# Analytical procedure for stress field solution in concrete gravity dams

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## Abstract

Analytical solutions are of great interest to designers of concrete dams. Even though it is possible to solve this problem with numerical methods, it is undeniable the contribution of a totally analytical interior stress solution of dams. The easiness of computational planning and interpretation of this procedure, besides the speed and quality of the results obtained when compared to numerical solutions, justify the use of this method for preliminary stress analysis in concrete gravity dams.

The analytical proposed development is based on classical formulations of solid mechanics and equilibrium of cross sections along the structure – assimilated to a deep beam – for a 2-D plane strain problem. The seismic loads associated to inertia and hydrodynamic forces are evaluated by a Pseudo-Static procedure, and are applied as static equivalent forces, which makes easier the analytical solution of this problem.

The procedure development will be presented in this work, besides several examples and applications obtained through a computational tool developed by the Dynamics and Fluid-Structure Interaction Group (DFSG) at University of Brasilia. The results show a great potential of this solution applied to solid mechanics problems in engineering of dams, and indicates good results when compared to formulations and design examples obtained from USBR (United States Bureau of Reclamation).

Keywords: analytical, concrete, dams, seismic, stress

## 1 Introduction

It is well known that a closed form stress field solution for concrete gravity dams is a challenge to structural engineers. The arbitrary shape and loads applied in this geometry turns the design into a multi-variable task. Numerical solutions for this problem are available, but it is undeniable that an analytical procedure is of great interest to structural designers.

The Gravity Method [1] is a solution proposed by the United States Bureau of Reclamation (USBR) for design of concrete gravity dams. In-plane stresses are results of this procedure. They can be easily combined for principal stress analysis at any interior point of the given geometry. It is a great design

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tool, but the mid-steps of this procedure are not very clear in literature, and that implies that the valid boundaries of this solution can be become a problem to designers.

Past works developed by the DFSG have made important achievements in an attempt to solve analytically this problem [2–4]. A complete rebuild of this method was proposed by Ribeiro [5]. It has been shown that final formulations, including Static and Pseudo-Static procedures, are identical to the original ones given by USBR. A forward step, including a new type of analysis is now available, and it gives this method additional seismic solution, including a modified pseudo-dynamic approach for stress field determination. A computer code called SAGDAM (Stress Analysis of Gravity Dams) developed by the DFSG is able to solve the stress field including both static and seismic actions in a dam.

The easiness of computational planning of this procedure and quality of the results obtained when compared to finite element method models makes this solution a very powerful tool for initial design of concrete gravity dams. The step-by-step procedure and some practical examples are presented in this work.

## 2 Exterior actions in a dam

Dams are usually subjected, in normal operation conditions, to three major exterior actions: self-weight, hydrostatic and uplift pressures (Figure 1). During an earthquake seismic loads must be also taken in account, and these include the hydrodynamic and inertia effects (estimated under a specified level of seismic analysis; Pseudo-Static in example).

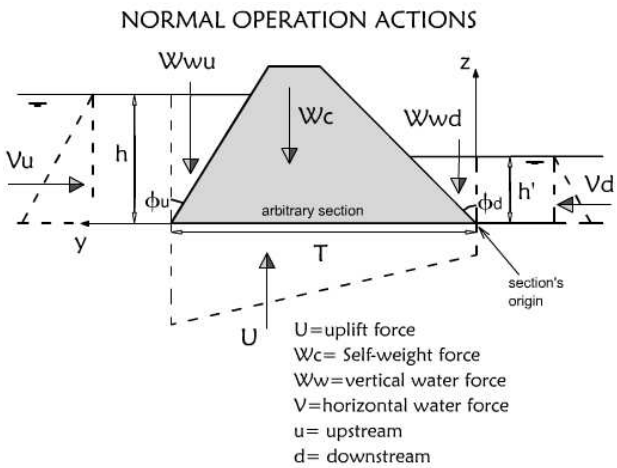


Figure 1: Arbitrary section over normal operation actions (static analysis)

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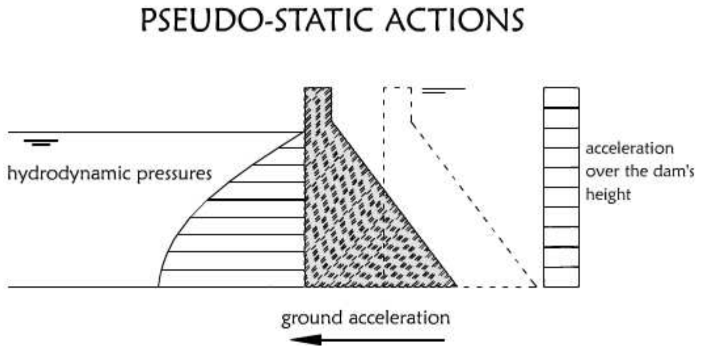


Figure 2: Pseudo-Static actions in a dam (seismic analysis)

## 3 Analytical equations (normal operation actions)

Stresses formulations on arbitrary sections are listed in the next items. This analytical development is based on classical Theory of Elasticity 2D, under some simplified hypotheses, and the idea of global equilibrium of forces used in Strength of Materials.

