

Generalized Finite Element Method with global-local enrichments for nonlinear fracture analysis

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Abstract

The main feature of partition of unity methods such as *hp*-cloud, generalized and extended finite element methods is their ability of utilizing a priori knowledge about the solution of a problem in the form of enrichment functions. Linear combination of partition of unity shape functions can reproduce exactly the enrichment functions and thus their approximation properties are preserved. This class of shape functions enables, for example, the approximation of solutions in the neighborhood of cracks [1], inclusions, voids [2], etc. using finite element meshes that do not fit them. However, analytical derivation of enrichment functions with good approximation properties is mostly limited to two-dimensional linear problems. In this paper, we present a procedure to numerically generate enrichment functions for problems with localized nonlinearities. The procedure is based on the so-called generalized finite element method (GFEM) with global-local enrichment functions [3]. It involves the solution of boundary value problems around regions exhibiting nonlinear behavior and the enrichment of the global solution space with local solutions through the partition of unity framework used in the GFEM. This approach was originally proposed to efficiently handle crack propagation simulation [4], but it also has the potential to solve nonlinear problems accurately and with a reduced computational cost compared with the conventional finite element method. An illustrative nonlinear problem of the J_2 -based plasticity theory with linear isotropic hardening is solved using the proposed procedure to demonstrate its robustness, accuracy and computational efficiency.

1 Introduction

Several technological advances attempted today require computational methods able to model complex, highly localized phenomena. A representative example is the case of mechanically short cracks. These cracks are much smaller than any dimension of a structural component, but larger than the

details of the material microstructure (polycrystalline grains in the case of metals) [5]. Non-linear process zones in these cracks, while relevant, are several orders of magnitude smaller than any structural dimension. The analysis of small cracks is critically important for structures subjected to high frequency excitations, like acoustic loads. The majority of the life of these components corresponds to the growth and incubation of small cracks. The analysis of this class of problems can not be handled simply by using available finite element methods (FEMs) and massive computer power. The mesh density required by the FEM leads to prohibitively small time steps in the case of time dependent problems. In addition, the adaptive construction of FEM discretizations may be quite costly, since it demands several adaptive cycles on very large computational models.

In this paper, we propose a novel generalized finite element method for non-linear fracture analysis of small cracks. The key ideas of the proposed method are the computation of the so-called global-local enrichment functions and the use of the partition of unity concept to build conforming approximation spaces on a *coarse* finite element mesh.

The Generalized FEM (GFEM) has had a significant impact on the solution of problems in which there is an a-priori knowledge about the behavior of the solution. By numerically constructing enrichment functions, the proposed GFEM brings the benefits of existing GFEMs to a much broader class of problems and, in particular, to non-linear fracture problems. High-order generalized finite elements and adaptivity are explored in the computation of enrichment functions able to robustly represent small scale features, like cracks, using FE meshes that are straightforward to generate. These meshes are in practice generated during the design of the component and are able to model its global response. By enriching these meshes with the proposed global-local enrichments, the resulting discretization is able to capture the interactions between the global (structural) and small scale (near crack) response. The outline of the paper is as follows. Sections 2 and 2.1 briefly review the main concepts of the GFEM and the so-called global-local enrichments, respectively. Section 2.2 presents the formulation of global-local enrichments for localized non-linearities with emphasis on fracture mechanics problems. The plasticity model used in this paper is described in Section 3. A numerical example illustrating the performance of the proposed method is presented in Section 4. The conclusions are outlined in Section 5.

2 Generalized FEM: A summary

The generalized FEM [6–9] is an instance of the partition of unity method which has its origins in the works of Babuška et al. [6, 10, 11] (under the names “special finite element methods,” “generalized finite element method” and “finite element partition of unity method”) and Duarte and Oden [8, 12–15] (under the names “*hp* clouds” and “cloud-based *hp* finite element method”). Several meshfree methods proposed in recent years can also be formulated as special cases of the partition of unity method. The key feature of these methods is the use of a partition of unity (POU), which is a set of functions whose values sum to the unity at each point in a domain. Additional methods based on the partition of unity concept are, for example, the extended finite element method [16], the method of finite spheres [17] and the particle partition of unity method [18], to name just a few. In these methods, discretization spaces for a Galerkin method are defined using the concept of a partition of unity and local spaces

that are built based on a-priori knowledge about the solution of a problem. A shape function, $\phi_{\alpha i}$, in the GFEM is computed from the product of a linear finite element shape function, φ_{α} , and an enrichment function, $L_{\alpha i}$,

$$\phi_{\alpha i}(\mathbf{x}) = \varphi_{\alpha}(\mathbf{x})L_{\alpha i}(\mathbf{x}) \quad (\text{no summation on } \alpha) \quad (1)$$

where α is a node in the finite element mesh. Figure 1 illustrates the construction of GFEM shape functions.

The linear finite element shape functions φ_{α} , $\alpha = 1, \dots, N$, in a finite element mesh with N nodes constitute a partition of unity, i.e., $\sum_{\alpha=1}^N \varphi_{\alpha}(\mathbf{x}) = 1$ for all \mathbf{x} in a domain Ω covered by the finite element mesh. This is a key property used in partition of unity methods. Linear combinations of the GFEM shape functions $\phi_{\alpha i}$, $\alpha = 1, \dots, N$, can represent *exactly* any enrichment function $L_{\alpha i}$ [14, 15].

