Some variational formulations for non-associated hardening plasticity

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Abstract

The aim of this paper is to present some generalized variational formulations that find application in incremental or limit analysis of an elastic-plastic continuum including hardening effects and non-associated plastic flow laws.

The class of variational principles focused here are obtained with the aid of a generalized potential function, named bipotential by [1]. This pseudo-potential is used to derive, in a non-standard form, the constitutive relation describing the particular elastic-plastic behavior considered. The device is applicable, for instance, to represent non-associated hardening behaviors. This is the case of the Modified Cam-clay model, widely used in soil mechanics and other areas dealing with frictional materials.

The main result in the paper is a criterion, for this class of generalized material models, that is able to determine whether the phenomenon of plastic collapse will occur or not when the loading is proportionally increased. This criterion is based on a mixed minimum principle.

Keywords: plastic collapse, non-associated plasticity, hardening.

1 Introduction

This paper is focused on some variational formulations that find application in a class of elastoplastic materials that includes hardening behavior and nonassociated flow rules, such as frictional materials. This class is characterized by mathematical models whose flow laws are derived, by means of partial subdifferentiation, of a generalized potential, called bipotential. The arguments of the bipotential are generalized stresses and plastic strain rates, so that partial subdifferentiation ends up with implicit equations for the fluxes; this justifies the name Implicit Standard Materials (ISM) given by G. de Saxcé and coworkers [1–3]. The class of implicit standard materials includes the generalized standard materials (GSM) introduced by [4], which only consider associated flow laws.

The main result in the paper is a criterion for this class of generalized material models, based on a minimum principle and aimed to determine whether the phenomenon of plastic collapse will occur or

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not when the loading is proportionally increased.

The scope of the paper includes a brief presentation of: (i) the constitutive equations of plasticity with internal variables allowing for non-associated evolution equations, (ii) the GSM approach and the bipotential generalization, (iii) a minimum principle characterizing the flow law of GSM materials, and (iv) plastic collapse for non-associated elastic-plastic materials and a related minimum principle, first obtained by [5]. New theoretical results, concerning the principles introduced in the aforementioned reference, are developed in this work.

We use the notation: $\boldsymbol{\sigma}$ for stress tensors, $\boldsymbol{\varepsilon}$ for strain tensors and $\boldsymbol{d}$ for strain rate tensors. Mean and deviatoric parts of the stress are denoted by

$$\sigma_m := \frac{1}{3}\mathrm{tr}\,\boldsymbol{\sigma} \qquad \boldsymbol{S} := \boldsymbol{\sigma} - \sigma_m \boldsymbol{1} \tag{1}$$

where $\boldsymbol{1}$ is the identity, tr is the trace operator and dev denotes the deviatoric part of a tensor. Then, the volumetric components of the strain and strain rate are defined as

$$\varepsilon_v := \mathrm{tr}\,\boldsymbol{\varepsilon} \qquad \varepsilon_v := \mathrm{tr}\,\boldsymbol{d} \tag{2}$$

Superscripts e and p identify elastic and plastic parts of strain or strain rates.

2 Generalized Standard Materials (GSM)

In this section we consider associated elastic-plastic materials with internal variables, representing hardening, included in the framework of generalized standard materials [6, 7].

Let the plastic admissibility domain be denoted by

$$P = \{(\boldsymbol{\sigma}, \boldsymbol{A}) \;\mid\; f(\boldsymbol{\sigma}, \boldsymbol{A}) \leqslant 0\} \tag{3}$$

where $\boldsymbol{A}$ is a list of thermodynamic forces associated to hardening mechanisms of the material. This domain is convex and contains the origin $(\boldsymbol{\sigma}, \boldsymbol{A}) = (0, 0)$. The yield function f is assumed regular here for the sake of simplicity. Accordingly, the dissipation potential is defined as