### 3.1 Simplified hypotheses

The stress field solution is based on the following properties:

1. Linear stress distribution for vertical normal and parabolic shear stress distributions over the section in analysis;
2. The dam is built of a homogeneous, isotropic and linear elastic concrete;
3. All applied loads are transferred to foundation under beam action. This means that the dam is analyzed as a cantilever beam with unitary width;
4. The dam's body is divided in concrete joints that have uniform properties along its length, and are uniformly elastic;
5. There is no interaction between adjacent joints. Each one is treated independently.

### 3.2 Vertical normal stress formulation ($\sigma_Z$)

From classical beam theory, assuming that normal stresses are linear distributed over a section, it is shown that:

$$\sigma_Z = \frac{\Sigma W}{A} + \frac{\Sigma M \cdot y}{I} \tag{1}$$

Equation (1) can be rearranged to a coordinate system fixed at the dam's downstream position ($y$). That implies on the following equation:

$$\sigma_Z\left(y\right) = \left(\frac{\Sigma W}{T} - \frac{6 \cdot \Sigma M}{T^2}\right) + \left(\frac{12 \cdot \Sigma M}{T^3}\right) \cdot y \tag{2}$$

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where:
$\Sigma W$ = sum of all vertical forces over an analyzed section;
$\Sigma M$ = sum of all moments over an analyzed section;
$T$ = length of the section being analyzed;
$y$ = distance measured from section's downstream side;

Figure 3 illustrates the signal convention for stresses and forces over a section.

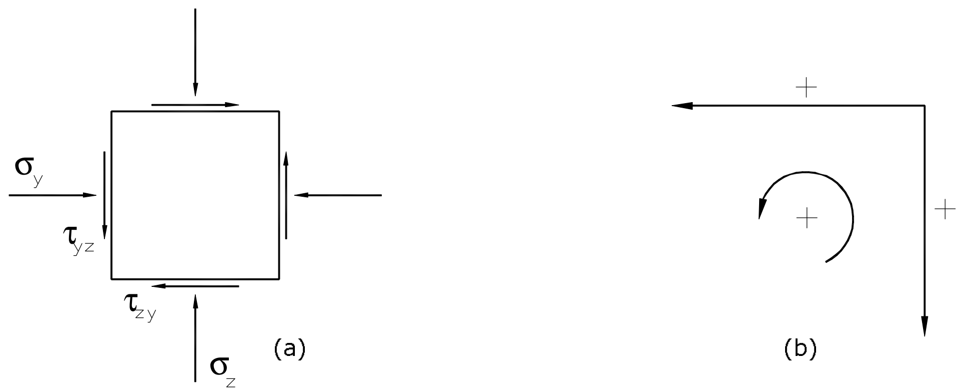


Figure 3: Positive stresses (a) and forces (b) over a section

### 3.3 Shear stress formulation ($\tau_{YZ}$ or $\tau_{ZY}$)

It is assumed that shear stresses follow a second degree parabolic equation:

$$\tau_{ZY}(y) = a_1 + b_1 \cdot y + c_1 \cdot y^2 \tag{3}$$

where $a_1$,$b_1$ and $c_1$ are specific constants for every section in analysis.

Three boundary conditions are needed for this problem. Two of them can be obtained at the dam's upstream and downstream sides. The last one is given by the shear force acting on a section. The solution of these constants leads to the following final equation for shear stresses:

$$\tau_{ZY}\,(y) = \tau_{ZYD} - \frac{1}{T} \cdot \left[ \frac{6 \cdot \Sigma V}{T} + 2 \cdot \tau_{ZYU} + 4 \cdot \tau_{ZYD} \right] \cdot y + \frac{1}{T^2} \cdot \left[ \frac{6 \cdot \Sigma V}{T} + 3 \cdot \tau_{ZYD} \cdot +3 \cdot \tau_{ZYU} \right] \cdot y^2 \tag{4}$$

where:
$\Sigma V$ = sum of all horizontal forces over an analyzed section;
$\tau_{ZYU}$ = shear stress at dam's upstream side, given by Equation (5);

