Multi-rigid-body contact problems with Coulomb friction: complementarity and equivalent formulations

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Abstract. *This work addresses the problem of predicting the possible accelerations and contact forces of a set of rigid, three-dimensional bodies in contact in the presence of Coulomb friction. Using mathematical programming algorithms, the solution of the contact problem is obtained either by means of a mixed nonlinear complementarity problem (original Coulomb law) or of a linear complementarity problem (pyramidal friction law). Sufficient conditions for obtaining equivalence with formulations as bound-constrained minimization problems are provided. Numerical experiments are presented, using the software BOX-QUACAN, developed by the Optimization group of the Applied Mathematics department of the State University of Campinas. We conclude that the approach is effective for solving the model with friction in both variants, with the approximate pyramidal friction law and with the original Coulomb cone.*

Keywords*. Multi-rigid-body contact problem, Coulomb friction, complementarity problem, bound-constrained minimization*

1. Introduction

Planning the motion of several rigid bodies in contact is a challenging task. Posed as a nonlinear algebraic differential system (see, for instance, Brenan et all (1996) p. 150), after a time discretization, it can be cast as a sequence of problems, one for each time frame. The papers of Pang and Trinkle (1996) and Trinkle et all (1995, 1997) provide the theory and framework for the single time frame problem, concerned with the computation of contact forces and accelerations of a set of rigid three-dimensional bodies, in the presence of friction. The contact model is formulated in its most general version with the Coulomb friction law. Nevertheless, relaxations (pyramid law) or special cases give rise to simpler problems, namely, linear complementarity (LCP) ones. In the aforementioned papers, only the LCP relaxation is addressed in the numerical experiments. Recently, Tzitzouris (2001), in his thesis work, developed a fully implicit time-stepping scheme for the simulation of the multi-rigid-body contact problem with Coulomb friction. A central feature of the algorithm presented therein is the solver employed at each time step for solving the nonlinear complementarity problems.

The Newton-Euler equations governing the motion of the objects, together with the dynamic equations of the manipulator, the Signorini condition, the Coulomb friction law and the requirement that the contact forces be compressive compose a mixed nonlinear complementarity problem, shortly denoted by MNCP, in the following format:

$$\begin{aligned}&\text{Given } f:\Re^{n+m}\to\Re^{n} \text{ and } g:\Re^{n+m}\to\Re^{p}\text{, find } u\in\Re^{n} \text{ and } v\in\Re^{m} \text{ such that}\\ &u\geq 0,\ f(u,v)\geq 0,\ u^{T}f(u,v)=0 \quad\text{and}\quad g(u,v)=0.\end{aligned} \tag{1}$$

The versatility of the MNCP format comprises well-known classes of problems: nonlinear equations, nonlinear programming, nonlinear complementarity problems, and variational inequalities, as well as new problem classes, such as extended linear-quadratic programming and general equilibrium models.

Roughly speaking, complementarity problems might be addressed either in a straightforward way, by algorithms specifically developed for this kind of problem, or by the minimization of a merit function created to represent the problem, in the sense that it will be zero only at the solutions of the complementarity problem. Ferris and Pang (1996) and references therein provide further details regarding this classification.

In this work we adopt the second approach, casting the MNCP as an equivalent bound-constrained minimization problem. The philosophy behind such a strategy is surveyed by Andreani and Friedlander (2002), with a comprehensive discussion of the important features involved, focusing mainly on variational inequalities and related problems.

This paper is organized as follows: the equivalent formulation is stated in Section 2. Specific details on the algorithms used to solve the bound-constrained minimization problem are given in Section 3. The numerical experiments are presented in Section 4. First, a brief description of three study cases from MCPLIB is given, used to validate our approach in other scenarios. Next, results on the multi-rigid-body contact problem are presented for both the pyramidal approximation and the original Coulomb friction law. Finally, Section 5 contains our comments and conclusions.

