, 28, 36, 44, 52, 60 and 5, 13, 21, 29, 37, 45, 53, 61, respectively (Fig.(1)). Variables X_3 and X_4 control the heights of the two stiffeners in the y direction defined by the top elements 59 to 62; similarly X_5 and X_6, for stiffeners corresponding to top elements 58 and 63; finally, X_7 and X_8 control the stiffeners located along the edges.

Having defined the optimization problem we are able to comment on the results obtained with the three strategies presented. Figure (3) shows the results with the three methods. In the present case, the active modes set includes the modes which are contained in a 10% tolerance above the fundamental eigenvalue.

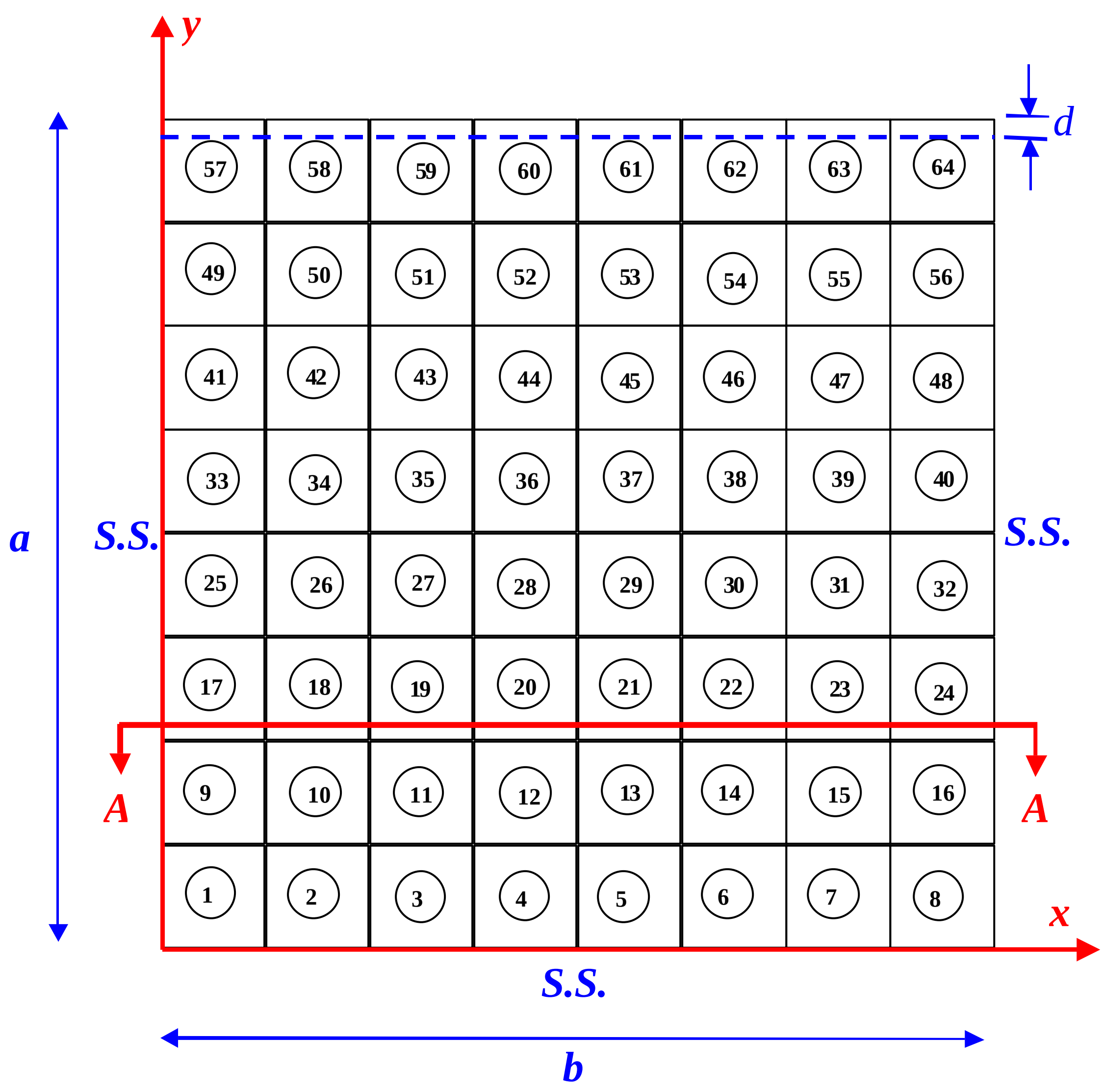

Figure 1 – Finite element plate mesh with 64 elements and subject to prescribed displacement, *d*.