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$\tau_{ZYD}$ = shear stress at dam's downstream side, given by Equation (6);

$$\tau_{ZYU} = -[\sigma_{ZU} - p] \cdot \tan(\phi_U) \tag{5}$$

$$\tau_{ZYD} = [\sigma_{ZD} - p'] \cdot \tan(\phi_D) \tag{6}$$

$\sigma_{ZU}$ = vertical normal stress at dam's upstream side ($y = T$), given by Equation (2);
$\sigma_{ZD}$ = vertical normal stress at dam's downstream side ($y = 0$), given by Equation (2);
$p$ or $p'$ = hydrostatic pressure at section's upstream or downstream side;
$\phi_U$ or $\phi_D$ = angle between upstream or downstream side and a vertical line.

### 3.4 Horizontal normal stress formulation ($\sigma_Y$)

Unlike $\sigma_Z$ and $\tau_{ZY}$, the horizontal normal stress formulation is not a hypothesis of this procedure. It is the result of element equilibrium of forces (as shown on Figure 4; using previously defined equations), and it is the most difficult to obtain analytically. Solution of this problem provides a third degree parabolic equation:

$$\sigma_Y(y) = a_2 + b_2 y + c_2 y^2 + d_2 y^3 \tag{7}$$

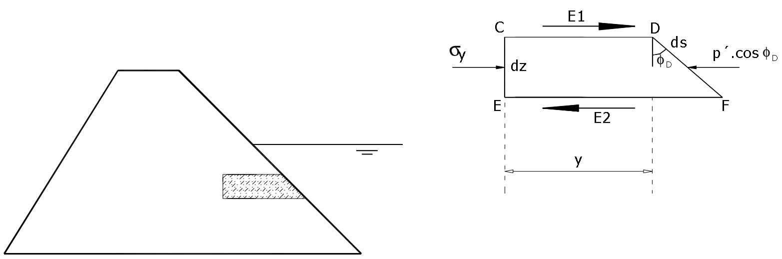


Figure 4: Element equilibrium at dam's downstream side

The solution of these constants leads to the following final equation for horizontal normal stresses [5]:

$$\sigma_y = (a_1 \cdot \tan(\phi_D) + p') + \left(b_1 \cdot \tan(\phi_D) + \frac{\partial a_1}{\partial z}\right) \cdot y + \left(c_1 \cdot \tan(\phi_D) + \frac{1}{2} \cdot \frac{\partial b_1}{\partial z}\right) \cdot y^2 + \left(\frac{1}{3} \cdot \frac{\partial c_1}{\partial z}\right) \cdot y^3 \tag{8}$$

where the partial derivatives in $z$ are given by Equations (9), (10) and (11).

$$\partial a_1/\partial z = \frac{\partial \sigma_{ZD}}{\partial z} \cdot \tan(\phi_D) - \omega^{\otimes} \cdot \tan(\phi_D) \tag{9}$$

$\omega^{\otimes}$ = unitary water weight; omitted if $p' = 0$;

$$\frac{\partial \sigma_{ZD}}{\partial z} = \omega_c + \tan(\phi_U) \cdot \left(\frac{12\Sigma M}{T^3} + \frac{2\Sigma W}{T^2} - \frac{2 \cdot p}{T}\right) + \tan(\phi_D) \cdot \left(\frac{12\Sigma M}{T^3} - \frac{4\Sigma W}{T^2} + \frac{4 \cdot p'}{T}\right) - \frac{6\Sigma V}{T^2}$$

$\omega_c$ = unitary concrete weight;

$$\frac{\partial b_1}{\partial z} = -\frac{1}{T^2} \cdot \left[6 \cdot \left(\frac{\partial \Sigma V}{\partial z}\right) - \frac{\partial T}{\partial z} \cdot \left(\frac{12 \cdot \Sigma V}{T} + 2 \cdot \tau_{ZYU} + 4 \cdot \tau_{ZYD}\right)\right] - \frac{1}{T} \cdot \left[2 \cdot \left(\frac{\partial \tau_{ZYU}}{\partial z}\right) + 4 \cdot \left(\frac{\partial \tau_{ZYD}}{\partial z}\right)\right] \quad (10)$$

$$\frac{\partial \Sigma V}{\partial z} = p' - p$$

$$\frac{\partial T}{\partial z} = \tan(\phi_U) + \tan(\phi_D)$$

$$\frac{\partial \tau_{ZYD}}{\partial z} = \frac{\partial a_1}{\partial z}$$

$$\frac{\partial \tau_{ZYU}}{\partial z} = -\frac{\partial \sigma_{ZU}}{\partial z} \cdot \tan(\phi_U) + \omega^{\otimes} \cdot \tan(\phi_U)$$