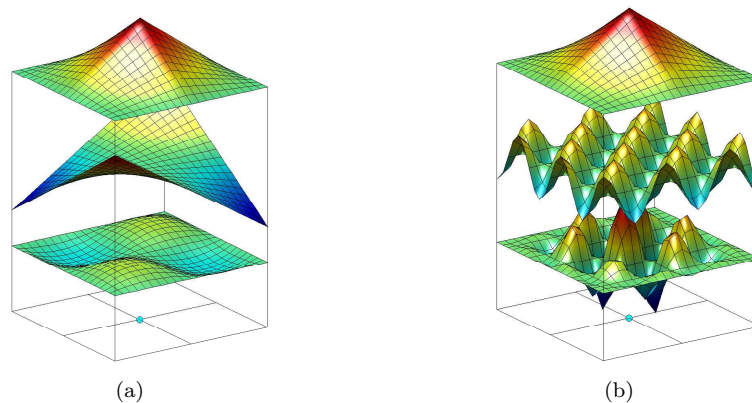


Figure 1: Construction of a generalized FEM shape function using a polynomial (a) and a non-polynomial enrichment (b). Here, φ_{α} are the functions at the top, the enrichment functions, $L_{\alpha i}$, are the functions in the middle, and the generalized FE shape functions, $\phi_{\alpha i}$, are the resulting bottom functions.

Enrichment functions The GFEM has been successfully applied to the simulation of boundary layers [19], dynamic propagating fractures [20], line singularities [7], acoustic problems with high wave number [21, 22], polycrystalline microstructures [23], porous materials [24], etc. All these applications have relied on closed-form enrichment functions that are known to approximate well the physics of the problem. These so-called custom enrichment functions are able to provide more accurate and robust simulations than the polynomial functions traditionally used in the standard FEM, while relaxing some meshing requirements. However, for many classes of problems—like those involving material

non-linearities–enrichment functions with good approximation properties are not amenable to analytical derivation. In this paper, this limitation is removed using *global-local enrichment functions*, as described below.

2.1 Global-local enrichment functions

In this section, we present a global-local approach to numerically build enrichment functions for localized non-linearities. The approach is based on the global-local formulation presented in [3, 4, 25]. We focus on three-dimensional non-linear fracture problems. The formulation is, however, applicable to other classes of problems as well.

2.1.1 Formulation of Global Problem

Consider a domain $\bar{\Omega}_G = \Omega_G \cup \partial\Omega_G$ in \mathbb{R}^3 . The boundary is decomposed as $\partial\Omega_G = \partial\Omega_G^u \cup \partial\Omega_G^\sigma$ with $\partial\Omega_G^u \cap \partial\Omega_G^\sigma = \emptyset$.

The equilibrium equations are given by

$$\nabla \cdot \boldsymbol{\sigma} = \mathbf{0} \quad \text{in } \Omega_G, \quad (2)$$

The constitutive relations may be given by Hooke's law, $\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}$, where \mathbf{C} is Hooke's tensor, or by a non-linear stress-strain relation as discussed in Section 3.

The following boundary conditions are prescribed on $\partial\Omega_G$

$$\mathbf{u} = \bar{\mathbf{u}} \text{ on } \partial\Omega_G^u \quad \boldsymbol{\sigma} \cdot \mathbf{n} = \bar{\mathbf{t}} \text{ on } \partial\Omega_G^\sigma, \quad (3)$$

where \mathbf{n} is the outward unit normal vector to $\partial\Omega_G^\sigma$ and $\bar{\mathbf{t}}$ and $\bar{\mathbf{u}}$ are prescribed tractions and displacements, respectively.

Let \mathbf{u}_G^0 denote the generalized FEM solution of the problem (2), (3). This is hereafter denoted as the *initial global problem*. The GFEM approximation \mathbf{u}_G^0 is the solution of the following problem:

Find $\mathbf{u}_G^0 \in \mathbf{X}_G^{hp}(\Omega_G) \subset H^1(\Omega_G)$ such that, $\forall \mathbf{v}_G^0 \in \mathbf{X}_G^{hp}(\Omega_G)$

$$\int_{\Omega_G} \boldsymbol{\sigma}(\mathbf{u}_G^0) : \boldsymbol{\varepsilon}(\mathbf{v}_G^0) d\mathbf{x} + \eta \int_{\partial\Omega_G^u} \mathbf{u}_G^0 \cdot \mathbf{v}_G^0 ds = \int_{\partial\Omega_G^\sigma} \bar{\mathbf{t}} \cdot \mathbf{v}_G^0 ds + \eta \int_{\partial\Omega_G^u} \bar{\mathbf{u}} \cdot \mathbf{v}_G^0 ds \quad (4)$$

where, $\mathbf{X}_G^{hp}(\Omega_G)$ is a discretization of $H^1(\Omega_G)$, a Hilbert space defined on Ω_G , built with generalized or standard FEM shape functions and η is a penalty parameter. We use the penalty method due to its simplicity and generality. Detailed discussion and analysis of methods for the imposition of Dirichlet boundary conditions in generalized FEMs can be found in, e.g., [26]. Problem (4) leads to a system of linear or non-linear equations for the unknown degrees of freedom of \mathbf{u}_G^0 , depending on the material model used. The mesh used to solve problem (4) is typically a coarse quasi-uniform mesh.

