Micromechanical approach to the strength properties of geocomposites with a Drucker-Prager matrix

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Abstract

The present paper describes a micromechanical-based approach to the strength properties of composite materials with a Drucker-Prager matrix in the situation of non associated plasticity. The concept of limit stress states for such materials is first extended to the context of homogenization. It is shown that the macroscopic stress states can theoretically be obtained from the solution to a sequence of viscoplastic problems stated on the representative elementary volume. The strategy of resolution implements a non-linear homogenization technique based on the modified secant method. Finally, the procedure is applied to the determination of the macroscopic strength properties and plastic flow rule of materials reinforced by rigid inclusions, as well as for porous media. The role of the matrix dilatancy coefficient is in particular discussed in both cases.

1 Introduction

Assessment of the macroscopic properties of composites is one of the major concern in material and structural engineering. As far as the elastic properties of the composites are concerned, the micromechanical techniques are nowadays widely applied and proved successful.

Nevertheless, the application of the micromechanical tools to the modeling of the non-linear behavior of composites is relatively recent and several issues are still open [1]. In particular, the determination of the strength properties of randomly heterogeneous materials remains an important task. The main contributions in this domain have been dedicated to the case of purely cohesive constituents described by a von Mises strength condition (see for instance [2–4]). In theses approaches, the local response of a perfectly plastic behavior to a monotonic loading process is simulated by means of a fictitious non-linear elastic behavior.

Few works have dealt with materials exhibiting frictional properties, which is common for geomaterials such as concrete, mortar, rocks or soils. The approach based on the fictitious non-linear elasticity, originally developed in the context of purely cohesive materials, have been transposed in [5] to the case of a Drucker-Prager matrix with a plastic flow rule described by a von Mises potential. However, such a method cannot be extended in order to take into account the local dilatancy of the Drucker-Prager matrix.

In a recent work, [6] presented a theoretical approach to the strength criterion of composite materials with a Drucker-Prager matrix complying with the normality rule. This approach is based on the mathematical equivalence between the limit analysis problem defining the macroscopic limit stress states and a fictitious non-linear viscous problem.

This contribution aims at extending the latter work to the situation where the assumption of normality rule is not valid, which is commonly encountered for geomaterials. The methodology adopted herein may be summarized as follows:

- the local non associated plastic behavior is viewed formally as the limit of a sequence of viscous behaviors with isotropic prestress;
- the limit state problem is formulated as a sequence of viscoplastic problems;
- the determination of the macroscopic strength properties is achieved by implementing a non-linear homogenization technique based on the modified secant method.

2 The framework of limit analysis theory

Some aspects related to the definition of the macroscropic strength properties of composites are briefly recalled in this section.

2.1 Macroscopic strength criterion of two-phase composites

A representative elementary volume (r.e.v) Ω of a randomly two-phase material is considered in the subsequent analysis (Fig. 1). Ω^m and Ω^h denote the domains occupied by the matrix and the set of heterogeneities in the r.e.v., respectively. The volume fraction of heterogeneities is defined by the ratio $\phi = |\Omega^h|/|\Omega|$.

The analysis will particularly focus on two kinds of heterogeneities:
- rigid inclusions with a prefect bonding interface between the matrix and inclusions;
- voids.

At the microscopic description, the first situation refers for instance to mortar, concrete (matrix=cement past, inclusions=aggregate/sand grains) or to some coarse soils like sandy gravel, whereas the second situation corresponds to ordinary porous media.

In both cases, the strength properties of the matrix are defined by a Drucker-Prager failure criterion:

$$\underline{\underline{\sigma}} \in G^m \iff F^m(\underline{\underline{\sigma}}) = \sigma_d + T(\sigma_m - h) \leq 0 \qquad (1)$$

where G^m is the convex of admissible stress states $\underline{\underline{\sigma}}$. $\sigma_m = \text{tr}\,\underline{\underline{\sigma}}/3$ is the local mean stress, $\underline{\underline{\sigma}}_d = \underline{\underline{\sigma}} - \sigma_m \underline{\underline{1}}$
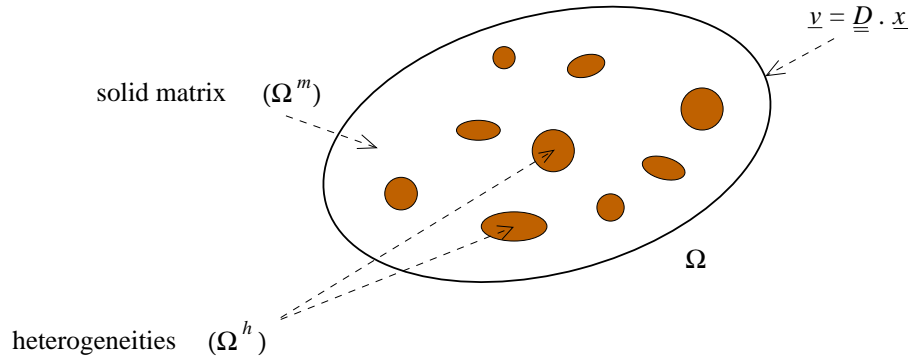
Figure 1: Problem geometry and loading mode.