$\omega^{\otimes}$ = unitary water weight; omitted if $p = 0$;

$$\frac{\partial \sigma_{ZU}}{\partial z} = \omega_c + \tan(\phi_U) \cdot \left(\frac{4 \cdot p}{T} - \frac{4\Sigma W}{T^2} - \frac{12\Sigma M}{T^3}\right) + \tan(\phi_D) \cdot \left(\frac{2\Sigma W}{T^2} - \frac{2 \cdot p'}{T} - \frac{12\Sigma M}{T^3}\right) + \frac{6\Sigma V}{T^2}$$

$$\frac{\partial c_1}{\partial z} = \frac{1}{T^3} \cdot \left[6 \cdot \left(\frac{\partial \Sigma V}{\partial z}\right) - \frac{\partial T}{\partial z} \cdot \left(\frac{18 \cdot \Sigma V}{T} + 6 \cdot \tau_{ZYU} + 6 \cdot \tau_{ZYD}\right)\right] + \frac{1}{T^2} \cdot \left[3 \cdot \left(\frac{\partial \tau_{ZYU}}{\partial z}\right) + 3 \cdot \left(\frac{\partial \tau_{ZYD}}{\partial z}\right)\right] \quad (11)$$

Equations (2) and (4), combined with (7) and its constants, gives the stress field solution for a dam under normal operation condition. That is: under self-weight plus hydrostatic pressures acting on upstream and downstream sides. Uplift pressures contribution is not included.

## 4 Analytical equations (pseudo-static actions)

Comments on seismic action of inertia and hydrodynamic effects using pseudo-static approach on previous formulations are listed in the next items.

### 4.1 Procedure hypotheses

The pseudo-static approach for analysis of dams follows the following hypotheses:

1. Rigid dam movement (uniformly accelerated over its height);
2. Incompressible fluid.

### 4.2 Additional forces

In this procedure additional forces appear due to inertia and hydrodynamic effects on a rigid body movement in an incompressible fluid. Inertia contribution is given by $I_F = m \cdot v_H$ , where $m$ is the structural mass over an analyzed section, and $v_H$ is the rigid body acceleration. Hydrodynamic effects are calculated though a simplified Westergaard [6] formula, given by Equation (12).

$$p_e\,(h) = \frac{0,543}{0,583} \cdot \frac{7}{8} \cdot \gamma \cdot V_g \cdot \sqrt{H \cdot h} \tag{12}$$

where:

| | | |
|---|---|---|
| $p_e$ | = | hydrodynamic pressure along the fluid-structure interface; |
| $\gamma$ | = | unitary fluid weight; |
| $V_g$ | = | horizontal seismic acceleration in terms of gravity acceleration $(v_H/g)$; |
| $H$ | = | reservoir height; |
| $h$ | = | vertical distance between analyzed section and reservoir surface. |

Hydrodynamic pressure distribution is illustrated on Figure 2.
Equation's (12) integral over $h$ provides the hydrodynamic force over an analyzed section.

$$F_{HD} = \int_0^h \frac{0,543}{0,583} \cdot \frac{7}{8} \cdot V_g \cdot \gamma \cdot \sqrt{H \cdot h}\; dh = \frac{2}{3} \cdot \frac{0,543}{0,583} \cdot \frac{7}{8} V_g \cdot \gamma \cdot \sqrt{H} \cdot h^{1,5} \tag{13}$$

### 4.3 Vertical normal stress formulation ($\sigma_Z$)

Equation (2) remains the same in this analysis. But moments and forces are increased by additional inertia and hydrodynamic effects.

### 4.4 Shear stress formulation ($\tau_{YZ}$ or $\tau_{ZY}$)

Equation (4) remains valid, but $\tau_{ZYU}$ and $\tau_{ZYD}$ are modified by the presence of hydrodynamic pressure terms ($p_e$ and $p'_e$) on its equations (Figure 5):

$$\tau_{YZU} = \tau_{ZYU} = -[\sigma_{ZU} - p \pm \dagger\dagger p_e] \cdot \tan\,(\phi_U) \tag{14}$$

$$\tau_{YZD} = \tau_{ZYD} = [\sigma_{ZD} - p' \pm \dagger p'_e] \cdot \tan\,(\phi_D) \tag{15}$$

where:
$p_e$ = upstream hydrodynamic pressure on a section, given by Westergaard formula;
$p'_e$ = downstream hydrodynamic pressure on a section, given by Westergaard;
†† = positive for downstream acceleration, otherwise negative;
† = positive for upstream acceleration, otherwise negative.