2.1.2 Formulation of local Problem

The proposed approach involves the solution of a local boundary value problem defined in a neighborhood Ω_{loc} of a crack or other small scale feature, and subjected to boundary conditions provided

by the global solution \mathbf{u}_G^0 . The following *local problem* is solved on Ω_{loc} after the global solution \mathbf{u}_G^0 is computed as described above:

Find $\mathbf{u}_L \in \mathbf{X}_{loc}^{hp}(\Omega_{loc}) \subset H^1(\Omega_{loc})$ such that, $\forall \mathbf{v}_L \in \mathbf{X}_{loc}^{hp}(\Omega_{loc})$

$$\int_{\Omega_{loc}} \boldsymbol{\sigma}(\mathbf{u}_L) : \boldsymbol{\varepsilon}(\mathbf{v}_L) dx + \eta \int_{\partial\Omega_{loc} \setminus (\partial\Omega_{loc} \cap \partial\Omega_G^\sigma)} \mathbf{u}_L \cdot \mathbf{v}_L ds =$$

$$\eta \int_{\partial\Omega_{loc} \setminus (\partial\Omega_{loc} \cap \partial\Omega_G)} \mathbf{u}_G^0 \cdot \mathbf{v}_L ds + \eta \int_{\partial\Omega_{loc} \cap \partial\Omega_G^u} \bar{\mathbf{u}} \cdot \mathbf{v}_L ds + \int_{\partial\Omega_{loc} \cap \partial\Omega_G^\sigma} \bar{\mathbf{t}} \cdot \mathbf{v}_L ds \quad (5)$$

where, $\mathbf{X}_{loc}^{hp}(\Omega_{loc})$ is a discretization of $H^1(\Omega_{loc})$ using the GFEM shape functions presented in [27, 28].

A key aspect of problem (5) is the use of the generalized FEM solution of the (coarse) global problem, \mathbf{u}_G^0 , as Dirichlet boundary condition on $\partial\Omega_{loc} \setminus (\partial\Omega_{loc} \cap \partial\Omega_G)$. Exact boundary conditions are prescribed on portions of $\partial\Omega_{loc}$ that intersect either $\partial\Omega_G^u$ or $\partial\Omega_G^\sigma$. Cauchy (spring) boundary conditions can also be used on $\partial\Omega_{loc} \setminus (\partial\Omega_{loc} \cap \partial\Omega_G)$.

2.2 GFEM^{g-1} for Non-Linear Problems

The proposed GFEM^{g-1} non-linear analysis algorithm is described with the aid of the three-dimensional model problem shown in Figure 2. In the first step of procedure, the initial global solution \mathbf{u}_G^0 is computed on a coarse global mesh as illustrated in Figure 2. The formulation of this problem is given by (4). Cracks are *not* discretized when solving this problem. A *linear elastic material law is used at this step*.

In the second step, a local problem is defined around each crack in the global domain. These local problems are automatically built using elements extracted the coarse global discretization around the cracks. Details on this procedure can be found in Section A.1 of [3]. Figure 3 illustrates the procedure. The formulation of the local problems is given by (5). In this case, a non-linear material law (cf. Section 3) is used and the local solution \mathbf{u}_L is found with the aid of Newton's method. The *hp* GFEM presented in [27, 28] is used to discretize the local problems. Figure 4 illustrates an *hp* adapted discretization and the corresponding local solution \mathbf{u}_L computed around a surface crack.

Global-local enrichment functions The procedure to compute the local solution \mathbf{u}_L is based on a global-local finite element analysis, broadly used in many practical applications of the FEM [29]. The error of \mathbf{u}_L depends not only on the discretization used in the local domain Ω_{loc} , but, it is also affected by the quality of the boundary conditions used on $\partial\Omega_{loc} \setminus (\partial\Omega_{loc} \cap \partial\Omega_G)$, i.e., the global (coarse) solution \mathbf{u}_G^0 . As a result, the error of \mathbf{u}_L may be large *even if a very fine mesh is used on Ω_{loc}* . Furthermore, the global-local FEM can *not* account for possible interactions of local (near crack, for example) and global (structural) behavior and, thus, is not suitable for problems involving multiscale phenomena. In the proposed GFEM^{g-1}, these issues are addressed by going one step further in a global-local analysis; the global problem is solved using \mathbf{u}_L as *enrichment functions*. The local solution, \mathbf{u}_L , of the local problem defined in (5) is used to build generalized FEM shape functions for the coarse global mesh. Equation (1) is used with the partition of unity function, φ_α , provided by the global, *coarse*, FE mesh