the deviatoric part of $\underline{\underline{\sigma}}$ and $\sigma_d = \sqrt{\underline{\underline{\sigma}}_d : \underline{\underline{\sigma}}_d}$. The parameters h and T respectively characterize the tensile strength and the friction coefficient. If $\underline{v}$ denotes the velocity field in the r.e.v and $\underline{\underline{d}} = (\mathrm{grad}\,\underline{v} + {}^t\,\mathrm{grad}\,\underline{v})/2$ the associated strain rate, the convex G^m can equivalently be characterized by its support function $\pi^m(\underline{\underline{d}}) = \sup\{\underline{\underline{\sigma}} : \underline{\underline{d}}\,|\,F^m(\underline{\underline{\sigma}}) \leq 0\}$, with [7]

$$\pi^m(\underline{\underline{d}}) = \begin{cases} h\,d_v & \text{if } X = d_v - T\,d_d \geq 0 \\ +\infty & \text{if } X = d_v - T\,d_d < 0 \end{cases} \tag{2}$$

where $d_v = \mathrm{tr}\,\underline{\underline{d}}$ is the volume strain rate, $\underline{\underline{d}}_d = \underline{\underline{d}} - d_v\underline{\underline{1}}$ the deviatoric part of $\underline{\underline{d}}$ and $d_d = \sqrt{\underline{\underline{d}}_d : \underline{\underline{d}}_d}$.

The loading of the r.e.v is defined by means of uniform strain rate boundary conditions

$$\underline{v} = \underline{\underline{D}} \cdot \underline{x} \qquad \forall \underline{x} \in \partial\Omega \tag{3}$$

$\underline{\underline{D}}$ denoting the macroscopic strain rate. It is can be readily shown that for a velocity field $\underline{v}$ complying with (3), $\underline{\underline{D}}$ is the average of the microscopic strain rate over the domain Ω

$$\underline{\underline{D}} = \langle\,\underline{\underline{d}}\,\rangle_\Omega = \frac{1}{|\Omega|} \int_\Omega \underline{\underline{d}}\,\mathrm{d}\Omega \tag{4}$$

The macroscopic strength domain G^{hom}, that is, the set of admissible macroscopic states of stress $\underline{\underline{\Sigma}}$ is defined as [8]

$$G^{hom} = \left\{\underline{\underline{\Sigma}} = \langle\,\underline{\underline{\sigma}}\,\rangle_\Omega\,\,|\,\,\mathrm{div}\,\underline{\underline{\sigma}} = 0\,,\,\,\forall\underline{x} \in \Omega^m\,\,\,F^m(\underline{\underline{\sigma}}) \leq 0\right\} \tag{5}$$

The convexity of G^m implies the same property for G^{hom}. The dual kinematic definition of the latter may be expressed through its support function π^{hom} defined as

$$\pi^{hom}(\underline{\underline{D}}) = \sup\{\,\underline{\underline{\Sigma}} : \underline{\underline{D}}\,|\,\,\underline{\underline{\sigma}} \in G^{hom}\,\} \tag{6}$$

It can be proved that

$$\pi^{hom}(\underline{\underline{D}}) = \min_{\underline{v}} \left\langle \pi(\underline{\underline{d}}) \right\rangle_\Omega \tag{7}$$

where the minimization is performed over the velolcity fields complying with the boundary condition (3).

Classical results of convex analysis indicate that
• G^{hom} is the convex envelope of tangent hyperplanes

$$G^{hom} = \bigcap_{\underline{\underline{D}}} \left\{ \underline{\underline{\Sigma}} \mid \underline{\underline{\Sigma}} : \underline{\underline{D}} - \pi^{hom}(\underline{\underline{D}}) \leq 0 \right\} \tag{8}$$

• The boundary ∂G^{hom} of the macroscopic strength domain is defined by the stress states $\underline{\underline{\Sigma}}$ satisfying

$$\underline{\underline{\Sigma}} : \underline{\underline{D}} = \pi^{hom}(\underline{\underline{D}}) = \min_{\underline{v}} \left\langle \pi(\underline{\underline{d}}) \right\rangle_\Omega \tag{9}$$

where $\underline{\underline{D}}$ is oriented following the outward normal to ∂G^{hom} at point $\underline{\underline{\Sigma}}$.

It must be kept in mind that the classical limit analysis framework described above, is implicitly dedicated to the strength propertie of materials with associated plastic flow rule.

2.2 viscoplastic formulation

The determination of ∂G^{hom} through the kinematic approach requires to solving problem (9), which turn to be an uneasy task for randomly heterogeneous materials. As far as the limit states of stress are concerned, the behavior of the matrix may formally be modeled as rigid plastic with normality rule. An alternative viscoplastic formulation to determine the solution of (9) consists in adopting the support function π^m of the convex G^m as a potential for the matrix state equation [9]

$$\underline{\underline{\sigma}} = \frac{\partial \pi^m}{\partial \underline{\underline{d}}} \tag{10}$$

A classical result of convex analysis states that a stress $\underline{\underline{\sigma}}$ defined in this way lies on the boundary of G^m (i.e. $F^m(\underline{\underline{\sigma}}) = 0$) at the point where the outward normal is parallel to $\underline{\underline{d}}$. Since, for Drucker-Prager materials, $\pi^m(\underline{\underline{d}})$ is not differentiable at $\underline{\underline{d}} = \underline{\underline{0}}$, (10) should therefore be replaced by $\underline{\underline{\sigma}} \in \partial \pi^m(\underline{\underline{0}})$, where $\partial \pi^m(\underline{\underline{0}})$ is the sub-differential of π^m at $\underline{\underline{d}} = \underline{\underline{0}}$.