Forces and moments are increased by additional inertia and hydrodynamic effects.

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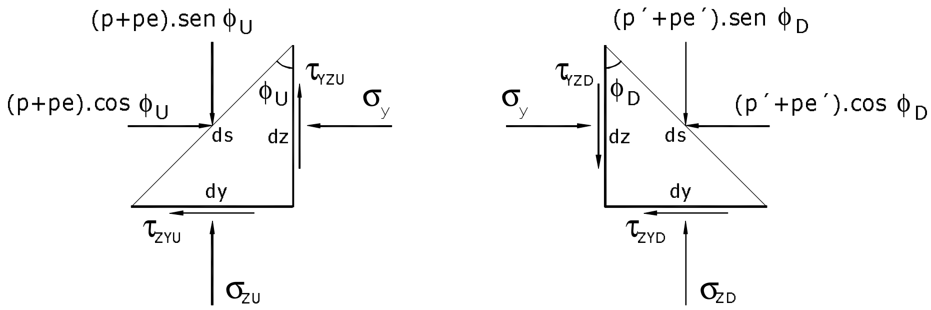


Figure 5: Element equilibrium at upstream and downstream sides

### 4.5 Horizontal normal stress formulation ($\sigma_Y$)

Element equilibrium shown on Figure 5 receives two additional components: inertia and hydrodynamic forces. Constants $a_2$ and $b_2$ suffer the following modifications:

$$a_2 = a_1 \tan(\phi_D) + p' \pm \dagger\dagger p'_e \tag{16}$$

$$b_2 = b_1 \tan(\phi_D) + \partial a_1/\partial z \pm \dagger\dagger\lambda \cdot \omega_c \tag{17}$$

where:
$\lambda = v_h/g =$ horizontal acceleration modulus divided by gravity acceleration;

Constants $c_2$ and $d_2$ do not receive additional terms on its formulations. However the partial derivatives in $z$are affected by seismic action. The first one is given by:

$$\frac{\partial a_1}{\partial z} = \left(\frac{\partial \sigma_{ZD}}{\partial z} - \omega^{\otimes} \pm \dagger \frac{\partial p'_e}{\partial z}\right) \cdot \tan(\phi_D) \tag{18}$$

$$\begin{aligned}\frac{\partial \sigma_{ZD}}{\partial z} =& \omega_c + \tan(\phi_U) \cdot \left(\frac{12\Sigma M}{T^3} + \frac{2\Sigma W}{T^2} - \frac{2 \cdot p}{T} \pm \dagger\dagger \frac{2 \cdot p_e}{T}\right) \\ &+ \tan(\phi_D) \cdot \left(\frac{12\Sigma M}{T^3} - \frac{4\Sigma W}{T^2} + \frac{4 \cdot p'}{T} \pm \dagger\dagger \frac{4 \cdot p'_e}{T}\right) - \frac{6\Sigma V}{T^2}\end{aligned} \tag{19}$$

where $\partial p'_e/\partial z$ is the first hydrodynamic downstream pressure derivative.

Equation (10) remains the same for $\partial b_1/\partial z$, but inside terms suffer the following modifications:

$$\frac{\partial \Sigma V}{\partial z} = -\left(p - p' \pm \dagger\lambda \cdot \omega_c \cdot T \pm \dagger p_e \pm \dagger p'_e\right) \tag{20}$$

$$\frac{\partial \tau_{ZYD}}{\partial z} = \frac{\partial a_1}{\partial z} = \left(\frac{\partial \sigma_{ZD}}{\partial z} - \omega^{\otimes} \pm \dagger \frac{\partial p'_e}{\partial z}\right) \cdot \tan(\phi_D) \tag{21}$$

$$\frac{\partial \tau_{ZYU}}{\partial z} = \left(\omega^{\otimes} - \frac{\partial \sigma_{ZU}}{\partial z} \pm \dagger \frac{\partial p_e}{\partial z}\right) \cdot \tan(\phi_U) \tag{22}$$

$$\begin{aligned}\frac{\partial \sigma_{ZU}}{\partial z} =& \omega_c + \tan(\phi_U) \cdot \left(\frac{4 \cdot p}{T} \pm \dagger \frac{4 \cdot p_e}{T} - \frac{4\Sigma W}{T^2} - \frac{12\Sigma M}{T^3}\right) \\ &+ \tan(\phi_D) \cdot \left(\frac{2\Sigma W}{T^2} \pm \dagger \frac{2 \cdot p'_e}{T} - \frac{2 \cdot p'}{T} - \frac{12\Sigma M}{T^3}\right) + \frac{6\Sigma V}{T^2}\end{aligned} \tag{23}$$

Equation (11) remains the same for $\partial c_1/\partial z$. Modifications on inside terms have already been shown on above equations. It is important to notice that all equations on seismic effects action must include forces and moments with inertia and hydrodynamic contributions. Except when commented, all terms remains the same and have the same meaning as in the static case.