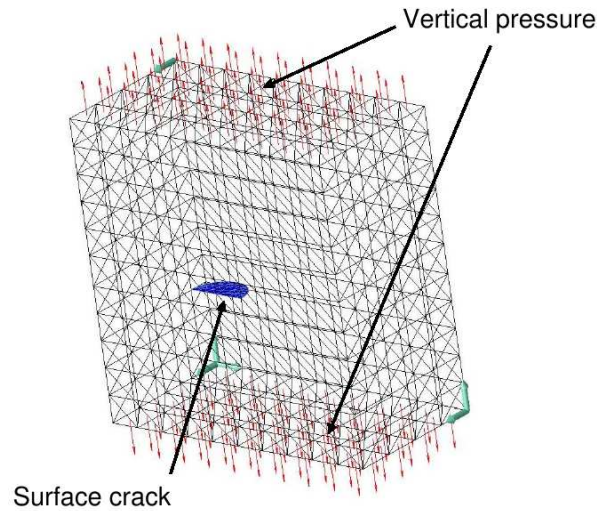


Figure 2: Model problem used to illustrate the main ideas of the $GFEM^{g-1}$. The figure shows a small surface crack in a 3-D domain. The solution computed on this coarse global mesh provides boundary conditions for the extracted local domain in a neighborhood of the crack. *The crack is shown in the global domain for illustration purposes only.* In the proposed $GFEM^{g-1}$, small scale cracks are *not* discretized in the global problem. Instead, global-local enrichment functions are used.

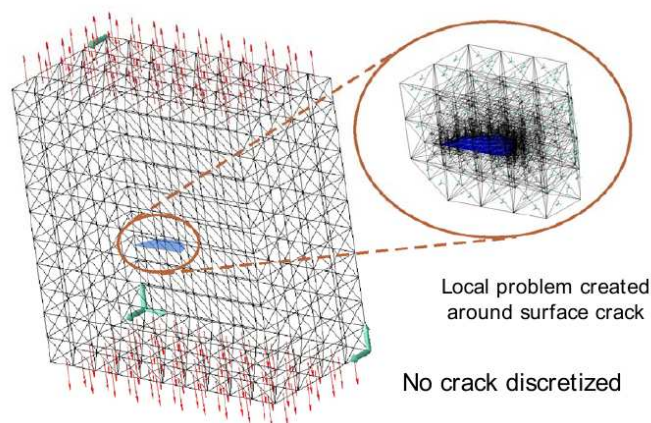


Figure 3: Definition of a local problem around a crack using elements extracted from the coarse global discretization.

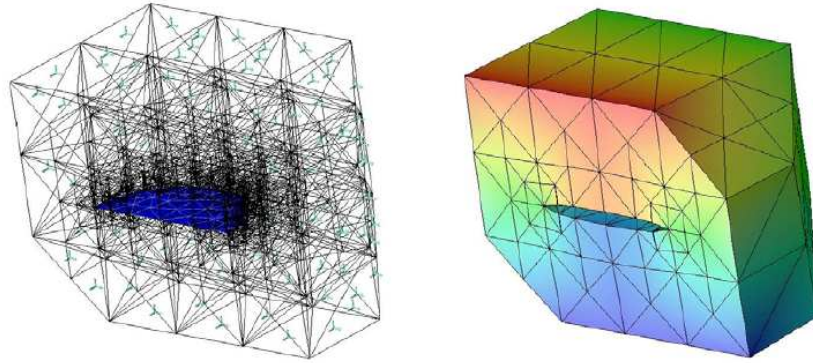


Figure 4: H_p adapted local discretization used for the computation of the non-linear solution \mathbf{u}_L of a local problem.

and the enrichment function given by the solution of the local problem, i.e.,

$$\phi_\alpha(\mathbf{x}) = \varphi_\alpha(\mathbf{x})\mathbf{u}_L \quad (6)$$

Hearafter, \mathbf{u}_L is denoted a *global-local enrichment function*.

Only a few degrees of freedom are added to the global (structural scale) discretization even if the computation of the local solution requires several thousands of degrees of freedom since \mathbf{u}_L is a known function. This is in contrast with, e.g., the S-FEM [30, 31] which adds a large number of enrichments to a global coarse mesh in order to capture small scale behavior in the global domain. The global problem is solved on the *coarse* global mesh enriched with the shape functions defined in (6). Like in the case of the local problem, a non-linear material law is used in the computation of this so-called *enriched global problem*. These shape functions (6) are hierarchically added to the FE discretization, and thus, a few entries are added to element matrices while keeping existing ones associated with standard FE shape functions. Figure 5 illustrates the enrichment of the global coarse mesh with the solution of a local problem defined in a neighborhood of a crack.

3 Plasticity Model

In order to accounting for irreversible response and hardening effects of the medium, a classical rate-independent plasticity model suitable for three-dimensional analysis in the global-local GFEM framework was considered. A detailed description of it can be found in Simo and Hughes [32].

The main features of the adopted model are: von Mises yield function, isotropic hardening and associativity assumption assumed for both hardening law and normality rule for plastic flow.