$$D(\boldsymbol{d}^p, \dot{\boldsymbol{\beta}}) := \sup_{(\boldsymbol{\sigma}^*, \boldsymbol{A}^*)\in P} \left(\boldsymbol{\sigma}^* \cdot \boldsymbol{d}^p + \boldsymbol{A}^* \cdot \dot{\boldsymbol{\beta}}\right) \tag{4}$$

where $\dot{\boldsymbol{\beta}}$ denotes the list of internal fluxes corresponding by duality to the internal forces $\boldsymbol{A}$. Then, the associative constitutive equations relating inelastic fluxes and internal forces are equivalently stated as one of the following equations

$$(\boldsymbol{d}^p, \dot{\boldsymbol{\beta}}) \in \partial\mathcal{I}_P(\boldsymbol{\sigma}, \boldsymbol{A}) \tag{5}$$

$$(\boldsymbol{\sigma}, \boldsymbol{A}) \in \partial D(\boldsymbol{d}^p, \dot{\boldsymbol{\beta}}) \tag{6}$$

$$D(\boldsymbol{d}^p, \dot{\boldsymbol{\beta}}) + \mathcal{I}_P(\boldsymbol{\sigma}, \boldsymbol{A}) = \boldsymbol{\sigma} \cdot \boldsymbol{d}^p + \boldsymbol{A} \cdot \dot{\boldsymbol{\beta}} \tag{7}$$

where the symbol ∂ denotes subdifferential and the indicator function $\mathcal{I}_P(\boldsymbol{\sigma}, \boldsymbol{A})$ equals 0 if $(\boldsymbol{\sigma}, \boldsymbol{A}) \in P$, or $+\infty$ otherwise. The subdifferential set $\partial\mathcal{I}_P(\boldsymbol{\sigma}, \boldsymbol{A})$ contains all the outward normals to the boundary of P at $(\boldsymbol{\sigma}, \boldsymbol{A})$ (see for instance: [8, p. 285]; [6, p. 48]; [2, p. 397]; [9]; [10]).

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In terms of the plastic function, this flow law is more often expressed by the system

$$(\boldsymbol{d}^p, \dot{\boldsymbol{\beta}}) = \dot{\lambda} \nabla f(\boldsymbol{\sigma}, \boldsymbol{A}) \tag{8}$$

$$\dot{\lambda} f(\boldsymbol{\sigma}, \boldsymbol{A}) = 0 \qquad f(\boldsymbol{\sigma}, \boldsymbol{A}) \leqslant 0 \qquad \dot{\lambda} \geqslant 0 \tag{9}$$

where ∇f is the gradient of f. Notice that (8) may also be written as $\boldsymbol{d}^p = \dot{\lambda} \nabla_{\boldsymbol{\sigma}} f(\boldsymbol{\sigma}, \boldsymbol{A})$ and $\dot{\boldsymbol{\beta}} = \dot{\lambda} \nabla_{\boldsymbol{A}} f(\boldsymbol{\sigma}, \boldsymbol{A})$, where $\nabla_{\boldsymbol{\sigma}} f(\boldsymbol{\sigma}, \boldsymbol{A})$ denotes partial differentiation.

The plastic admissibility of generalized stresses is enforced by (7) when it is taken into account that the right hand side is always finite. Hence, this equation reduces to $D(\boldsymbol{d}^p, \dot{\boldsymbol{\beta}}) = \boldsymbol{\sigma} \cdot \boldsymbol{d}^p + \boldsymbol{A} \cdot \dot{\boldsymbol{\beta}}$ together with $(\boldsymbol{\sigma}, \boldsymbol{A}) \in P$. Moreover, by the definition of the dissipation function it follows that

$$D(\boldsymbol{d}^p, \dot{\boldsymbol{\beta}}) + \mathcal{I}_P(\boldsymbol{\sigma}, \boldsymbol{A}) \geqslant \boldsymbol{\sigma} \cdot \boldsymbol{d}^p + \boldsymbol{A} \cdot \dot{\boldsymbol{\beta}} \qquad \forall \, \{(\boldsymbol{\sigma}, \boldsymbol{A}), (\boldsymbol{d}^p, \dot{\boldsymbol{\beta}})\} \tag{10}$$

and the equality holds if and only if the generalized stress and strain are related by the flow law that is been described in this section.