## 5 Analytical procedure valid boundaries

The proposed procedure was compared with results obtained in a finite element model. Valid boundaries of this solution are presented in the next items.

### 5.1 Dam geometry and finite element mesh properties

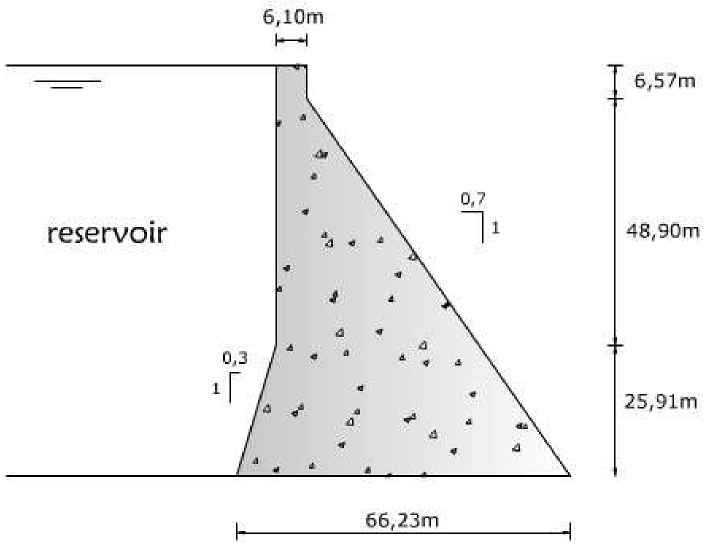


Figure 6: Friant Dam typical section geometry

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Figure 6 illustrates the analyzed dam. It is a real model of American Friant Dam under normal operation actions (except uplift forces). The finite element mesh was built using eight node plane strain quadrilaterals elements (Figure 7). In this analysis the foundation has the same material and element properties as the dam's body.

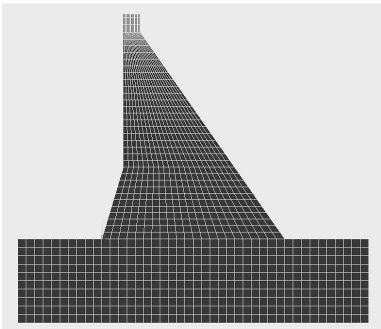

Figure 7: Friant Dam finite element model [7]

## 5.2 Overall comparison of results

Figures 8 and 9 illustrate an overall comparison of principal stress results under normal operation actions (except uplift forces).

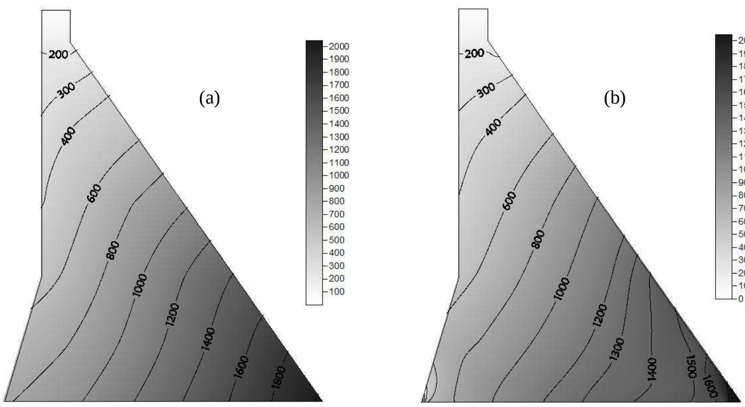


Figure 8: Maximum principal stresses (kPa) obtained analytically (a) and with finite element method (b)