When the normality rule is fulfilled, the fictitious viscoplastic problem reads as follows. For a given macroscopic strain rate $\underline{\underline{D}}$, we consider the microscopic fields $\underline{\underline{\sigma}}$ and $\underline{v}$ solutions of the mechanical problem defined on the r.e.v

$$\begin{cases} \operatorname{div} \underline{\underline{\sigma}} = 0 & \text{in } \Omega \\ \underline{\underline{d}} = (\operatorname{grad} \underline{v} +^{t} \operatorname{grad} \underline{v})/2 & \text{in } \Omega \\ \underline{\underline{\sigma}} = \partial \pi^{m}/\partial \underline{\underline{d}} & \text{in } \Omega^{m} \\ \underline{v} = \underline{\underline{D}} \cdot \underline{x} & \text{in } \partial \Omega \end{cases} \tag{11}$$

In addition to system (11), condition $\underline{\underline{d}} = \underline{\underline{0}}$ (resp. $\underline{\underline{\sigma}} = 0$) has to be satisfied in Ω^{h} if the heterogeneities are rigid inclusions (resp. pores). As regards the macroscopic strength properties, solving the problem (11) proves that $\underline{\underline{\Sigma}} = \left\langle \underline{\underline{\sigma}} \right\rangle_{\Omega}$ satisfies equality (9) [6, 9]. Consequently, the determination of ∂G^{hom} reduces to finding the effective behavior of medium made up of a viscoplastic matrix surronding heterogeneities (rigid inclusions or pores).

Unfortunately, the practical implementation of this approach turns to be impossible for composites with a Drucker-Prager matrix because of the high singularity of the support function π^{m} given by (11). Indeed, the latter takes infinite values if $X = d_{v} - T\,d_{d} < 0$. The technique proposed in [6, 10] to deal with this singularity, consists in introducing a sequence, indexed by the real scalars $a > 0$, of potentials $\psi_{a}(\underline{\underline{d}})$ on the form

$$\psi_{a}(\underline{\underline{d}}) = \psi_{a}(d_{v}, d_{d}) = f_{a}(X) + h\,d_{v} \tag{12}$$

where $f_{a}(X)$ is a sequence of C^{2}-class convex functions, decreasing on $]-a, +\infty]$, $f_{a}(X) = 0$ for $X \geq 0$ and such that $\lim\limits_{X \to -a^{+}} f_{a}(X) = +\infty$ (see Fig. 2).
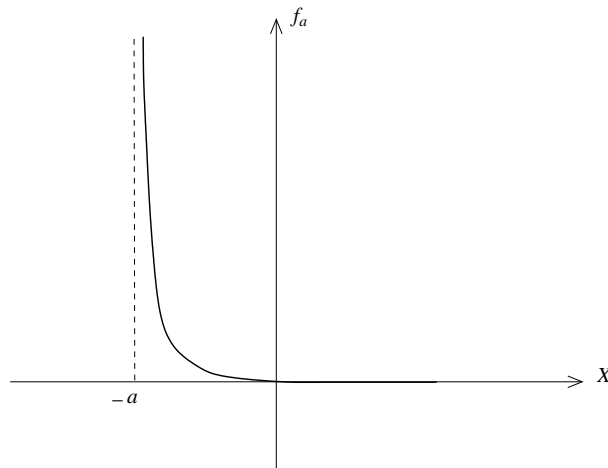

Figure 2: Shape of function f_{a}.

It can be easily seen from (12) that the sequence of functions $\psi_a(\underline{\underline{d}})$ tends towards the support function $\pi^m(\underline{\underline{d}})$ of the Drucker-Prager matrix when the scalar parameter a tends towards 0^+ (simple convergence). On the other hand, the stress state defined by

$$\underline{\underline{\sigma}} = \frac{\partial \psi_a}{\partial \underline{\underline{d}}} = f_a'(X) \left(\underline{\underline{1}} - T \, \underline{\underline{d}}_d/d_d \right) + h \, \underline{\underline{1}} \tag{13}$$

always meets condition $\underline{\underline{\sigma}} \in \partial G^m$ (i.e. $F^m(\underline{\underline{\sigma}}) = 0$).

Besides, condition (13) indicates that $f_a'(X) \neq 0$ if $\underline{\underline{\sigma}} \neq h \underline{\underline{1}}$, and thus $X \in]-a, 0[$. This property means that the normality rule $d_v = T \, d_d$ is asymptotically fulfilled when $a \to 0^+$ (provided that $\underline{\underline{\sigma}} \neq h \underline{\underline{1}}$).

The problem (11), which resolution leads to the determination of the macroscopic strength domain G^{hom} is thus replaced by the sequence of viscoplastic problems

$$\begin{cases} \operatorname{div} \underline{\underline{\sigma}} = 0 & \text{in } \Omega \\ \underline{\underline{d}} = (\operatorname{grad} \underline{v} +^t \operatorname{grad} \underline{v})/2 & \text{in } \Omega \\ \underline{\underline{\sigma}} = \partial \psi_a/\partial \underline{\underline{d}} & \text{in } \Omega^m \\ \underline{v} = \underline{\underline{D}} \cdot \underline{x} & \text{in } \partial \Omega \end{cases} \tag{14}$$

to which must be added the appropriate condition $\underline{\underline{d}} = \underline{\underline{0}}$ or $\underline{\underline{\sigma}} = 0$ in the domain Ω^h.

It is shown in [6] that a solution $(\underline{\underline{\sigma}}_a, \underline{v}_a)$ of the above system satisfies the following conditions:
- $\underline{\underline{d}} = \lim\limits_{a \to 0^+} \underline{\underline{d}}_a$ is associated with a kinematically admissible velocity field;
- $\underline{\underline{\Sigma}} = \lim\limits_{a \to 0^+} \left\langle \underline{\underline{\sigma}}_a \right\rangle_\Omega \in \partial G^{hom}$.