2. An equivalent formulation of the MNCP

Given the MNCP defined in Eq. (1), consider the following optimization problem:

$$\begin{aligned}&\text{Minimize} \quad \frac{1}{2}\left(\|f(u,v)-z\|_2^2+\|g(u,v)\|_2^2+(u^T z)^2\right)\\ &\text{subject to} \quad u,z\geq 0.\end{aligned} \tag{2}$$

The result below states sufficient conditions such that stationary points of problem (2) are also global solutions thereof, with zero objective function value, and thus solutions of problem (1).

Theorem 1. *If* (u^*,v^*,z^*) *is a stationary point of problem* (2) *and the Schur complement of* $g_v(u^*,v^*)$ *in the Jacobian*

$$J(u^*,v^*)=\begin{pmatrix} f_u(u^*,v^*) & f_v(u^*,v^*)\\ g_u(u^*,v^*) & g_v(u^*,v^*)\end{pmatrix} \tag{3}$$

is a row sufficient S-matrix, then (u^*,v^*) *is a solution of problem* (1).

Proof. See Andreani et all (2003).

The required condition of row-sufficiency is weaker than asking the Schur complement of $g_v(u^*,v^*)$ in the Jacobian $J(u^*,v^*)$ to be a positive semidefinite matrix, which would be related to the monotony of the operator associated to this Schur complement. If the functions f and g are affine, we obtain the following stronger result:

Theorem 2. *Let* f *and* g *be affine functions, problem* (1) *be feasible and the Jacobian given in Eq.* (3) *be a row sufficient matrix. Then, if* (u^*,v^*,z^*) *is a stationary point of problem* (2), (u^*,v^*) *is a solution of problem* (1).

Proof. See Andreani et all (2003).

3. Mathematical programming algorithms

Let us consider the bound-constrained minimization problem, namely

$$\begin{aligned}&\text{Minimize} \quad F(x)\\ &\text{subject to} \quad \ell_b \leq x \leq u_b\end{aligned} \tag{4}$$

where $F:\Re^N \rightarrow \Re$ is differentiable on the feasible set $\mathcal{B}=\{x\in\Re^N \mid \ell_b \leq x \leq u_b\}$ and any component of the bounds ℓ_b, u_b may be infinite. Friedlander et all (1994) proposed a matrix-free trust-region type algorithm, called `BOX-QUACAN`, to solve problem (4). At each iteration, `BOX-QUACAN` has to handle the subproblem of minimizing a (not necessarily convex) quadratic on a box which is the intersection of the feasible set $\mathcal{B}$ with a trust region defined by the infinity norm.

General large-scale nonlinear programming problems may be efficiently solved using augmented Lagrangian techniques, as long as a good method for solving problem (4) is available. This is the main idea behind the package `LANCELOT`, from Conn et all (1992). Employing a similar philosophy, the Optimization Group at the State University of Campinas has developed the code `EASY` (available at http://www.ime.unicamp.br/~martinez), a double-precision Fortran 77 implementation of a trust-region Augmented Lagrangian method for large-scale nonlinear programs (see Krejić et all (2000) for further details).

4. Numerical experiments

The bound-constrained minimization problems that arise from the formulation of the complementarity problems of Eq. (1) by means of problem (2) were solved with the trust-region algorithm `BOX-QUACAN`, by performing a single outer iteration of the augmented Lagrangian solver `EASY`. Before addressing the contact problems, three preliminary families of problems were investigated, and the results are summarized below.

4.1. Three study cases from MCPLIB

Dirkse and Ferris (1995) describe the origin and structure of complementarity test problems in the MCPLIB library, expressed in the GAMS modeling language. The problems are formulated as mixed complementarity ones.

In the following we provide the computational results, including a summary of the essential elements, of three classes of problems selected for numerical investigation using our merit function and the bound constrained minimization approach. The first two classes came from economics, whereas the third falls into the optimal control category.