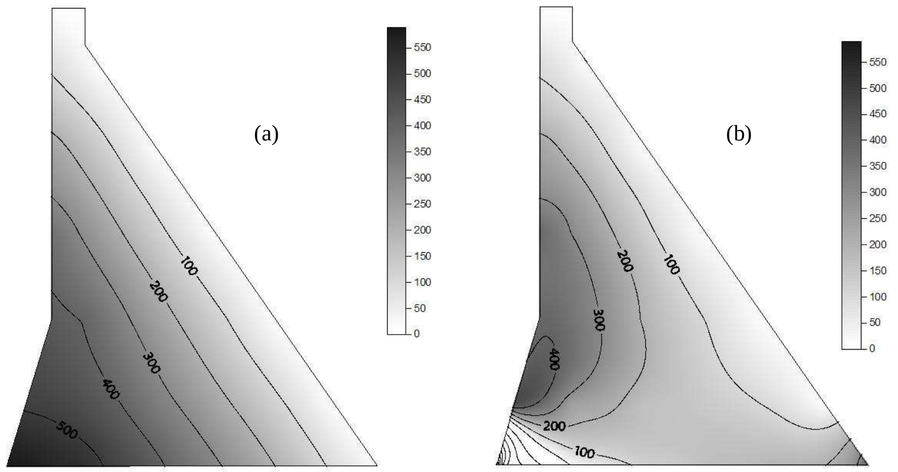


Figure 9: Minimum principal stresses (kPa) obtained analytically (a) and with finite element method (b)

## 5.3 Section results

Section results are presented on Figures 10 through 13.