3 Implicit Standard Materials (ISM)

The variational inequality given by (10) is the starting formulation to introduce the concept of bipotentials, which is intended to enhance the field of application of pseudo-potentials formulations for constitutive relations.

For convenience, we define generalized stress and strain rate as follows

$$\boldsymbol{\Sigma} := (\boldsymbol{\sigma}, \boldsymbol{A}) \qquad \boldsymbol{D}^p := (\boldsymbol{d}^p, \dot{\boldsymbol{\beta}}) \tag{11}$$

with scalar product

$$\boldsymbol{\Sigma} \cdot \boldsymbol{D}^p = \boldsymbol{\sigma} \cdot \boldsymbol{d}^p + \boldsymbol{A} \cdot \dot{\boldsymbol{\beta}} \tag{12}$$

meaning dissipated power.

A function $b(\boldsymbol{\Sigma}, \boldsymbol{D}^p)$, with values in $[-\infty, \infty]$, is said to be a bipotential if it is separately convex with respect to $\boldsymbol{\Sigma}$ and $\boldsymbol{D}^p$, not identically $+\infty$, and satisfies the following condition

$$b(\boldsymbol{\Sigma}, \boldsymbol{D}^p) \geqslant \boldsymbol{\Sigma} \cdot \boldsymbol{D}^p \qquad \forall \, (\boldsymbol{\Sigma}, \boldsymbol{D}^p) \tag{13}$$

For a function $b(\boldsymbol{\Sigma}, \boldsymbol{D}^p)$ complying with the above definition, it is proven that any pair $(\boldsymbol{\Sigma}, \boldsymbol{D}^p)$ satisfying one of the three conditions below will also satisfy the remaining two conditions:

$$\boldsymbol{D}^p \in \partial_{\boldsymbol{\Sigma}} b(\boldsymbol{\Sigma}, \boldsymbol{D}^p) \tag{14}$$

$$\boldsymbol{\Sigma} \in \partial_{\boldsymbol{D}^p} b(\boldsymbol{\Sigma}, \boldsymbol{D}^p) \tag{15}$$

$$b(\boldsymbol{\Sigma}, \boldsymbol{D}^p) = \boldsymbol{\Sigma} \cdot \boldsymbol{D}^p \tag{16}$$

For a proof of the above equivalence see, for instance, Appendix A in [5].

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We are using in (14) the notation $\partial_{\boldsymbol{\Sigma}} b(\boldsymbol{\Sigma}, \boldsymbol{D}^p)$ for the partial subdifferential of b with respect to $\boldsymbol{\Sigma}$. This is, by definition, the set of all $\boldsymbol{D}^p$ such that

$$b(\boldsymbol{\Sigma}^*, \boldsymbol{D}^p) - b(\boldsymbol{\Sigma}, \boldsymbol{D}^p) \geqslant (\boldsymbol{\Sigma}^* - \boldsymbol{\Sigma}) \cdot \boldsymbol{D}^p \qquad \forall \boldsymbol{\Sigma}^* \tag{17}$$

Likewise, (15) is equivalent, by definition, to

$$b(\boldsymbol{\Sigma}, \boldsymbol{D}^{p*}) - b(\boldsymbol{\Sigma}, \boldsymbol{D}^p) \geqslant \boldsymbol{\Sigma} \cdot (\boldsymbol{D}^{p*} - \boldsymbol{D}^p) \qquad \forall \boldsymbol{D}^{p*} \tag{18}$$

Moreover, the above conditions are also equivalent to the following

$$\boldsymbol{\Sigma} \in \arg \inf_{\boldsymbol{\Sigma}^*} \{ b(\boldsymbol{\Sigma}^*, \boldsymbol{D}^p) - \boldsymbol{\Sigma}^* \cdot \boldsymbol{D}^p \} \tag{19}$$

$$\boldsymbol{D}^p \in \arg \inf_{\boldsymbol{D}^{p*}} \{ b(\boldsymbol{\Sigma}, \boldsymbol{D}^{p*}) - \boldsymbol{\Sigma} \cdot \boldsymbol{D}^{p*} \} \tag{20}$$

with $\arg\inf$ denoting the set of solutions of the minimization problem.