Being C the isotropic elastic constitutive tensor, S the deviatoric part of the stress tensor, σ_t the yield stress and $K(\alpha)$ the hardening function, the local governing equations of the model are:

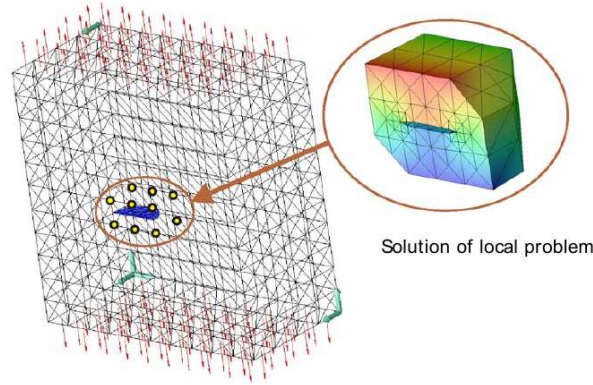


Figure 5: Enrichment of the coarse global mesh with a local solution. Only three degrees of freedom are added to nodes with yellow glyphs. The crack is shown in the global domain for illustration purposes only.

$$\sigma = C(\varepsilon - \varepsilon^p) \quad (\text{linear elastic stress-strain relationship}) \quad (7)$$

$$f = \|S\| - \sqrt{2/3} K(\alpha) \quad (\text{yield criterion}) \quad (8)$$

$$K(\alpha) = \sigma_t + h\alpha + (\sigma_\infty - \sigma_t) [1 - \exp(-\omega\alpha)] \quad (\text{nonlinear hardening function}) \quad (9)$$

$$\dot{\varepsilon}^p = \dot{\lambda} \frac{\partial f}{\partial \sigma} = \dot{\lambda} n; \quad n = \frac{S}{\|S\|} \quad (\text{flow rule}) \quad (10)$$

$$\dot{\alpha} = \dot{\lambda} \sqrt{\frac{2}{3}} \quad (\text{hardening rule}) \quad (11)$$

$$\dot{\lambda} \geq 0, \quad f(\sigma, \alpha) \leq 0, \quad f \dot{\lambda} = 0, \quad \dot{f} \dot{\lambda} = 0 \quad (\text{Kuhn-Tucker and consistency conditions}) \quad (12)$$

It must be noted that the hardening law adopted, Düster [33], combines a linear and an exponential function, being h the linear hardening parameter, ω the hardening exponent and σ_∞ the saturation stress.

An implicit-type returning mapping algorithm (essentially, the 'trial' elastic stress vector, 'predictor', is projected onto the yield surface, 'plastic corrector') was carried out in order to integrate the incremental form of the constitutive model. Moreover, a consistent tangent elastoplastic constitutive tensor was adopted when using a Newton-Raphson iterative and incremental process aiming to compute a sequence of equilibrated and constitutive compatible responses at the solid structural analysis level.

4 Numerical Example

We perform numerical experiment to show the effectiveness of the proposed $GFEM^{g-1}$ in this section. Numerical examples are analyzed by both the GFEM and $GFEM^{g-1}$ and the result obtained by the $GFEM^{g-1}$ is compared with that obtained by the GFEM to show the accuracy of the $GFEM^{g-1}$ solution and its computational cost.

In the analysis using the $GFEM^{g-1}$, no crack is discretized in the global domain as discussed in Section 2.2. A single local problem is defined around a crack in the domain. Plastic strain is assumed to be confined to the interior of the local domain. This justifies the use of elastic material law in the initial global problem.

4.1 Small surface crack

As an example to demonstrate the effectiveness of the $GFEM^{g-1}$, we analyze a small surface crack example introduced in Section 2.2. Figure 6 shows the geometry of this model problem and applied tension at the top and bottom of the domain. This problem has a small half-circular surface breaking crack at the center of the front face. The following geometrical and loading parameters are assumed: in-plane dimensions $2b = 2.0$, $2h = 2.0$, $2r = 0.4$; domain thickness $t = 1.0$; vertical traction $t_y = 115.0$ (MPa). The domain is discretized with a uniform mesh of $6^*(10 \times 11 \times 4)$ tetrahedral elements. Figure 2 shows this mesh.

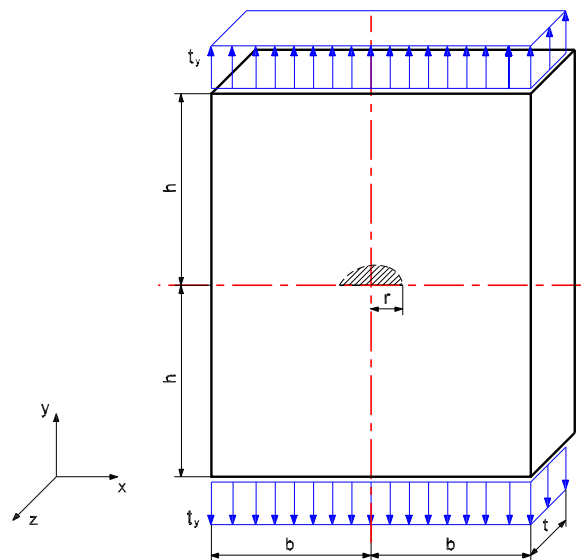


Figure 6: Domain with a small single surface crack.