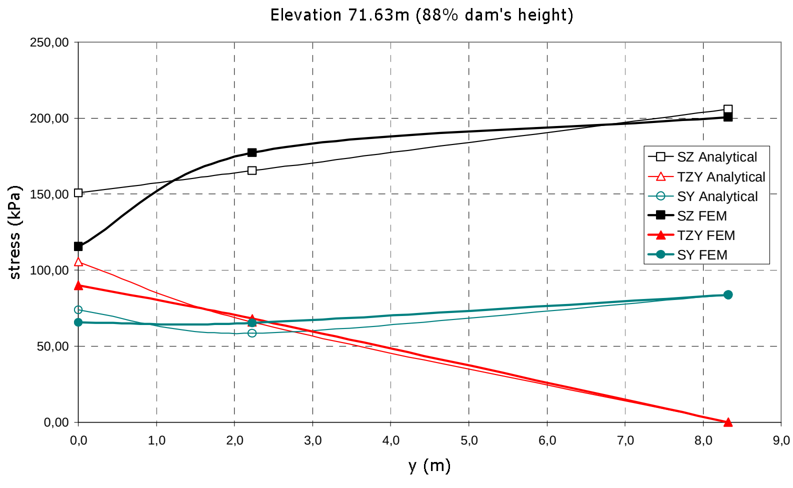


Figure 10: Elevation 71.63m stress distribution

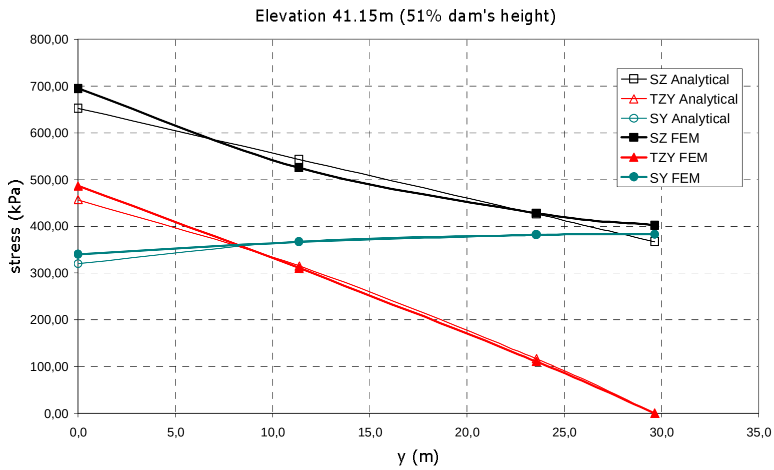


Figure 11: Elevation 41.15m stress distribution

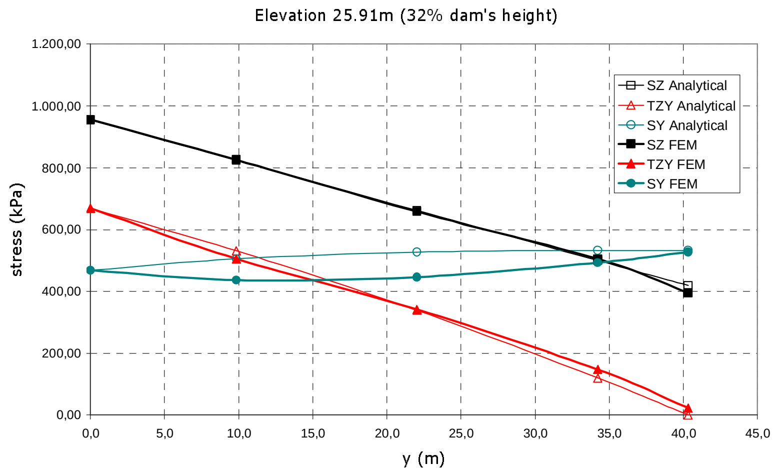


Figure 12: Elevation 25.91m stress distribution


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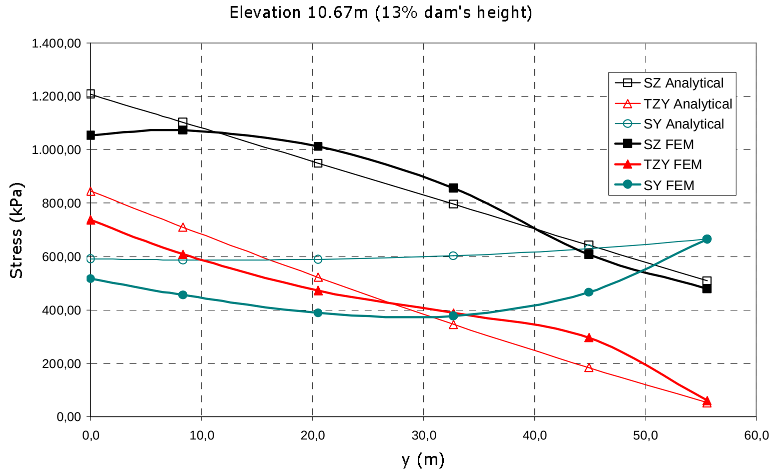


Figure 13: Elevation 10.67m stress distribution

### 5.4 Results analysis

From the above figures and graphics it is clear that the analytical procedure provides very good results for stress field distribution when compared to finite element solution. Figures 8a and 8b are almost identical. Very good results are also shown by Figures 9a and 9b. Section analysis reveals that analytical stress functions are very close to numerical results on a great part of the dam's elevations. Sections near the base or close to abrupt changes in geometry provide poor results.

## 6 Some seismic results samples

Seismic action is able to make significant changes on stress field magnitudes during earthquakes. Figures 14a and 14b illustrate some sample results. The first one indicates the stress field on Pine Flat Dam under normal operation actions. The second one indicates the stress field during a 0.2g ($1.962m/s^2$) peak ground acceleration (upstream direction), using the pseudo-static approach.

## 7 Conclusions

Analysis of previous results indicates that:

1. Analytical results are globally very similar to the ones obtained trough numerical methods as seen in Figures 8 and 9;
2. Section analysis justifies the above results. Analytical stress equations cover most of the dam's

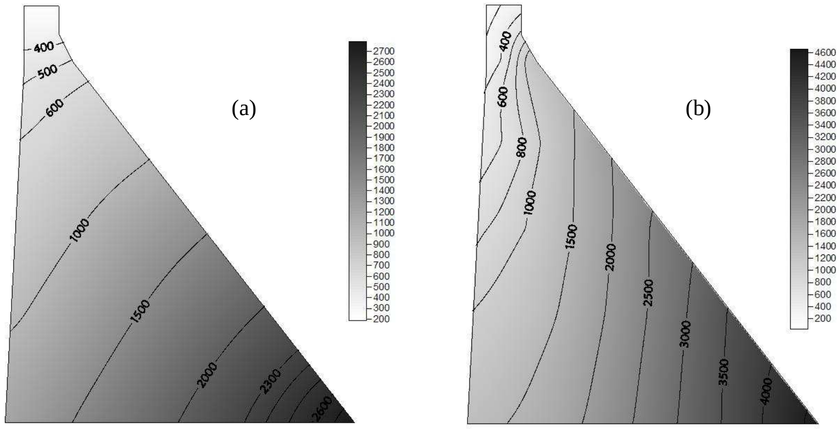


Figure 14: Pine Flat Dam maximum principal stresses (kPa) under normal (a) and upstream seismic action (b)

stress field with good accuracy. Singular points on geometry or near boundary conditions (dam's base) provide poor results;

3. The simple nature of this procedure and the good results provided makes it an ideal tool for preliminary design of concrete dams. Computational cost when compared to the output's quality is minimal for automatic calculations, and that is one the method's great advantage.
4. Seismic action can make significant changes on stress field during earthquakes. When Pine Flat Dam is accelerated at upstream direction, maximum principal stresses magnitudes at downstream side can increase up to 67%. The same effect could be expected in a downstream direction, generating overstressed regions on upstream side.

## Acknowledgements

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