Then, we use the bipotential to characterize a particular flow law in the following manner: a generalized stress $\boldsymbol{\Sigma}$ and a generalized plastic strain rate $\boldsymbol{D}^p$ are related by the flow law, and we denote this by

$$(\boldsymbol{\Sigma}, \boldsymbol{D}^p) \in \mathsf{FlowLaw} \tag{21}$$

if and only if this pair is extremal for the bipotential, i.e. this pair satisfies one of the equivalent conditions (16) to (20). Implicit standard materials (ISM) are characterized by evolution equations of this particular kind.

We introduce now a key concept for the present paper. The gap function $\psi(\boldsymbol{\Sigma}, \boldsymbol{D}^p)$, associated to the flow law emanating from the bipotential, is defined as follows

$$\psi(\boldsymbol{\Sigma}, \boldsymbol{D}^p) := b(\boldsymbol{\Sigma}, \boldsymbol{D}^p) - \boldsymbol{\Sigma} \cdot \boldsymbol{D}^p \tag{22}$$

For an arbitrary choice of $(\boldsymbol{\Sigma}, \boldsymbol{D}^p)$ the gap function gives the difference between the available power of dissipation to the actual dissipated power of the arguments.

The following proposition gives the relation of this gap function to the flow law.

Proposition 1. Let ψ be given by (22) and FlowLaw defined in (21), then

$$\psi_0 := \inf_{\boldsymbol{\Sigma}, \boldsymbol{D}^p} \psi(\boldsymbol{\Sigma}, \boldsymbol{D}^p) \tag{23}$$

is finite and nonnegative. Further

1. If $\psi_0 = 0$ then

$$(\boldsymbol{\Sigma}, \boldsymbol{D}^p) \in \mathsf{FlowLaw} \qquad \Leftrightarrow \qquad (\boldsymbol{\Sigma}, \boldsymbol{D}^p) \in \arg \inf_{\boldsymbol{\Sigma}, \boldsymbol{D}^p} \psi(\boldsymbol{\Sigma}, \boldsymbol{D}^p) \tag{24}$$

2. If $\psi_0 > 0$ then FlowLaw is empty.

Proof. The gap function $\psi(\boldsymbol{\Sigma}^*, \boldsymbol{D}^{p*})$ is proper [10] and nonnegative for all $(\boldsymbol{\Sigma}^*, \boldsymbol{D}^{p*})$, as a consequence of the definition of the bipotential. Further, if $(\boldsymbol{\Sigma}, \boldsymbol{D}^p) \in \mathsf{FlowLaw}$ then, using (16), it follows that $\psi(\boldsymbol{\Sigma}, \boldsymbol{D}^p) = 0$, which leads directly to the thesis. □

We assume from now on that the constitutive model is such that FlowLaw is not empty. However, we will show that, even under this hypothesis for the material behavior, a proposition analogous to the one stated above describes the situation of an ISM body with respect to the occurrence of plastic collapse.

3.1 An ISM model for frictional materials

As an example of a material that can be cast in the framework of implicit standard materials, i.e. with an evolution equation derived froma a bipotential, we give in this subsection a very brief description of the Modified Cam-clay model (MCC) widely used in soil mechanics [9, 11, 12]. A bipotential for this type of material was given in [1] (see also [13, p. 65]) and it is considered in the following.