Linear isotropic hardening model is used by including only a linear function from Equation (9) ($\sigma_\infty = \sigma_t$) and material properties are listed in Table 1. Newton-Rhapson method is used with load control to obtain nonlinear solutions since globally hardening behavior is expected in this analysis. We use the relative norm of residual as a tolerance criterion and the tolerance is given as 0.0001. Eight loading steps are provided for this nonlinear analysis. As a measure for accuracy of the solution, we choose the crack mouth opening displacement (CMOD), which is frequently used in fracture analysis. The CMOD is measured at the center of the surface crack boundary on the front face.

Table 1: Material data of alluminum alloy.

Young's modulus (GPa)	Poisson's ratio	σ_Y (MPa)	K (MPa)
71	0.33	135.3	576.79

Cubic shape functions are used in the global and local domains for both analyses using the GFEM and $GFEM^{g-1}$. The surface crack is discretized by Heavyside and singular Westergaard functions obtained from two-dimensional linear elastic fracture mechanics solutions. Although singular Westergaard functions do not describe nonlinear stress distribution around the crack front accurately, they can at least describe a crack opening in the element partially cut by the crack surface. In the global problem enriched with nonlinear local solutions, the surface crack is represented by the local solution.

Figure 7 shows the nonlinear load-displacement history obtained by the GFEM and $GFEM^{g-1}$. The linear elastic GFEM solution is also included in the figure for comparison purpose. Nonlinear solutions deviate from the linear elastic solution as loading increases. The solution obtained by the $GFEM^{g-1}$ exhibits a good agreement with the one by GFEM. The small differences between the nonlinear GFEM and $GFEM^{g-1}$ solutions at intermediate loading steps are expected because the nonlinear local solution enrichment is customized for the final loading step as discussed in Section 2.2. Figure 8 illustrates the deformed shape of the domain with a small surface crack obtained by the GFEM and $GFEM^{g-1}$ at the final loading step. The color in the figure represents the vertical displacement distribution. It demonstrates that the $GFEM^{g-1}$ can achieve almost the same level of accuracy as the GFEM even though much coarser global mesh is employed and linear elastic material law is used in the initial global problem to provide boundary conditions to local problems.

Table 2 lists the number of degrees of freedom used in the global and local problem and the number of iterations taken at each loading step. Abbreviation G and L in the table represent global and local problem, respectively. This result sheds some light on the possible cost reduction that can be achieved by using the $GFEM^{g-1}$. In the analysis by the $GFEM^{g-1}$, 19,836 and 18,744 degrees of freedom are used in the global and local problem respectively, while GFEM requires 37,104 degrees of freedom. The table indicates that the total numbers of iterations required for the two approaches are 19 and 20, which are almost the same. Thus, the $GFEM^{g-1}$ requires less amount of computational cost than GFEM and the difference in the cost between the two approaches would be more significant as more local problems are created in the domain.

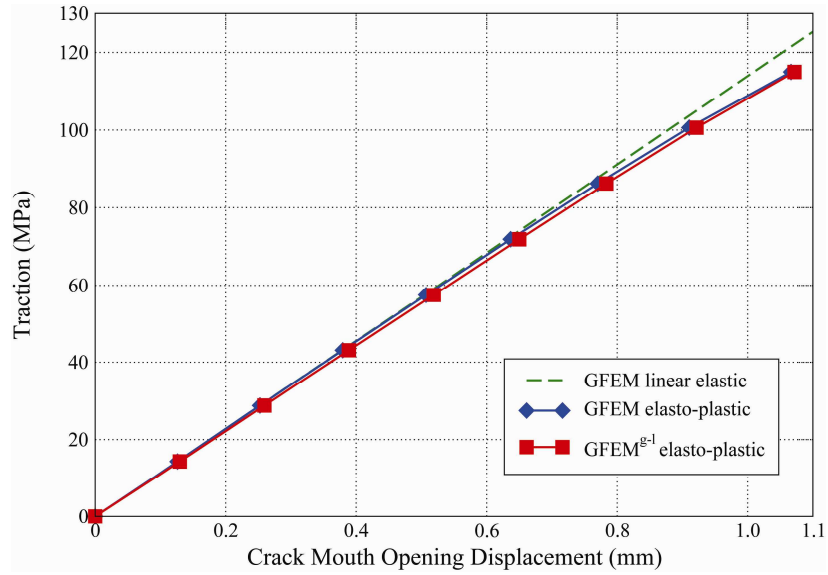


Figure 7: Nonlinear load-displacement history of the surface crack example.

