Simplified analysis of stress and strain histories in notched metallic parts undergoing cyclic inelastic deformations using projection techniques

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Abstract

The load on notched mettalic parts are often so high that the local stress calculated from the theory of Elasticity, using nominal stress and notch factor is considerably above the yield strength. This is especially true for parts designed for only a few thousands load cycles, but may be true for others too. Uniformly repeated load cycles impose uniformly repeated strain cycles on the metal at the root of the notch, as long as most of the part remains elastic. If the metal at the notch root is strained beyond the yield strain it may strain harden and cyclically harden or soften. This work presents a simple technique to estimate the inelastic stress and strain histories at the root of the notch of a thin metallic plate undergoing a non monotonic loading. An internal variables theory that accounts for the hardening effects induced by cyclic plasticity is considered. Since the stress state is one-dimensional at the root of the notch, the main idea is to compute the elastic stress and strain from the classical theory of Elasticity and to approximate the real stress and strain using projection techniques (Linear rule and Neuber rule) and an adequate set of elasto-plastic constitutive equations. The predictions of the simplified theory are compared to finite elements simulations showing an excellent agreement. The finite elements simulations of a few cycles of loading with a refined mesh can take a long time in a ordinary PC while the simulation with the simplified theory take a few seconds. An user friendly code was developed to simulate different kinds of problems. Examples concerning cyclic loadings in different notched plates are presented and analyzed showing the main features of the proposed methodology..

Keywords: Neuber rule, Linear rule, cyclic loading, elasto-platicity, elasto-viscoplasticity

1 Introduction

More sophisticated and realistic inelastic constitutive equations are often considered too complex to be used in simple engineering problems. In the last years, a reasonable effort was done to develop simple

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techniques to solve engineering problems concerning the inelastic behavior of metallic structures – a "Inelastic Strength of Materials" [1–13].

The present work is concerned with a simplified technique for the analysis of stress concentration in notched metallic plates under tension undergoing elastic or inelastic deformations.

For thin plates under traction, it is reasonable to suppose that the structure is submitted to a two-dimensional state of stress. In this case (see figure 1), it is possible to verify that the state of stress is one-dimensional at the root of the notch. Hence, simplified procedures can be used to determine the stress and strain at this point even if the material is undergoing nonlinear inelastic deformations.

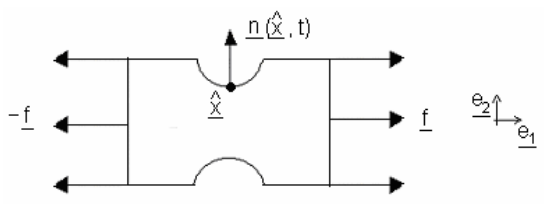

Figure 1: Thin plate under tensile loading.

The proof that there is only one nonzero stress component at the root of the notch is very simple and is based on the fact that, in a given system that occupies a region Ω with boundary Γ of the Euclidean space, if $\underline{f}$ is the external surface force applied on Γ, then, the stress tensor $\underline{\underline{\sigma}}$ is such that

$$\underline{\underline{\sigma}}(\underline{x},t)\underline{n}(\underline{x},t) = \underline{f}(\underline{x},t) \forall \underline{x} \in \Gamma, \forall t \tag{1}$$

where $\underline{n}(\underline{x},t)$ is the unit outward normal to Γ in $\underline{x}$. Taking the coordinate system presented in figure 1 it follows that:

$$\underline{n}(\widehat{\underline{x}},t) = 0\underline{e_1}+1\underline{e_2}; \quad \underline{f}(\widehat{\underline{x}},t) = 0\underline{e_1}+0\underline{e_2}; \quad \underline{\underline{\sigma}}(\widehat{\underline{x}},t) = \sum_{i=1}^{2}\sum_{j=1}^{2}\left[\sigma_{ij}\left(\underline{e_i}\otimes\underline{e_j}\right)\right] \tag{2}$$