Plastic admissibility of generalized stresses is represented by the following constraint

$$f(\boldsymbol{\sigma},\rho) := \sqrt{\frac{3}{2M^2}\|\boldsymbol{S}\|^2 + (\sigma_m+\rho)^2} - \rho \leqslant 0 \tag{25}$$

where the scalar hardening variable ρ is the only stress-like internal variable of the model. That is, with reference to our general notation, we have in this particular case $\boldsymbol{A} \equiv \rho$ and $\boldsymbol{\sigma} \equiv (\boldsymbol{S}, \sigma_m)$.

The usual flow law for MCC is written

$$\boldsymbol{d}^{p\ \mathrm{dev}} = \dot{\lambda}\nabla_{\boldsymbol{S}} f = \frac{3\dot{\lambda}}{2M^2 R(\boldsymbol{\sigma},\rho)}\boldsymbol{S} \tag{26}$$

$$d_v^p = \dot{\lambda}\nabla_{\sigma_m} f = \frac{\dot{\lambda}(\sigma_m+\rho)}{R(\boldsymbol{\sigma},\rho)} \tag{27}$$

$$\dot{\beta} = \frac{\dot{\lambda}(\sigma_m+\rho)}{R(\boldsymbol{\sigma},\rho)} \tag{28}$$

with the notation

$$R(\boldsymbol{\sigma},\rho) := \sqrt{\frac{3}{2M^2}\|\boldsymbol{S}\|^2 + (\sigma_m+\rho)^2} \tag{29}$$

The Cam-clay model described here is not a generalized standard material in the sense of [6] since this model adopts the non-associated hardening evolution equation (28). The hardening part (28) of the flow law is justified by experimental observation on these materials indicating that hardening is dependent on the volumetric plastic strain. Accordingly, (28) produces the appropriate link between the hardening internal variable and the volumetric plastic strain.

The hardening flux $\dot{\beta}$ is maintained, for convenience, formally distinct to the volumetric plastic strain rate $\boldsymbol{d}^p$, although both are identical in any actual evolution process. This equality is enforced by defining a set K such that $(\boldsymbol{d}^p, \dot{\beta}) \in K$ if and only if $d_v^p = \dot{\beta}$. The indicator of K is the function $\mathcal{I}_K(\boldsymbol{d}^p, \dot{\beta})$ that equals 0 if $d_v^p = \dot{\beta}$ and $+\infty$ otherwise.

With the notation introduced above the bipotential is written as follows

$$b(\boldsymbol{\sigma},\rho,\boldsymbol{d}^p,\dot{\beta}) = M\rho\sqrt{\frac{2}{3}\|\boldsymbol{d}^{p\ \mathrm{dev}}\|^2 + \frac{1}{M^2}(d_v^p)^2} + \mathcal{I}_P(\boldsymbol{\sigma},\rho) + \mathcal{I}_K(\boldsymbol{d}^p,\dot{\beta}) \tag{30}$$

that is

$$b(\boldsymbol{\sigma},\rho,\boldsymbol{d}^p,\dot{\beta}) = \begin{cases} M\rho\sqrt{\frac{2}{3}\|\boldsymbol{d}^{p\ \mathrm{dev}}\|^2 + \frac{1}{M^2}(d_v^p)^2} & \text{if} \quad d_v^p = \dot{\beta} \quad \text{and} \quad f(\boldsymbol{\sigma},\rho) \leqslant 0 \\ +\infty & \text{otherwise} \end{cases} \tag{31}$$

4 Plastic collapse of solids obeying the ISM model

Elastic-plastic bodies or structures may collapse under a constant system of loads in a process where the stress distribution remains constant while the body undergoes unbounded purely plastic deformation. This phenomenon is eventually experienced by solids made of materials with associated flow rules or even with non-associated laws.

In classical associative plasticity it is proven that the load producing plastic collapse is the maximum load that the body can sustain in equilibrium. It is well known that for non-associated plastic behavior these two concepts, namely: plastic collapse and limit load, are not necessarily linked.