Spatial price competition models

Harker (1986) presents four alternative models for spatial competition, all based on the same data and coded in the GAMS file harkmcp.gms. The first model is the classical spatial price equilibrium one, with perfectly competitive producers and suppliers facing average cost pricing of transportation. Next, two monopoly models are given. In the first, the firm owns both means of production and distribution network (hence, marginal cost pricing prevails at both the factory and the railhead), whereas in the second the firm uses the distribution network with average cost pricing. Last, it comes a multi-producer oligopoly model, with average cost pricing of transportation links.

For each model, two examples were addressed, differing on the transportation network topology: example 1, with solely regional centroids and origin-destination pairs and example 2, with inclusion of transshipment nodes, that represent transportation facilities such as rail yards or ports.

The dimension of the reformulated problems varied between 27 and 129. All of them were successfully solved, achieving an objective function value smaller that 10^{-10} for problem (2). Originally formulated as nonlinear programs, the modeling of market behavior by means of the equilibrium conditions that come from the first order optimality conditions seems convenient, as the objective functions include integrals, which might be difficult or expensive. In Harker's models, however, the integrands are simple enough, so one approach is not preferable to the other.

Economic equilibria

Scarf (1973), in the chapter 5 of his book, adresses the computation of equilibria in a general Walrasian model, among other applications. Example I, given in section 5.3, involves six commodities, five consumers and eight activities (sectors of production). The optimal commodity prices in this model are determined up to a positive factor. The price system can be normalized by fixing a numeraire, or by fixing the sum of the prices. Such normalization choices are respectively coded in GAMS files scarfanum.gms and scarfasum.gms.

The dimension of problem (2) with the model that normalizes the price system by fixing the sum of the prices is 30. With the normalization of the price system by fixing the numeraire, the dimension of problem (2) is 28. For each normalization, four runs were performed, with different initializations. Average results are summarized in Tab. (1), where ItBox, FE, ItQua, MVP and CPU indicate, respectively, number of trust-region iterations, number of functional evaluations, number of iterations of the quadratic model solver, number of matrix-vector products performed, and cpu time spent in seconds.

Table 1. Average results of the numerical performance of BOX-QUACAN for Scarf's models.

Type of normalization	ItBox	FE	ItQua	MVP	CPU
Fixing sum of prices	25.3	29.8	848.5	956.0	32.0
Fixing numeraire	26.3	30.3	700.0	778.9	30.3

Optimal control

An application in discrete-time optimal control was analised by Bertsekas (1982). It came from the discretization of a continuous-time problem of minimizing an objective function expressed as an integral, and constraints that express the evolution of the state variables subject to the control variables (first order differential system). Discretization turns the problem into the minimization of a quadratic function, with linear constraints.The author formulates and solves the complementarity problem obtained from the first order optimality conditions. This is also the approach that underlies the GAMS file bert_oc.gms. Four test families were created, varying the given initial states and the initialization of the control trajectories. Dimension of problem (2) varied from 70 to 70000 and, although the outer iterations performed by BOX and the number of functional evaluations remain quite the same as dimension increases, the effort grows exponentially with the grid, as depicted in Fig. 1. For the largest dimension, it took an average of 8.8 hours of CPU for solving the problem. All the runs were successfully terminated, reaching objective function values between 10^{-13} and 10^{-4}.

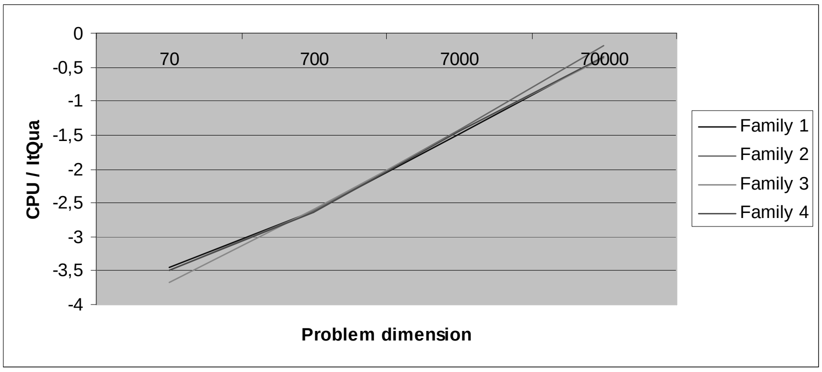

Figure 1. Log-log plot of problem dimension against the rate `CPU/ItQua` for each test family of Bertseka's control problems.