The above viscoplastic regularization of the matrix state equation is on the basis of the technique implemented in [6] in order to evaluate, under the assumption of normality rule, the macroscopic strength of Drucker-Prager materials reinforced by rigid inclusions.

The object of the following sections is to extend the above approach to the situation where the normality rule is not valid.

3 Limit analysis homogenization in the context of non associated plasticity

The question of macroscopic strength properties of a composite made up of a Drucker-Prager matrix with embedded heterogeneities (rigid inclusions or pores) is examined in the general situation where the normality rule is not fulfilled.

More precisely, it is assumed that the matrix plastic yielding is characterized by

$$\underline{\underline{d}} = \dot{\chi} \, \frac{\partial \mathcal{G}^m}{\partial \underline{\underline{\sigma}}} \quad , \quad \dot{\chi} \geq 0 \quad \text{with} \quad \mathcal{G}^m(\underline{\underline{\sigma}}) = \sigma_d + t \, \sigma_m \tag{15}$$

$\mathcal{G}^m$ being the plastic potential and $t \in [0, T]$ the dilatancy coefficient. If $\underline{\underline{\sigma}} \neq h\underline{\underline{1}}$, the non associated flow rule (15) actually reads

$$d_v = t\, d_d \tag{16}$$

3.1 State equation of the solid matrix

In order to generalize the approach described in section **2.2** to the case $t \neq T$, the idea consists in defining the stress state by means of a potential $\psi_a(d_v, d_d) = f_a(Y) + h\, d_v$ and an isotropic prestress $\sigma_0(Y)\underline{\underline{1}}$, with $Y = d_v - t\, d_d$:

$$\underline{\underline{\sigma}} = \frac{\partial \psi_a}{\partial \underline{\underline{d}}}(Y) + \sigma_0(Y)\underline{\underline{1}} \tag{17}$$

(17) implies that

$$\sigma_m = f_a'(Y) + h + \sigma_0(Y) \quad ; \quad \underline{\underline{\sigma}}_d = -t\, f_a'(Y)\, \underline{\underline{d}}_d/d_d \tag{18}$$

In order to meet condition $F^m(\underline{\underline{\sigma}}) = 0$, the prestress reads

$$\sigma_0(Y) = f_a'(Y)\left(\frac{t}{T} - 1\right) \tag{19}$$

Furthermore, if the stress state $\underline{\underline{\sigma}}$ differs from $h\underline{\underline{1}}$, equations (18) and (19) shows that $f_a'(Y) \neq 0$, which implies that $Y \in\,]-a, 0[$ and consequently that the non associated plastic flow rule $d_v = t\, d_d$ is fulfilled asymptotically when $a \to 0^+$.

Interestingly, the state equation (17) can conveniently be written on the form

$$\underline{\underline{\sigma}} = \mathbb{C}^m(\underline{\underline{d}}) : \underline{\underline{d}} + \sigma^p(\underline{\underline{d}})\underline{\underline{1}} \tag{20}$$

The secant isotropic fourth-order tensor $\mathbb{C}^m(\underline{\underline{d}})$ is given by

$$\mathbb{C}^m(\underline{\underline{d}}) = 3k^m\mathbb{J} + 2\mu^m\mathbb{K} \quad \text{with} \quad k^m = \frac{f_a'(Y)}{Y} \quad \mu^m = -t\frac{f_a'(Y)}{2d_d} \tag{21}$$

where $\mathbb{J} = \frac{1}{3}\underline{\underline{1}} \otimes \underline{\underline{1}}$ and $\mathbb{K} = \mathbb{I} - \mathbb{J}$, while the isotropic prestress reads

$$\sigma^p(\underline{\underline{d}}) = h + \frac{f_a'(Y)}{Y}\left(\frac{t}{T}Y - d_v\right) \tag{22}$$

It should be observed that, except for $\underline{\underline{\sigma}} = h\underline{\underline{1}}$, Y is a stricly negative number, which ensures the positivity of the moduli k^m and μ^m.

3.2 Limit states of the composite

We seek now the limit states of the composite at the macroscopic scale within the framework of a micromechanical reasoning. For a given value of the dilatancy coefficient $t \in [0, T]$, the set of macroscopic limit states $\underline{\underline{\Sigma}}$ is denoted in the the sequel by $\mathcal{E}_t$. The micromechanical definition of the latter will be specified below.

The limit analysis theory is implicitly dedicated to the strength propertie of materials with associated plastic flow rule $t = T$. Clearly enough, $\mathcal{E}_t$ can not be derived from the classical theorems of limit analysis if the normality rule is not fulfilled (i.e. $t \neq T$).

We herein extend the definition of the limit stress states proposed in [11, 12] to the context of homogenization.

We first proceed at a fixed value of parameter $a > 0$. A macroscopic stress tensor $\underline{\underline{\Sigma}}_a$ is said to be a limit state of the composite made up of the matrix defined by f_a and the heterogeneities, if there exists a microscopic stress field $\underline{\underline{\sigma}}_a$ statically admissible with $\underline{\underline{\Sigma}}_a$ (i.e. div $\underline{\underline{\sigma}}_a = 0$ and $\underline{\underline{\Sigma}}_a = \langle\, \underline{\underline{\sigma}}_a\, \rangle_\Omega$) and a velocity field $\underline{v}_a$ meeting uniform strain boundary conditions (3), such that $\underline{\underline{\sigma}}_a$ and the strain rate $\underline{\underline{d}}_a = (\mathrm{grad}\, \underline{v}_a +^t \mathrm{grad}\, \underline{v}_a)/2$ are related by the state equation (18).