Accordingly, the first concern in this section is to give a precise mathematical formulation characterizing plastic collapse for materials with hardening and represented by means of bipotentials. Afterwards, we propose a variational problem and prove a proposition relating, in a non-standard manner, the solution of this minimization problem with the existence of collapse solutions.

Some additional notation is introduced now: The symbol $\mathcal{D}$ denotes the linear deformation operator mapping velocities (displacements) into compatible strain rates (respectively strains). The internal power associated with a stress field $\boldsymbol{\sigma}$ and a velocity distribution w is denoted

$$\langle \boldsymbol{\sigma}, \mathcal{D}\boldsymbol{w} \rangle := \int_{\mathcal{B}} \boldsymbol{\sigma} \cdot \mathcal{D}\boldsymbol{w}\, d\mathcal{B} \tag{32}$$

Likewise, the external power associated with a load system $\boldsymbol{F}$ is

$$\langle \boldsymbol{F}, \boldsymbol{w} \rangle := \int_{\mathcal{B}} \boldsymbol{b} \cdot \boldsymbol{w}\, d\mathcal{B} + \int_{\Gamma_u} \boldsymbol{\tau} \cdot \boldsymbol{w}\, d\Gamma \tag{33}$$

where $\boldsymbol{b}$ and $\boldsymbol{\tau}$ are volume and surface load densities, Γ denotes area and Γ_u is part of the boundary where null displacements are prescribed. Accordingly, the set of stress fields in equilibrium with a given load system $\boldsymbol{F}$ is defined as

$$\mathcal{S}(\boldsymbol{F}) := \{\boldsymbol{\sigma} \ \mid \ \langle \boldsymbol{\sigma}, \mathcal{D}\boldsymbol{w} \rangle = \langle \boldsymbol{F}, \boldsymbol{w} \rangle \ \forall\ \boldsymbol{w}\} \tag{34}$$

where the virtual velocity $\boldsymbol{w}$ varies in the linear space of admissible velocities.

4.1 The equations of plastic collapse

When the body undergoes plastic collapse the stress and the internal thermodynamical forces remain constant. In view of the elastic state equations, a constant stress field induces an elastic deformation distribution that is also constant in time. This, in turn, means that the strain rate field is at the same time compatible and purely plastic.

Likewise, according to the state equation relative to internal variables, internal hardening forces being constant during plastic collapse implies that the kinematical internal variables are also constant, that is, $\dot{\boldsymbol{\beta}} = 0$ at all points in the body.

This description of plastic collapse is implemented in the sequel. The computation of the critical factor α that amplifies a prescribed load system $\boldsymbol{F}$ so as to produce unbounded purely plastic deformation, when superposed to a fixed (non-amplified) load $\boldsymbol{F}^0$, can be formulated as follows:

PC - *The plastic collapse problem.* Find $(\alpha, \boldsymbol{\sigma}, \boldsymbol{A}, v)$ such that

$$\boldsymbol{\sigma} \in \mathcal{S}(\boldsymbol{F}^0 + \alpha \boldsymbol{F}) \tag{35}$$

$$\langle \boldsymbol{F}, \boldsymbol{v} \rangle = 1 \tag{36}$$

$$b(\boldsymbol{\sigma}, \boldsymbol{A}, \mathcal{D}\boldsymbol{v}, 0) = \boldsymbol{\sigma} \cdot \mathcal{D}\boldsymbol{v} \quad \text{in } \mathcal{B} \tag{37}$$

$$\alpha \geqslant 0 \tag{38}$$

Notice that (36) has only been included to select one normalized velocity distribution, so discarding all its multiples by a scalar, which would also be solutions otherwise.

4.2 Variational plastic collapse analysis for ISM

We propose in this subsection a minimization problem and then explain how it can be used for the purpose of determining the existence of collapse solutions to PC, i.e. $(\alpha, \boldsymbol{\sigma}, \boldsymbol{A}, v)$ satisfying (35-38).