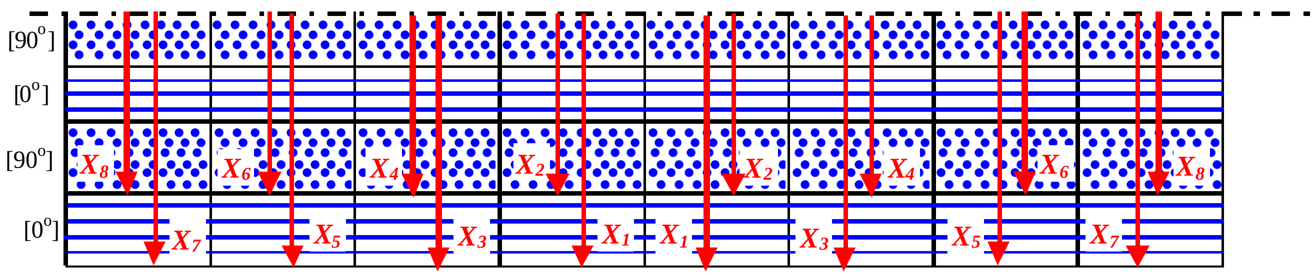

Figure 2 – Design variables used for the longitudinal stiffeners.

From Fig.(3) it is observed that the basic strategy and the CMIS could reach comparable optimum objectives values, but the CMIS converged sooner and presented no oscillation with the design optimization cycles. With the CMIS after 10 optimization cycles, the optimum was reached and is 4,2% below the optimal result with the basic strategy (#1) which took 14 cycles. With the strategy #2 the convergence was stable, but to a lower optimum value. The mode shape is the same observed in the initial design, different from the optimal solutions of strategies #1 and #3. Although in this problem the differences between strategies #1 and # 3 were not so pronounced, it still shows the superiority of the CMIS strategy.

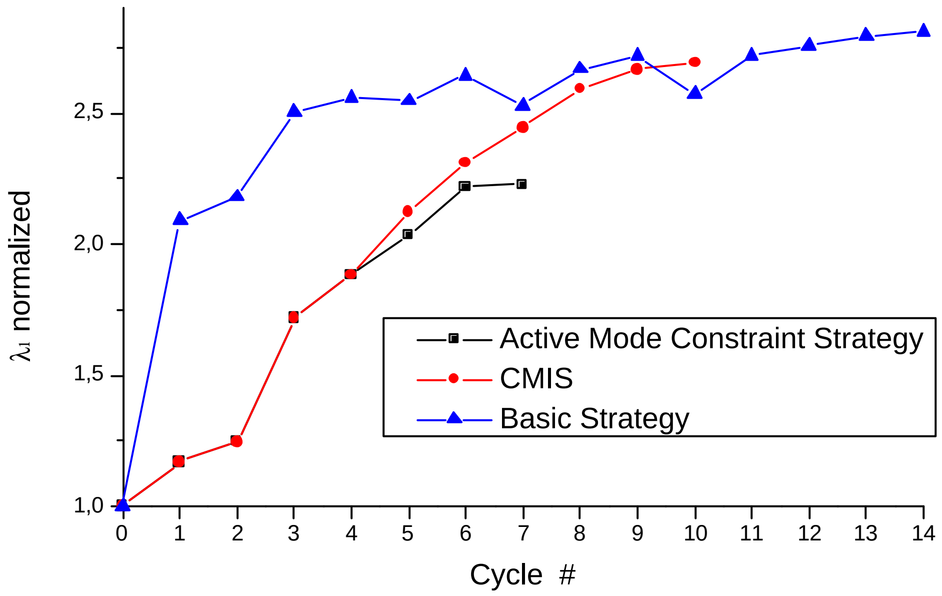

Figure 3 – Iteration history of the strategies applied to a laminated plate with thermal residual stresses.

4.2 Buckling Parameter Optimization of a Square Orthotropic Framed Plate with Central Circular Hole

This example is a more sophisticated problem in terms of the structural behavior of the laminated plate as well from an optimization point of view. The laminated plate in Fig.(4) has a circular hole in its center. There are eight design variables that control the thickness of the ten frame shaped regions shown in Fig.(1) (Andrade, Almeida and Hernandes, 2001b). The strategies #1 and #3 are applied to solve this optimization problem where the buckling load factor is maximized. For the CMIS a 2% tolerance was used in order to define the modal active set.

In Figure (1) the direction of fibers are indicated, which will affect the distribution of inplane thermal stresses and associated stiffening effects. As before, the objective is the tailoring of composite patches that will take advantage of inplane thermal residual stresses and increase the laminate buckling load. In Figure (5) the optimal configurations are represented.