In a second step, the macroscopic limit states of the heterogeneous material with a Drucker-Prager matrix are defined as limits of the stress states $\underline{\underline{\Sigma}}_a$ when $a \to 0^+$:

$$\mathcal{E}_t = \left\{ \underline{\underline{\Sigma}} = \lim_{a \to 0^+} \underline{\underline{\Sigma}}_a \right\} \tag{23}$$

In the particular case of normality rule $t = T$, this definition coincides with that derived in the framework of limit analysis (see Eq. (5)): $\mathcal{E}_t = \partial G^{hom}$. In contrast, when $t < T$, the set defined by $\mathcal{E}_t$ is actually a subset of the domain G^{hom} determined from limit analysis.

Such a reasoning associates to each macroscopic stress $\underline{\underline{\Sigma}} \in \mathcal{E}_t$ the macroscopic strain rate $\underline{\underline{D}} = \langle\, \underline{\underline{d}}_a\, \rangle_\Omega$ corresponding to the velocity field $\underline{v}_a$ ($\underline{\underline{D}}$ actually does not depend on the value of a). Analysing the properties of $\underline{\underline{D}}$ provides the structure of the plastic flow rule for the homogenized material.

3.3 Implementation of the non-linear homogenization technique

The practical determination of the set $\mathcal{E}_t$ requires to solve the sequence of viscoplastic problems

$$\begin{cases} \mathrm{div}\, \underline{\underline{\sigma}} = 0 & \text{in } \Omega \\ \underline{\underline{d}} = (\mathrm{grad}\, \underline{v} +^t \mathrm{grad}\, \underline{v})/2 & \text{in } \Omega \\ \underline{\underline{\sigma}} = \mathbb{C}^m(\underline{\underline{d}}) : \underline{\underline{d}} + \sigma^p(\underline{\underline{d}})\underline{\underline{1}} & \text{in } \Omega^m \\ \underline{v} = \underline{\underline{D}} \cdot \underline{x} & \text{in } \partial\Omega \end{cases} \tag{24}$$

The non-linearity of the matrix behavior is decribed by de dependence of k^m, μ^m and σ^p, given in (21), on the local volume and deviatoric strain rates d_v and d_d. The strain rate field $\underline{\underline{d}}^1$ solution of problem (24). Consequently, the moduli (k^m, μ^m) and prestress σ^p are non-uniform as well. The classical homogenization schemes cannot thus be directly implemented since they only deal with composites made up of homogeneous phases. The idea is then to resort to the so-called *non-linear homogenization technique* (see Reference [4]). This strategy is based on the concept of effective strain rate and on the implementation of linear homogenization schemes in which the local behavior non-linearly depends on the loading parameters.

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[1]For sake of clarity, the subscript a will be omitted in the sequel.

The concept of effective strain rate aims at capturing in a simplified way the effect of the loading $\underline{\underline{D}}$ on the non-linear stiffness. The effective strain tensor $\underline{\underline{d}}^e$, depending on the loading $\underline{\underline{D}}$, should be an appropriate average of $\underline{\underline{d}}$ over the matrix. The procedure consists in adopting for any $\underline{x} \in \Omega^m$: $k^m\left(\underline{\underline{d}}(\underline{x})\right) = k^m\left(\underline{\underline{d}}^e\right) = k_{eq}^m$, $\mu^m\left(\underline{\underline{d}}(\underline{x})\right) = \mu^m\left(\underline{\underline{d}}^e\right) = \mu_{eq}^m$ and $\sigma^p\left(\underline{\underline{d}}(\underline{x})\right) = \sigma^p\left(\underline{\underline{d}}^e\right) = \sigma_{eq}^p$. The question of choosing an appropriate effective strain rate has been extensively discussed in the litterature (see [4, 13] for instance). As regards the problem considered herein, it appears that the modified secant method [4, 14] provides a relevant framework.

Given a macroscopic strain rate $\underline{\underline{D}}$, the effective volume and deviatoric strain rates in the matrix are defined in the modified secant method as

$$d_v^e = \sqrt{\langle\, d_v^2\, \rangle_{\Omega^m}} \quad ; \quad d_d^e = \sqrt{\langle\, d_d^2\, \rangle_{\Omega^m}} \tag{25}$$

In a consistent manner with these definitions, we also introduce the effective estimate Y^e of $Y = d_v - t\, d_d$

$$Y^e = d_v^e - t\, d_d^e \tag{26}$$

Reasoning at a fixed a, the method amounts then to replacing the "real matrix" with state equation (20) by an equivalent material, i.e. the linear viscoelastic effective material defined by

$$\underline{\underline{\sigma}} = \mathbb{C}^m(d_v^e, d_d^e) : \underline{\underline{d}} + \sigma^p(d_v^e, d_d^e)\,\underline{\underline{1}} \quad \text{with} \quad \mathbb{C}^m(d_v^e, d_d^e) = 3k_{eq}^m\,\mathbb{J} + 2\mu_{eq}^m\,\mathbb{K} \;;\; \sigma^p(d_v^e, d_d^e) = \sigma_{eq}^p \tag{27}$$

Regarding k_{eq}^m, μ_{eq}^m and σ_{eq}^p as constants, a micromechanical reasoning based on the implementation of linear homogenization and the Levin theorem shows that the homogenized behavior of the composite made up of the matrix decribed by (27) and the embedded heterogeneities (rigid inclusions or pores) takes the form

$$\underline{\underline{\Sigma}} = \mathbb{C}^{hom} : \underline{\underline{D}} + \Sigma^p\,\underline{\underline{1}} \tag{28}$$

where $\mathbb{C}^{hom}(k_{eq}^m, \mu_{eq}^m)$ is the tensor of homogeneized viscous moduli

$$\mathbb{C}^{hom} = 3k^{hom}(k_{eq}^m, \mu_{eq}^m)\,\mathbb{J} + 2\mu^{hom}(k_{eq}^m, \mu_{eq}^m)\,\mathbb{K} \tag{29}$$

and $\Sigma^p = \Sigma^p(k_{eq}^m, \mu_{eq}^m, \sigma_{eq}^p)$ is the prestress at the macroscopic scale.