In order to obtain the desired variational formulation, given below, we impose (37) in the form of a minimum principle as given in (22), but now we adjoin (35), (36) and (38) as constraints.

MP - *A minimum principle*

$$\Upsilon = \inf_{\alpha, \boldsymbol{\sigma}, \boldsymbol{A}, \boldsymbol{v}} \left\{ \Psi(\boldsymbol{\sigma}, \boldsymbol{A}, \mathcal{D}\boldsymbol{v}, 0) \quad | \quad \boldsymbol{\sigma} \in \mathcal{S}(\boldsymbol{F}^0 + \alpha \boldsymbol{F});\ \langle \boldsymbol{F}, \boldsymbol{v} \rangle = 1;\ \alpha \geqslant 0 \right\} \tag{39}$$

where $\Psi := \int_{\mathcal{B}} \psi d\mathcal{B}$.

A first remark on the use of this variational formulation is the following lemma, whose proof is straightforward.

Lemma. The infimum Υ of the minimum principle MP is finite and nonnegative if the feasible set is nonempty, otherwise it is $+\infty$.

Using the constraints in (39) we get

$$\Psi(\boldsymbol{\sigma}, \boldsymbol{A}, \mathcal{D}\boldsymbol{v}, 0) = \mathrm{b}(\boldsymbol{\sigma}, \boldsymbol{A}, \mathcal{D}\boldsymbol{v}, 0) - \langle \boldsymbol{\sigma}, \mathcal{D}\boldsymbol{v} \rangle = \mathrm{b}(\boldsymbol{\sigma}, \boldsymbol{A}, \mathcal{D}\boldsymbol{v}, 0) - \langle \boldsymbol{F}^0, \boldsymbol{v} \rangle - \alpha \tag{40}$$

where $\mathrm{b} := \int_{\mathcal{B}} b d\mathcal{B}$.

Thus, the minimum principle (39) can be rewritten as

$$\Upsilon = \inf_{\alpha,\boldsymbol{\sigma},\boldsymbol{A},\boldsymbol{v}} \Big\{ \mathrm{b}(\boldsymbol{\sigma}, \boldsymbol{A}, \mathcal{D}\boldsymbol{v}, 0) - \langle \boldsymbol{F}^0, \boldsymbol{v} \rangle - \alpha \quad \big| \quad \boldsymbol{\sigma} \in \mathcal{S}(\boldsymbol{F}^0 + \alpha \boldsymbol{F});\ \langle \boldsymbol{F}, \boldsymbol{v} \rangle = 1;\ \alpha \geqslant 0 \Big\} \tag{41}$$

Now, the main remarks and results: (i) MP has always a solution for Υ, and (ii) finding this solution can give the answer to the question about existence of solutions to PC. More precisely:

Proposition 2. The plastic collapse problem PC and the minimum principle MP are related by the following implications:

1. If there exists a solution $(\alpha, \boldsymbol{\sigma}, \boldsymbol{A}, \boldsymbol{v})$ of PC, then this set is a minimizer for MP and corresponds to $\Upsilon = 0$.
2. If MP has a minimizer $(\alpha, \boldsymbol{\sigma}, \boldsymbol{A}, \boldsymbol{v})$ such that $\Upsilon = 0$, then this set is a solution for PC.

Proof.

1. Collapse solutions satisfy all constraints in MP and give $\Upsilon = 0$ since $\psi = 0$ on $\mathcal{B}$.
2. $\Upsilon = 0$ implies $\psi = 0$ on $\mathcal{B}$ and this completes the set of equations in PC. □

In other words, if we solve the minimization problem MP, analytically or numerically, we can conclude that:

1. If $\Upsilon = 0$ then the computed minimal solution describes the plastic collapse for the obtained amplifcation factor α.
2. If $\Upsilon > 0$ then the body does not collapse for any loading of the form $\boldsymbol{F}^0 + \alpha \boldsymbol{F}$.