4.2. Multi-rigid-body contact problems with Coulomb friction

Simple instances with two SCARA (Selective Compliant Articulated Robot for Assembly) robots grasping a cubic object were generated. The parameters necessary to describe the initial configuration were defined as in Murray et all (1994). The matrices of the test set generation were prepared using the toolbox from Corke (2002), to preserve as much as possible the physical meaning of the data.

First, assuming that both contacts are of the *sliding* type, the complementarity model turns into an LCP and the equivalent nonlinear programming reformulation (2) has only four variables. Two hundred tests were run, with different seeds, to generate the problem data. All these problems were *successfully solved*, with final objective function value smaller than 10^{-10} and within a tolerance of 10^{-5} for the norm of the projected gradient.

Assuming next that both contacts are of the *rolling* type, two instances were generated for each problem: tackling the MNCP defined by Eq. (1) via problem (2), and considering the LCP originated by the pyramidal approximation to the Coulomb cone.

Original Coulomb friction law

The number of variables of the nonlinear reformulation (2) is 16. Matrices were computed using the toolbox from Corke (2002), and two hundred tests were obtained by randomly generating the vector data. Out of the two hundred tests, 183 were successfully solved (92%). The results are visually displayed in Fig.2, a bar chart of $\lfloor -\log_{10} | f_M^* | \rfloor$ *versus* number of tests. The high frequency of tests with $\lfloor -\log_{10} | f_M^* | \rfloor = 10$ is perhaps related with the chosen value for the tolerance used as stopping criterion.

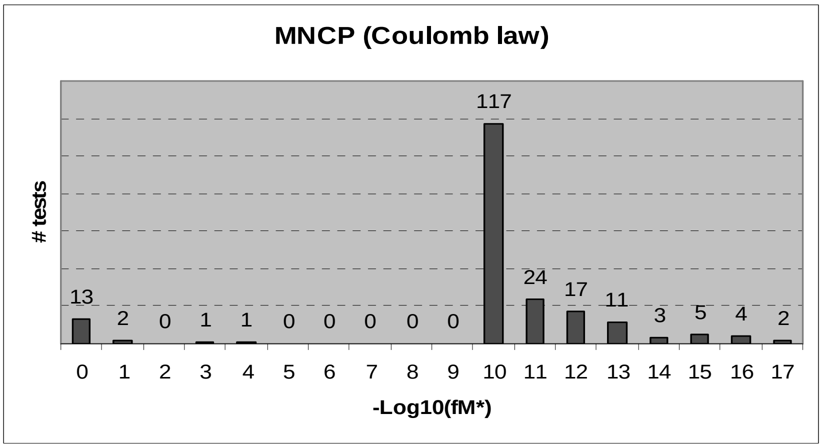

Figure 2. Objective function value (f_M^*) distribution of problem (2) for contact problem with original Coulomb friction law.

Pyramidal approximation

The nonlinear reformulation of the LCP that comes from replacing the Coulomb law with the pyramidal approximation has 20 variables. Again, the related matrix was computed using the toolbox from Corke (2002), and two hundred tests were obtained by randomly generating the vector data. The number of successfully solved problems was

194, which corresponds to 97%. Figure 3 contains a bar chart of $\lfloor -\log_{10} | f_L^* | \rfloor$ *versus* number of tests. As before, the threshold class characterized by $\lfloor -\log_{10} | f_L^* | \rfloor = 10$ is associated with the highest frequency of outcomes.