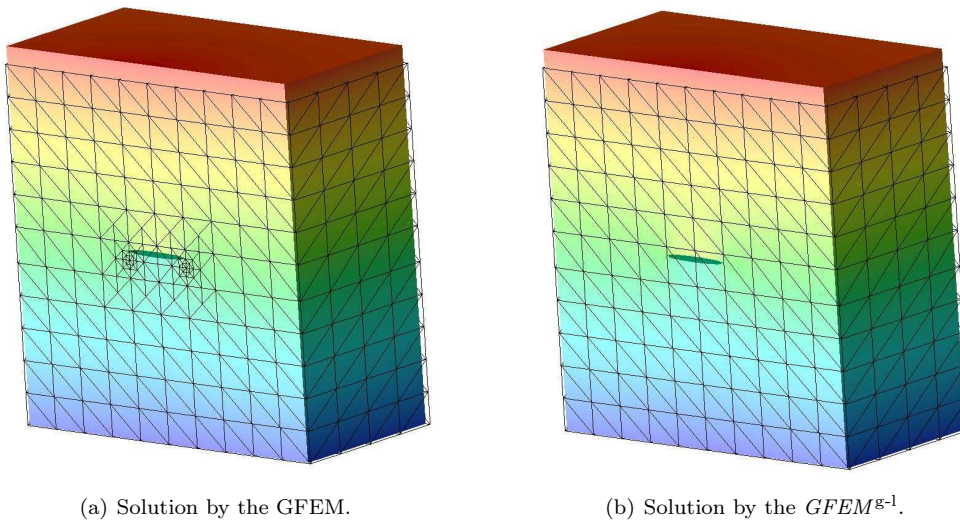


Figure 8: Deformed shape of the domain with a small surface crack.

Table 2: Number of iterations required at each loading step.

Load step	Number of degrees of freedom			Number of iterations		
	GFEM (G)	$GFEM^{g-1}$ (L)	$GFEM^{g-1}$ (G)	GFEM (G)	$GFEM^{g-1}$ (L)	$GFEM^{g-1}$ (G)
1				1		1
2				2		1
3				2		2
4	37,104	18,744	19.836	2	4	2
5				3		2
6				3		2
7				3		3
8				3		3
			Total	19	4	16

5 Conclusions

In this paper, we applied the GFEM with global-local enrichment function presented in [3, 25] to analyze nonlinear fracture mechanics problems based on J_2 plasticity theory. Our focus is on the problems where nonlinear behavior is confined to a small region of the domain and this class of problems can be very efficiently solved by the $GFEM^{g-1}$. The main conclusions of the analysis presented in this paper can be summarized as follows:

- The $GFEM^{g-1}$ allows the numerical construction of enrichment functions for problems exhibiting local nonlinear behavior where no a priori knowledge is known about their solutions, thus it brings the benefits of the existing generalized FEM to a broader class of problems.
- The $GFEM^{g-1}$ allows no crack discretization in the global domain while keeping the accuracy of solutions. A single coarse mesh can be used for any complex crack configuration.
- The numerical examples presented in Section 4 show that the $GFEM^{g-1}$ can provide nonlinear solutions almost as accurate as those obtained by the GFEM with reduced amount of computational cost.

References

- [1] Oden, J.T. & Duarte, C.A.M., Chapter: Solution of singular problems using hp clouds. *The Mathematics of Finite Elements and Applications– Highlights 1996*, ed. J.R. Whiteman, John Wiley & Sons: New York, NY, pp. 35–54, 1997.
- [2] Sukumar, N., Chopp, D., Moës, N. & Belytschko, T., Modeling holes and inclusions by level sets in the extended finite element method. *Computer Methods in Applied Mechanics and Engineering*, **190**, pp. 6183–6200, 2001.