Since

$$\underline{\underline{\sigma}}(\widehat{\underline{x}},t)\underline{n}(\widehat{\underline{x}},t) = \left(\sum_{i=1}^{2}\sum_{j=1}^{2}\left[\sigma_{ij}\left(\underline{e_i}\otimes\underline{e_j}\right)\right]\right)\underline{e_2} \text{and} \ \left(\underline{e_i}\otimes\underline{e_j}\right)\underline{e_k} = \underline{e_i}\left(\underline{e_j}\cdot\underline{e_k}\right) \forall i,j,k; \quad \underline{e_i}\cdot\underline{e_j} = \left\{\begin{array}{l} 1, \text{if } i = j \\ 0, \text{if } i \neq j \end{array}\right\} \tag{3}$$

it comes that

$$\sigma_{12}(\widehat{\underline{x}},t) = \sigma_{22}(\widehat{\underline{x}},t) = 0 \forall t \tag{4}$$

2 Elasto-plastic constitutive equations

The following set of elasto-plastic constitutive equations proposed by [14] is adequate to model the cyclic inelastic behavior of metallic material at room temperature. Since the purpose of this paper is to present a simplified methodology to estimate the inelastic stress and strain histories at the root of the notch, only the one-dimensional version of the constitutive theory is discussed on this paper.

$$\sigma_{11}= E(\varepsilon_{11}-\varepsilon_{11}^{P}) \tag{5}$$

$$\dot{\varepsilon}_{11}^{P}= \dot{p}S_g; \quad \varepsilon_{11}^{P}(t=0)=0 \tag{6}$$

$$S_g=\begin{cases} +1 \text{ if } (\sigma_{11}-X)\geq 0 \\ -1 \text{ if } (\sigma_{11}-X)<0 \end{cases} \tag{7}$$

$$\dot{X}= a\dot{\varepsilon}_{11}^{P}-bX\dot{p}; \quad X(t=0)=0 \tag{8}$$

$$\dot{Y}= v_2(v_1+\sigma_y-Y)\dot{p}; \quad Y(t=0)=\sigma_y \tag{9}$$

$$\Phi= |\sigma_{11}-X|-Y\leq 0; \quad \dot{p}\geq 0 \;;\Phi\dot{p}=0; p(t=0)=0 \tag{10}$$

where $<x> = \max\{0,x\}$. σ_{11}, ε_{11} and ε_{11}^{P} are, respectively, the stress, strain and plastic strain components in the loading direction. p is the cumulated plastic strain, X is an auxiliary variable related to the kinematic hardening and Y is an auxiliary variable related to the isotropic. $E, \sigma_y, a, b, v_1, v_2$ are material constants. E is the Young modulus, σ_y is the yielding stress for a virgin material, a and b are parameters that characterize the kinematic hardening and v_1, v_2 are parameters that characterize the isotropic hardening. These equations will not be discussed in detail in the present paper because a detailed analysis can be found in [15]. Simple numerical techniques for solving these equations with prescribed stress or strain histories can be found in [1].

3 Analysis of stress concentration in inelastic plates

The goal of this paper is to present a simplified methodology to estimate the redistribution of stress caused by plastic flow at the root of the notch of a thin metallic plates undergoing a non monotonic tensile loading. Due to the one-dimensional state of stress at the root of the notch, it is possible to approximate the local stress and plastic strain histories from the solution of the elastic problem using a projection technique. The condition of validity of the method is that the plastic zone must remain sufficiently contained (surrounded by an elastic zone).

Generally, the non-zero stress component σ_E at the root of the notch calculated on the basis of elasticity theory is easily obtained using the so-called elastic geometric stress-concentration factor K_T, that can be found in tables:

$$\sigma_E= K_T\sigma_N= K_T\frac{F}{A_N} \varepsilon_E=\frac{\sigma_E}{E} \tag{11}$$

where σ_N is the "nominal stress" (the external tensile resultant force F divided by a given "nominal" area A_N) and ε_E is the elastic strain component. related to σ_E.