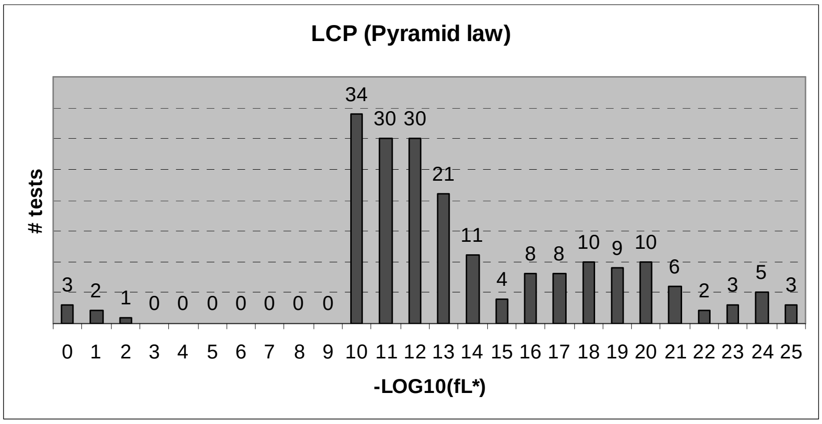

Figure 3. Objective function value (f_L^*) distribution of problem (2) for contact problem with pyramid law.

Coulomb cone *versus* pyramid law

Comparing the two approaches for solving the all rolling case with two contact points, the average computational results shown in Tables 2 and 3 indicate that, in terms of computational efforts, measured in terms of matrix-vector products and CPU time, the MNCP is around nine times more demanding than the LCP.

Table 2. CPU time in seconds.

	minimum	average	maximum
Reformulation of MNCP	0.000	0.200	8.740
Reformulation of LCP	0.000	0.030	0.170

Table 3. Average results of the numerical performance of `BOX-QUACAN` for contact problems.

	`ItBox`	`FE`	`ItQua`	`MVP`
Reformulation of MNCP	57.3	76.3	984.2	1150.8
Reformulation of LCP	7.5	8.9	90.8	126.3

With respect to the quality of the solution obtained in each case, a further analysis of the results is necessary. One desired feature of the solution of a problem with rolling contacts is to indicate a possible transition from rolling to sliding in a coherent fashion, that is, ensuring that the tangential accelerations oppose the tangential contact forces. The Coulomb friction cone model guarantee such conditions for the MNCP. The approximate model of the pyramid law, however, does not enforce the colinearity of the tangential acceleration and force. Thus, such colinearity was tested at each contact point for the successfully solved problems. Considering the MNCP approach, colinearity was obtained in 122 and 128 tests, for contact one (122/183 = 67%) and two (128/183 = 70%), respectively. For the pyramid model, the opposition holds in 97 and 153 tests, for contact one and two, respectively (97/194 = 50%, 153/194 = 79%).

Another observation concerning the quality of results is the following: the pyramid contains the Coulomb cone, but with the pyramid law the contact forces might lay outside the Coulomb cone. In fact, this was the case for several problems: 106 out of 200 for contact one (53%) and 53 out of 200 for contact two (27%).

Comparing the transitions, from rolling to rolling, sliding or breaking contacts, the agreement between both approaches occurred as follows: out of the 178 simultaneously successfully solved tests, 98% in contact one and 96% in contact two pointed to the same transition.

5. Final remarks

In this work we have presented an equivalent bound-constrained formulation for the mixed nonlinear complementarity problem, and provided a theoretical sufficient condition that ensures stationary points of the equivalent formulation will be solutions to the original complementarity problem. We are interested in solving MNCP's arising in the context of multi-rigid-body contact problems with Coulomb friction.

Although the problems we have solved are purely 'academic', we believe that the achieved results give some insight into the nature of our approach, mostly based on the reformulation of the original MNCP as a bound-constrained minimization problem, indicating its potentiality.

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7. References

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