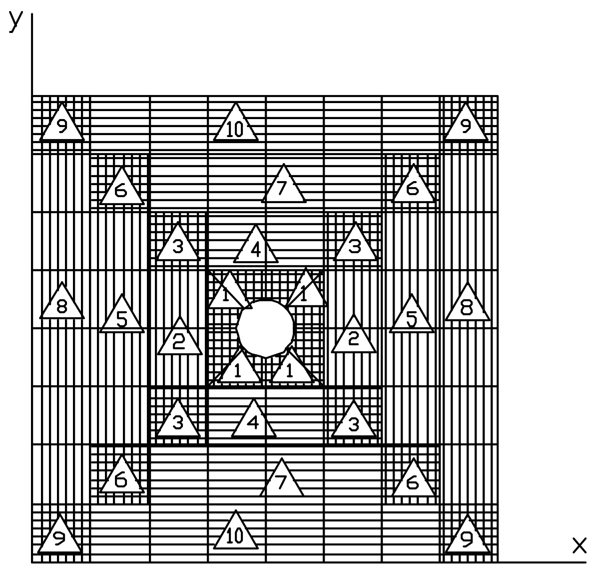

Figure 4 – Laminate patches for composite plate with hole.

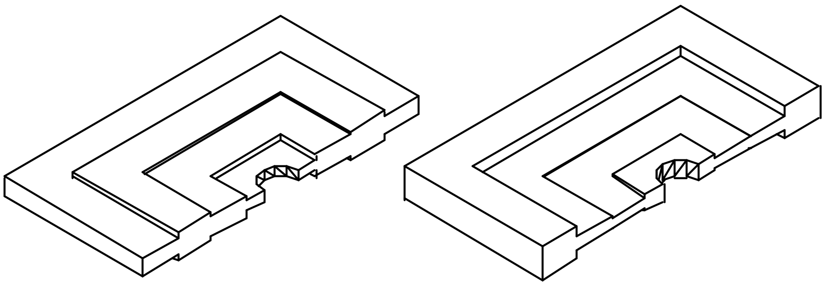

Figure 5 – Optimal thickness distribution for $DT = 0^{\circ}$C and $DT = -150^{\circ}$C.

Figure (6) presents the results obtained for plate optimization for two distinct temperature cases: $DT = -150^{\circ}$C and $DT = 0^{\circ}$C. For $DT = 0^{\circ}$C no stress stiffening effects exist and the solution is simpler, and so strategies #1 and #3 are both reasonably efficient. For the case of $DT = -150^{\circ}$C, a much harder case where the variable plate thickness has to be slowly tailored by the optimization, strategy #1 exhibits an oscillatory fruitless behavior very hard to deal with, demanding for convergence the use of thigh design variable move limits. On the other side, the CMIS exhibits an excellent behavior converging straight to the optimum design in only six optimization cycles.

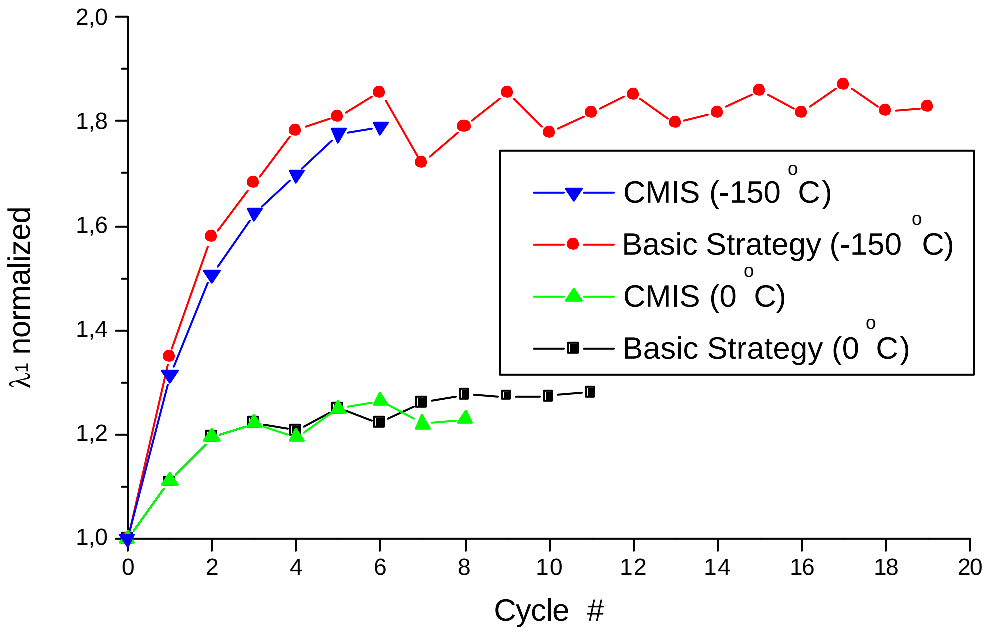

Figure 6 – Iterations for laminated plate with circular hole.

5. Conclusions

An strategy was presented to improve the efficiency of the solution of eigenvalue maximization problems in structural optimization in the context of finite element analysis. The *Critical Mode Identification Strategy* was successfully applied to two problems in which the buckling load factor of composite plates was maximized. The method belongs to the approximation concepts branch in structural optimization and therefore is extremely effective by keeping the number of finite element analysis and sensitivity analysis to a minimum. The method seems to be one more tool available to those interested in structural optimization, being proper to be applied to more complex and advanced problems of structures modeled with thousands of finite elements, where computational efficiency is really needed.

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