In this case, the study of the stress and plastic strain at the root of the notch can be reduced to the integration of the set of constitutive equations (5)-(10) considering a prescribed strain history. In order to understand how such simplification can be done, it is necessary to define a projection technique. There are basically two kinds of projections that can be used to approximate the stress and strain components at the root of the notch from σ_E and ε_E: the Linear Rule and the Neuber Rule. Those rules are often used in simple problems and a detailed discussion about then can be found in [16]. We strongly suggest the use of such approximated method only if the strain at the root of the notch is smaller than 0.01 (1%). It is important to remark that, at this strain level, plastic oligocyclic fatigue phenomena may occurs if the loading is non-monotonic. For a value of strain between 0.01 and 0.05 (1% to 5%) it is possible to use the theory but expression (11) may be inaccurate due to the large displacements. For bigger values of the strain, a non-linear theory that accounts for large deformations should be used.

4 Analysis of stress concentration in inelastic plates using Linear rule

The linear projection technique assumes that the strain component ε_{11} at the root of the notch is always equal to ε_E, regardless the material behavior, hence

$$\varepsilon_{11}(t) = \varepsilon_E(t) = \frac{\sigma_E(t)}{E} = \frac{K_T\sigma_N(t)}{E} = \frac{K_T F(t)}{E} \tag{12}$$

If the resultant external tensile force F(t) applied on the plate is known for all time instant t, it is possible to study the inelastic problem at the notch solving equations (5)-(10) with a prescribed strain history $\varepsilon_{11}(t)$ given by (12). Figure 2 shows schematically the main ideas of the linear projection technique for a monotonic loading and figure 3 shows the main ideas of the linear projection technique for a non-monotonic loading

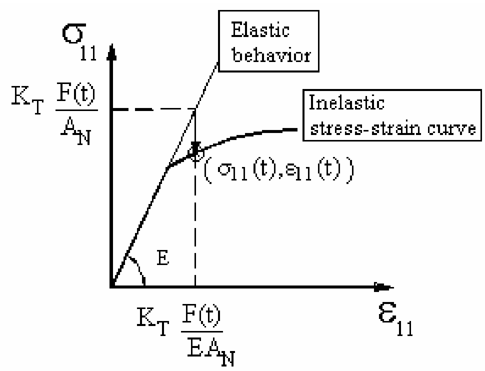

Figure 2: Linear projection technique for a monotonic loading.

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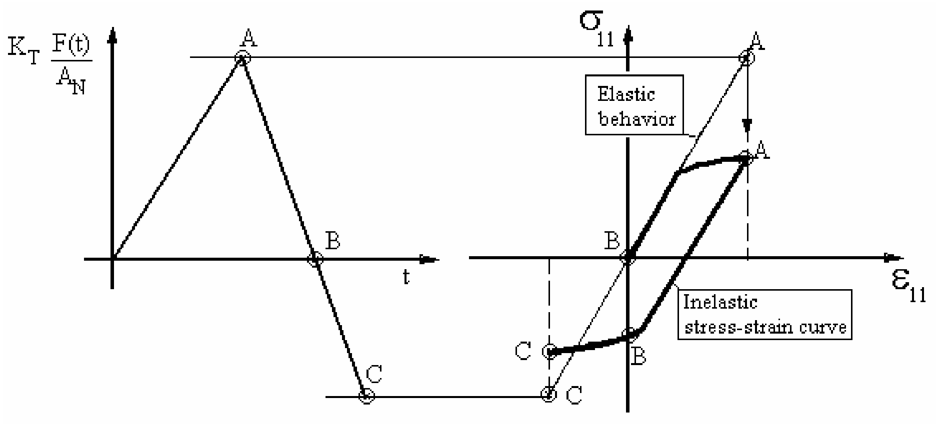

Figure 3: Linear projection technique for a non-monotonic loading.