- [3] Duarte, C.A. & Kim, D.J., Analysis and applications of a generalized finite element method with global-local enrichment functions. *197*, (6-8), pp. 487–504, 2008. [Http://dx.doi.org/10.1016/j.cma.2007.08.017](http://dx.doi.org/10.1016/j.cma.2007.08.017).
- [4] Duarte, C.A. & Babuška, I., A global-local approach for the construction of enrichment functions for the generalized fem and its application to propagating three-dimensional cracks. *ECCOMAS Thematic Conference on Meshless Methods*, eds. V.M.A. Leitão, C.J.S. Alves & C.A. Duarte, Lisbon, Portugal, 2005. 8 pages.
- [5] Tanaka, K., Mechanics and micromechanics of fatigue crack propagation. *Fracture Mechanics: Perspectives and Directions (Twentieth Symposium)*, eds. R.P. Wei & R.P. Gangloff, American Society for Testing and Materials: Philadelphia, ASTM STP 1020, pp. 151–183, 1989.
- [6] Babuška, I. & Melenk, J.M., The partition of unity finite element method. *International Journal for Numerical Methods in Engineering*, **40**, pp. 727–758, 1997.
- [7] Duarte, C.A., Babuška, I. & Oden, J.T., Generalized finite element methods for three dimensional structural mechanics problems. *Computers and Structures*, **77**, pp. 215–232, 2000.
- [8] Oden, J.T., Duarte, C.A. & Zienkiewicz, O.C., A new cloud-based *hp* finite element method. *153*, pp. 117–126, 1998.
- [9] Strouboulis, T., Copps, K. & Babuška, I., The generalized finite element method. *Computer Methods in Applied Mechanics and Engineering*, **190**, pp. 4081–4193, 2001.
- [10] Babuška, I., Caloz, G. & Osborn, J.E., Special finite element methods for a class of second order elliptic problems with rough coefficients. *SIAM Journal on Numerical Analysis*, **31(4)**, pp. 745–981, 1994.
- [11] Melenk, J.M. & Babuška, I., The partition of unity finite element method: Basic theory and applications. *Computer Methods in Applied Mechanics and Engineering*, **139**, pp. 289–314, 1996.
- [12] Duarte, C.A.M. & Oden, J.T., *Hp* clouds—A meshless method to solve boundary-value problems. Technical Report 95-05, TICAM, The University of Texas at Austin, 1995.
- [13] Duarte, C.A.M. & Oden, J.T., An *hp* adaptive method using clouds. *1996*, **139**, pp. 237–262.
- [14] Duarte, C.A.M. & Oden, J.T., *Hp* clouds—An *hp* meshless method. *1996*, **12**, pp. 673–705.
- [15] Duarte, C.A., *The hp Cloud Method*. PhD dissertation, The University of Texas at Austin, 1996. Austin, TX, USA.
- [16] Moës, N., Dolbow, J. & Belytschko, T., A finite element method for crack growth without remeshing. *International Journal for Numerical Methods in Engineering*, **46**, pp. 131–150, 1999.
- [17] De, S. & Bathe, K.J., The method of finite spheres. *Computational Mechanics*, **25**, pp. 329–345, 2000.
- [18] Griebel, M. & Schweitzer, M.A., A particle-partition of unity method for the solution of elliptic, parabolic and hyperbolic PDEs. *SIAM Journal on Scientific Computing*, **22(3)**, pp. 853–890, 2000.
- [19] Duarte, C.A. & Babuška, I., Mesh-independent directional *p*-enrichment using the generalized finite element method. *2002*, **55(12)**, pp. 1477–1492. [Http://dx.doi.org/10.1002/nme.557](http://dx.doi.org/10.1002/nme.557).
- [20] Duarte, C.A., Hamzeh, O.N., Liszka, T.J. & Tworzydło, W.W., A generalized finite element method for the simulation of three-dimensional dynamic crack propagation. *2001*, **190(15-17)**, pp. 2227–2262. [Http://dx.doi.org/10.1016/S0045-7825\(00\)00233-4](http://dx.doi.org/10.1016/S0045-7825(00)00233-4).
- [21] Babuška, I., Ihlenburg, F., Paik, E. & Sauter, S., A generalized finite element method for solving the Helmholtz equation in two dimensions with minimal pollution. *Computer Methods in Applied Mechanics and Engineering*, **128(3-4)**, pp. 325–360, 1995.
- [22] Melenk, J.M., *On Generalized Finite Element Methods*. Ph.D. thesis, The University of Maryland, 1995.
- [23] Simone, A., Duarte, C.A. & van der Giessen, E., A generalized finite element method for polycrystals with discontinuous grain boundaries. *2006*, **67(8)**, pp. 1122–1145. [Http://dx.doi.org/10.1002/nme.1658](http://dx.doi.org/10.1002/nme.1658).
- [24] Strouboulis, T., Copps, K. & Babuška, I., The generalized finite element method: An example of its

- implementation and illustration of its performance. *International Journal for Numerical Methods in Engineering*, **47(8)**, pp. 1401–1417, 2000.
- [25] Duarte, C.A., Kim, D.J. & Babuška, I., Chapter: A global-local approach for the construction of enrichment functions for the generalized fem and its application to three-dimensional cracks. *Advances in Mesh-free Techniques*, eds. V.M.A. Leitão, C.J.S. Alves & C.A. Duarte, Springer: The Netherlands, volume 5 of *Computational Methods in Applied Sciences*, 2007. ISBN 978-1-4020-6094-6.
- [26] Babuška, I., Banerjee, U. & Osborn, J.E., Survey of meshless and generalized finite element methods: A unified approach. *Acta Numerica*, **12**, pp. 1–125, 2003.
- [27] Pereira, J.P., Duarte, C.A., Guoy, D. & Jiao, X., *Hp*-Generalized FEM and crack surface representation for non-planar 3-D cracks. 2008. [Http://dx.doi.org/10.1002/nme.2419](http://dx.doi.org/10.1002/nme.2419).
- [28] Pereira, J.P., Duarte, C.A., Jiao, X. & Guoy, D., Generalized finite element method enrichment functions for curved singularities in 3D fracture mechanics problems. Submitted.
- [29] Felippa, C.A., Introduction to finite element methods., 2004. Course Notes. Department of Aerospace Engineering Sciences, University of Colorado at Boulder. Available at <http://www.colorado.edu/engineering/Aerospace/CAS/courses.d/IFEM.d>.
- [30] Fish, J., The s-version of the finite element method. *Computers and Structures*, **43**, pp. 539–547, 1992.
- [31] Park, J.W., Hwang, J.W. & Kim, Y.H., Efficient finite element analysis using mesh superposition technique. *Finite Elements in Analysis and Design*, **39**, pp. 619–638, 2003.
- [32] Simo, C., J & Hughes, T.J., *Computational Inelasticity*. Prentice-Hall, 1998.
- [33] Düster, A., *High order finite elements for three-dimensional thin walled nonlinear continua*. Ph.D. thesis, Technische Universität München, 2001.