The last step of the procedure, consists in observing that k_{eq}^m, μ_{eq}^m and σ_{eq}^p non-linearly depend on the macroscopic strain rate $\underline{\underline{D}}$. through relations to be described in the next sections. The relationship (28) actually reads

$$\underline{\underline{\Sigma}} = \mathbb{C}^{hom}(\underline{\underline{D}}) : \underline{\underline{D}} + \Sigma^p(\underline{\underline{D}})\,\underline{\underline{1}} \tag{30}$$

which asymptotically, as a tends to 0, provides the parametric equations of the set $\mathcal{E}_t$ defining the limit stress states of the composite at the macroscopic scale.

The non-linear homogenization technique implemented in the framework of the modified secant method will be applied in the next sections to the determination of the macroscopic strength criterion of materials reinforced by rigid inclusions and of porous media. For this purpose, the following property will be exploited

$$\lim_{a \to 0^+} Y^e = 0 \quad (\Longleftrightarrow d_v^e \approx_{a \to 0^+} t\, d_d^e) \tag{31}$$

4 Determination of the macroscopic strength criterion

4.1 Matrix reinforced by rigid inclusions

It is first observed that the macroscopic stress states $\underline{\underline{\Sigma}} = h \underline{\underline{1}} \in \mathcal{E}_t$ since the uniform microscopic fields $\underline{\underline{\sigma}} = h \underline{\underline{1}}$ and $\underline{v} = 0$ comply with the required conditions detailed in section **3.2**. The situation $\underline{\underline{\Sigma}} \neq h \underline{\underline{1}}$ is therefore examined in the sequel.

Observing that the microscopic strain rate in the rigid inclusions vanishes (i.e. $\underline{\underline{d}} = 0$ in Ω^h), it can readily be shown that $d_d^e \geq D_d/(1 - \phi)$, which implies that $d_d^e > 0$ for $D_d \neq 0$. Furthermore, the situation $Y^e \geq 0$ should be discarded. Indeed, this would imply that $\underline{\underline{\sigma}} = h \underline{\underline{1}}$ in the matrix (see (18-19)), which in turn implies that $\underline{\underline{\Sigma}} = h \underline{\underline{1}}$. Hence, the limit states $\underline{\underline{\Sigma}} \neq h \underline{\underline{1}}$ correspond to the situation $Y^e \in] - a, 0[$. For the latters, the flow rule (31) takes the asymptotic form

$$\lim_{a \to 0^+} d_v^e = t \, d_d^e \quad \Longrightarrow \quad \lim_{a \to 0^+} \frac{Y^e}{d_d^e} = 0 \tag{32}$$

In the case of rigid inclusions, the macroscopic prestress in the homogenized behavior (28) simply reads $\Sigma^p = \sigma_{eq}^p$. Hence, the macroscopic potential of the linear composite defined by the state equation (28) takes the form

$$\Psi^{hom}(\underline{\underline{D}}) = \frac{1}{2} \underline{\underline{D}} : \mathbb{C}^{hom} : \underline{\underline{D}} + \sigma_{eq}^p \, \text{tr} \, \underline{\underline{D}} \tag{33}$$

Reasoning as in [6], the effective strain rates in the matrix are related to the loading $\underline{\underline{D}}$ of the r.e.v through

$$\frac{1}{2} (1 - \phi) \, d_v^{e\,2} = \frac{1}{2} \frac{\partial k^{hom}}{\partial k_{eq}^m} D_v^2 + \frac{\partial \mu^{hom}}{\partial k_{eq}^m} D_d^2 \quad ; \quad (1 - \phi) \, d_d^{e\,2} = \frac{1}{2} \frac{\partial k^{hom}}{\partial \mu_{eq}^m} D_v^2 + \frac{\partial \mu^{hom}}{\partial \mu_{eq}^m} D_d^2 \tag{34}$$

To go further, we now need to evaluate the homogenized viscous moduli k^{hom} and μ^{hom}. The morphology examined in this paper (isotropic randomly reinforced composite) suggests to adopt the Mori-Tanaka Scheme, which conicides with the lower bound estimates of Hashin-Shtrikman in the case of rigid inclusions

$$k^{hom} = \frac{3k_{eq}^m + 4\phi\mu_{eq}^m}{3(1 - \phi)} \quad ; \quad \mu^{hom} = \mu_{eq}^m \frac{k_{eq}^m(6 + 9\phi) + \mu_{eq}^m(12 + 8\phi)}{6(1 - \phi)(k_{eq}^m + 2\mu_{eq}^m)} \tag{35}$$

It should be observed from (21) and (32) that

$$\lim_{a \to 0^+} \frac{\mu_{eq}^m}{k_{eq}^m} = \lim_{a \to 0^+} \frac{-t \, Y^e}{2 \, d_d^e} = 0 \tag{36}$$

This property can advantageously be used in order to simplify the asymptotic expressions of k^{hom} and μ^{hom} in (35), as well as of their derivatives in (34). It comes as $a \to 0^+$

$$d_v^e = \frac{D_v}{1 - \phi} \quad ; \quad d_d^e = \frac{1}{1 - \phi} \sqrt{\frac{2}{3} \phi \, D_v^2 + (1 + \frac{3}{2} \phi) \, D_d^2} \tag{37}$$