5 Analysis of stress concentration in inelastic plates using Neuber rule

For a monotonic loading, the Neuber projection technique assumes that the product of the stress and strain components ($\sigma_{11}\varepsilon_{11}$) at the root of the notch is independent of the plastic flow, hence

$$\sigma_{11}(t)\varepsilon_{11}(t) = \sigma_E(t)\varepsilon_E(t) = \frac{\sigma_E^2(t)}{E} = \frac{(K_T\sigma_N(t))^2}{E} = \frac{1}{E}\left[\frac{K_T F(t)}{A_N}\right]^2 \tag{13}$$

Since $\sigma_{11}(t) = E\,(\varepsilon_{11}(t) - \varepsilon_{11}^p(t))$ it comes that

$$E\,(\varepsilon_{11}(t) - \varepsilon_{11}^p(t))\,\varepsilon_{11}(t) = \frac{(K_T F(t))^2}{EA_N} \Rightarrow (\varepsilon_{11}(t))^2 - \varepsilon_{11}^p(t)\varepsilon_{11}(t) - \left[\frac{K_T F(t)}{EA_N}\right]^2 = 0$$

The above equation has two roots that, for a fixed t, gives the intersection of the hyperbola defined in (13) with the line defined by $\sigma_{11}(t) = E\,(\varepsilon_{11}(t) - \varepsilon_{11}^p(t))$. Since the hyperbola has two branches, the adequate root must be chosen considering the sign of the force F(t)

$$\varepsilon_{11}(t) = \frac{\varepsilon_{11}^p(t) + \sqrt{(\varepsilon_{11}^p(t))^2 + 4\left[\frac{K_T F(t)}{EA_N}\right]^2}}{2} \text{if} F(t) > 0; \varepsilon_{11}(t) = \frac{\varepsilon_{11}^p(t) - \sqrt{(\varepsilon_{11}^p(t))^2 + 4\left[\frac{K_T F(t)}{EA_N}\right]^2}}{2} \text{if} F(t) < 0 \tag{14}$$

If the resultant external tensile force F(t) applied on the plate is known for all time instant t, it is possible to study the inelastic problem at the notch by solving equations (5)-(10) with a strain history $\varepsilon_{11}(t)$ given by (14). Figure 4 shows schematically the main ideas of the linear projection technique for a monotonic loading. The approximate local inelastic solution is given by the intersection of the experimental curve with the hyperbola corresponding to the elastic solution (σ_E, ε_E).

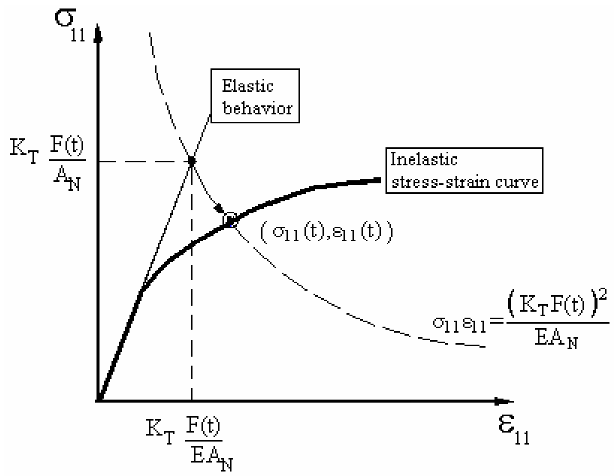

Figure 4: Neuber projection technique for a monotonic loading.

For cyclic loading, the method can be generalized branch by branch. Since the unloading is elastic, the same construction is repeated from the last loading point by using a change of axis (see figure 5)

$$(\sigma_{11}(\mathrm{t}) - \sigma_0)(\varepsilon_{11}(\mathrm{t}) - \varepsilon_0) = (\sigma_{\mathrm{E}}(\mathrm{t}) - \sigma_{\mathrm{E0}})(\varepsilon_{\mathrm{E}}(\mathrm{t}) - \varepsilon_{\mathrm{E0}}) =$$

$$= \frac{(\sigma_{\mathrm{E}}(\mathrm{t}) - \sigma_{\mathrm{E0}})^2}{\mathrm{E}} = \frac{(\mathrm{K_T}\sigma_{\mathrm{N}}(\mathrm{t}) - \mathrm{K_T}\sigma_{\mathrm{N0}})^2}{\mathrm{E}} = \frac{1}{E}\left[\frac{\mathrm{K_T F(t)} - \mathrm{K_T F_0}}{\mathrm{A_N}}\right]^2 \tag{15}$$

6 Comparison with finite element simulation

The goal of this section is to compare the approximate local inelastic solution with finite element simulations. It is considered a flat bar with transverse hole in axial tension as shown in figure 6 with d=20mm, w=100mm, h =5 mm. In this case the elastic stress concentration factor Kt is equal to 2.5. To make the analysis simpler, only elasto-plastic behavior with linear kinematic hardening is considered ($b=v_2=v_1=0$). The other material parameters are E = 70000 MPa, a=26600 MPa. A Poisson's ratio $\nu = 0.3$ is considered. The results using the projection techniques were obtained using the program CICLO-plast.