4.3 Back to generalized standard materials

In this section we analyze the particular form of the principle MP for the case of associated plasticity with hardening. In this case it holds that

$$\mathrm{b}(\boldsymbol{\sigma}, \boldsymbol{A}, \boldsymbol{d}^p, \dot{\boldsymbol{\beta}}) = D(\boldsymbol{d}^p, \dot{\boldsymbol{\beta}}) + \mathcal{I}_P(\boldsymbol{\sigma}, \boldsymbol{A}) \tag{42}$$

Then the formulation MP becomes

$$\Upsilon = \inf_{\alpha,\boldsymbol{\sigma},\boldsymbol{A},\boldsymbol{v}} \Big\{ D(\mathcal{D}\boldsymbol{v}, 0) - \langle \boldsymbol{F}^0, \boldsymbol{v} \rangle - \alpha \quad \big| \quad \boldsymbol{\sigma} \in \mathcal{S}(\boldsymbol{F}^0 + \alpha \boldsymbol{F});\ (\boldsymbol{\sigma}, \boldsymbol{A}) \in \mathsf{P};\ \langle \boldsymbol{F}, \boldsymbol{v} \rangle = 1;\ \alpha \geqslant 0 \Big\} \tag{43}$$

where $(\boldsymbol{\sigma}, \boldsymbol{A}) \in \mathsf{P}$ means that $(\boldsymbol{\sigma}(\mathbf{x}), \boldsymbol{A}(\mathbf{x})) \in P$ for all $\mathbf{x} \in \mathcal{B}$.

The optimization problem above can be separated as follows

$$\Upsilon = \mathsf{U} - \mathsf{L} \tag{44}$$

with

$$\mathsf{U} := \inf_{v} \Big\{ D(\mathcal{D}\boldsymbol{v}, 0) - \langle \boldsymbol{F}^0, \boldsymbol{v} \rangle \ \big| \ \langle \boldsymbol{F}, \boldsymbol{v} \rangle = 1 \Big\} \tag{45}$$

$$\mathsf{L} := \sup_{\alpha,\boldsymbol{\sigma},\boldsymbol{A}} \Big\{ \alpha \ \big| \ \boldsymbol{\sigma} \in \mathcal{S}(\boldsymbol{F}^0 + \alpha \boldsymbol{F});\ (\boldsymbol{\sigma}, \boldsymbol{A}) \in \mathsf{P};\ \alpha \geqslant 0 \Big\} \tag{46}$$

The optimization principles (45) and (46) are the kinematical and statical formulations of limit analysis. These are dual optimization principles representing the limit analysis problem, which in this case coincides with the plastic collapse problem under some conditions on the functional spaces describing the mechanical system. Dual optimization problems are the subject of minimax theory in convex analysis (see e.g. [14, p. 327]) where two basic results are called the weak and the strong duality theorems. The weak duality theorem, proven under mild hypotheses, when applied to the present situation, ensures that $\mathsf{U} \geqslant \mathsf{L}$. Strong duality demands more stringent conditions so as to prove that there exists $\mathsf{U} = \mathsf{L}$ and also that this number is a saddle value of the Lagrangian.

5 Conclusions

We analyzed two variational formulations, given in (23) and (41), restricted to a special class of materials, denoted ISM, whose evolution equations are derived from a bipotential by partial subdifferentiation.

The first minimization principle, (23), constitutes a variational formulation of the flow law that is used to obtain the second one.

The main result in the paper is precisely described in Proposition 2. It is a criterion, based on the minimum principle (41), aimed to determine whether the phenomenon of plastic collapse will occur or not when the loading is proportionally increased. This minimization principle (41), that is a mixed one in the sense that involves both kinematical and statical variables, is different to the statical and kinematical variational formulations proposed by [2]. The derivation of the minimization principle MP shown in the present paper is different, and more clearly motivated, compared to the presentation in [5].

Responsibility notice

The author is the only responsible for the printed material included in this paper

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