The local flow rule $d_v^e \approx_{a \to 0^+} t\, d_d^e$ together with (37) provide the plastic flow rule at the macroscopic scale

$$D_v = t^{hom} D_d \quad with \quad t^{hom} = t \sqrt{\frac{1 + \frac{3}{2}\phi}{1 - \frac{2}{3}\phi t^2}} \tag{38}$$

t^{hom} may be interpreted as the macroscopic dilatancy coefficient. Finally, the expression of the macroscopic strength criterion is derived from (28) and (38)

$$F^{hom}(\underline{\underline{\Sigma}}, \phi) = \Sigma_d - T^{hom}(h - \Sigma_m) = 0 \quad with \quad T^{hom} = T\sqrt{1 + \frac{3}{2}\phi}\,\frac{\sqrt{1 - \frac{2}{3}\phi t^2}}{1 - \frac{2}{3}\phi tT} \tag{39}$$

The coefficient T^{hom} plays the role of the macroscopic friction coefficient. (39) indicates that the composite is described at the macroscopic scale by a Drucker-Prager strength criterion.

The identity (38) characterizes the strain rates $\underline{\underline{D}}$ associated with the limit states $\underline{\underline{\Sigma}} \neq h\,\underline{\underline{1}}$. Since $t^{hom}/T^{hom} = (1 - \frac{2}{3}\phi tT)/(1 - \frac{2}{3}\phi t^2)$, the macroscopic plastic flow rule turns to be non associated (provided that $t \neq T$). It is also observed that the presence of rigid inclusions results in a higher dilatancy coefficient when compared to that of the matrix (i.e. $t^{hom} > t$). In the particular situations $t = T$ (normality rule) and $t = 0$ (no plastic volume strain), the macroscopic strength criterion (39) coincides with those presented in [6] and [5], respectively.
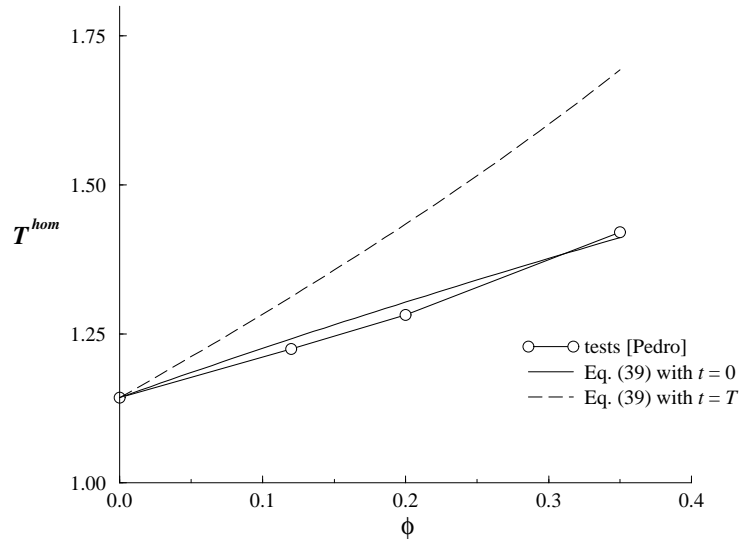

Figure 3: Evolution of the macroscopic friction coefficient with the inclusions volume fraction: theoretical predictions vs tests.

The micromechanical-based strength criterion defined by (39) is now compared to the exprimental results provided in [15]. The strength properties of sand samples reinforced by randomly distributed gravel inclusions have been investigated in [15] by means of triaxial tests. Fig. 3 displays the evolution of the macroscopic friction coefficient T^{hom} measured for gravel-reinforced sand samples, with respect to the gravel volume fraction ϕ. The theoretical predictions of T^{hom} obtained for $t = 0$ and $t = T$ are also represented in this figure. It clearly appears that the theoretical curve corresponding to $t = 0$ perfectly fits the experimental results. The sligth discrepancy observed from the comparison could be attributed for instance to the nature of the real interface gravel/sand. Indeed, perfect bonding has been assumed in the present micromechanical approach, while a frictional law should probably be more appropriate for the gravel/sand interface modeling.

4.2 Porous medium

We deal now with the situation of a matrix surrounding pores. The macroscopic prestress in the homogenized behavior (28) is now given by

$$\Sigma^p = \frac{k^{hom}}{k_{eq}^m}\,\sigma_{eq}^p \tag{40}$$

In the case of a porous medium, the effective strain rates in the matrix are related to $\underline{D}$ through [13]

$$\frac{1}{2}\,(1-\phi)\,d_v^{e\,2} = \frac{1}{2}\,\frac{\partial k^{hom}}{\partial k_{eq}^m}\,\left(D_v + \frac{\sigma_{eq}^p}{k_{eq}^m}\right)^2 - \frac{k^{hom}}{k_{eq}^{m\,2}}\,\sigma_{eq}^p\,D_v + \frac{1}{2}\,\left(1-\phi-2\frac{k^{hom}}{k_{eq}^m}\right)\,\left(\frac{\sigma_{eq}^p}{k_{eq}^m}\right)^2 + \frac{\partial \mu^{hom}}{\partial k_{eq}^m}\,D_d^2$$

$$(1-\phi)\,d_d^{e\,2} = \frac{1}{2}\,\frac{\partial k^{hom}}{\partial \mu_{eq}^m}\,\left(D_v + \frac{\sigma_{eq}^p}{k_{eq}^m}\right)^2 + \frac{\partial \mu^{hom}}{\partial \mu_{eq}^m}\,D_d^2$$

$$\tag{41}$$

As regards the tensor of homogenized viscous moduli, we herein adopt the Hashin-Shtrikman upper bounds which are known to reasonably model the linear (visco)elasic properties of isotropic porous media (see Reference [16]). Accordingly,

$$k^{hom} = k_{eq}^m\,\frac{4(1-\phi)\mu_{eq}^m}{3\phi k_{eq}^m + 4\mu_{eq}^m} \quad ; \quad \mu^{hom} = \mu_{eq}^m\,\frac{(1-\phi)(9k_{eq}^m + 8\mu_{eq}^m)}{k_{eq}^m(9+6\phi) + \mu_{eq}^m(8+12\phi)} \tag{42}$$

Observing that (28) still holds and proceeding in a similar way as in section **4.1**, the obtained set $\mathcal{E}_t$ of limit stress states is characterized by the following equations[2]

$$\Sigma_m = A\,\frac{D_v/D_d - t\,\sqrt{\frac{2}{3\phi}(D_v/D_d)^2 + \frac{1-\frac{2t^2}{3\phi}}{1+\frac{2}{3\phi}}}}{\sqrt{\frac{2}{3\phi}(D_v/D_d)^2 + \frac{1-\frac{2t^2}{3\phi}}{1+\frac{2}{3\phi}}} + C\,D_v/D_d} \qquad \Sigma_D = \frac{B}{\sqrt{\frac{2}{3\phi}(D_v/D_d)^2 + \frac{1-\frac{2t^2}{3\phi}}{1+\frac{2}{3\phi}}} + C\,D_v/D_d} \tag{43}$$

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[2]to avoid heavy mathematical developments, the calculation details are not provided here.

where

$$A = \frac{2hT(1-\phi)}{3\phi - 2t^2} \frac{1 - \frac{2t^2}{3\phi}}{1 - \frac{2Tt}{3\phi}} \;\; ; \;\; B = \frac{hT(1-\phi)}{1 + \frac{2}{3\phi}} \frac{1 - \frac{2t^2}{3\phi}}{1 - \frac{2Tt}{3\phi}} \;\; ; \;\; C = \frac{2T}{3\phi} \left(\frac{1 - \frac{2t^2}{3\phi}}{1 - \frac{2Tt}{3\phi}} - \frac{t}{T} \right) \qquad (44)$$

Expressions (43) characterize the parametric equations of the set $\mathcal{E}_t$. Eliminating D_v/D_d in these equations yields

$$F^{hom}(\underline{\underline{\Sigma}}, \phi) = \frac{1 + 2\phi/3}{T^2} \Sigma_d^2 + \left(\frac{3\phi}{2T^2} - 1 \right) \Sigma_m^2 + 2(1-\phi)\, h\, \Sigma_m - (1-\phi)^2\, h^2 = 0 \qquad (45)$$

which shows that the set $\mathcal{E}_t$ defines an elliptic domain independent on the value of the dilatancy coefficient t. This means that, unlike the situation of matrix reinforced by rigid inclusions, the macroscopic strength criterion could actually be derived from the classical theorems of limit analysis. This result is in accordance with those obtained by [10] in the case $t = T$ and by [17] in the case $t = 0$. The result derived in this paper extends the latters to the general case $0 < t < T$.

Furthermore, it can be established from (43) that the plastic flow rule is associated at the macroscopic scale. This normality rule means that the macroscopic strain rate $\underline{D}$ associated with $\underline{\underline{\Sigma}}$ through (43) is oriented following $\partial F^{hom}/\partial \underline{\underline{\Sigma}}$.

5 Concluding remarks

The macroscopic strength of Drucker-Prager matrix with non associated plastic flow rule, surronding randomly distributed heterogeneities, has been adressed within the framework of a non-linear homogenization technique. The definition of the limit stress states when the normality rule is not fulfilled, is first extended to the context of homogenization theory.

The methodology consists in replacing the limit analysis problem defining the macroscopic limit stress states by a sequence of viscplastic problems which solution leads asymptotically to the set of macroscopic limit stress states. The strategy for solving the viscoplastic problems stated on the representative elementary volume is based on the implementation of the modified secant method. Accordingly, the microscopic velocity solution of the non-linear viscous problem is characterized by an effective strain rate.

The procedure has been successively applied to assess the macroscopic strength of two particular composites: materials reinforced by embeded rigid inclusions and porous media. For the both types of composites, the macroscopic strength criterion has been determined. For a matrix reinforced by rigid inclusions, it is found that the strength domain takes the form of a Drucker-Prager criterion which inclination depend on the dilatancy coefficient of the matrix. Besides, the macroscopic plastic flow rule is characterized by a dilatancy coefficient higher than that of the matrix. The results prove to be in a good concordance with available laboratory tests.

In the case of a porous medium, the form of Drucker-Prager is not preserved at the macroscopic scale since the strength domain is elliptic. Unlike the situation of rigid inclusions, the macroscopic strength criterion is not affected by the matrix dilatancy coefficient. In addition, the normality rule

holds at the macroscopic scale.

To finish with, let us review some issues that still need to be addressed:

- A more realistic model for the interface matrix/rigid inclusions, such as a Coulomb interface law for instance, should be introduced.
- The solution of the viscoplastic problems obtained by means of the non-linear homogenization technique remains to be validated through finite element analyses of the same problems.
- In the case of porous media, the analysis should be extended to account for the presence of a saturating fluid. One should clarify in particular, how the fluid pressure affects the macroscopic strength criterion and the macroscopic plastic flow rule.

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