A two-dimensional four-nodes isoparametric element was used in the finite element simulations. The mesh was successively refined in order to give an error smaller than 0.5% in the elastic stress concentration factor K_T. In the analysis, the FEM prevision is considered closer to the real solution of the problem.

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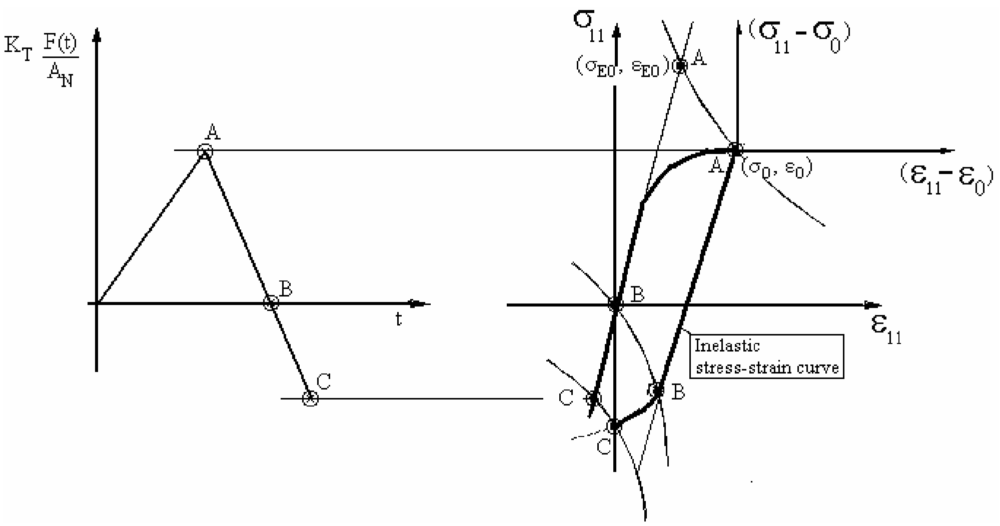

Figure 5: Neuber projection technique for a non-monotonic loading.

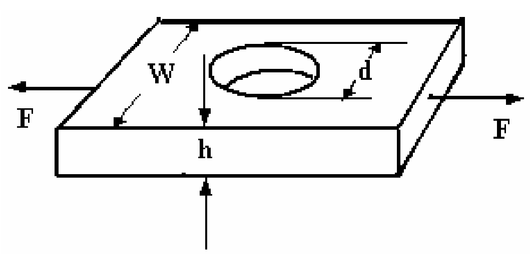

Figure 6: Flat bar with transverse hole in axial tension.

Figures 7 to 12 show the stress-strain curves for nominal stress σ_N ranging from zero to a maximum value max σ_N (max σ_N= 120 MPa in figure 7, max σ_N= 140 MPa in figure 8, max σ_N= 150 MPa in figure 9, max σ_N= 175 MPa in figure 10, max σ_N= 185.5 MPa in figure 11, max σ_N= 225 MPa in figure 12). The maximum plastic strain computed using Neuber rule are always bigger than the obtained by FEM simulations (in this sense it can be said that the Neuber rule is always conservative). The simulations using the linear rule are not always conservative, depending on the magnitude of the plastic strain. When the maximum plastic stress computed by FEM is smaller than 0.01 (0.1%), both projection techniques are conservative as it can be seen in figure 7, but the linear rule gives better approximations. The FEM simulations and the simulations using the are very close if the maximum plastic strain is around 0.01 (0.1%) as shown in figure 8. The maximum plastic strain computed using linear rule is smaller than the obtained by FEM if the strain obtained by FEM is bigger than 0.01 (0.1%) as it can be seen in figures 9 to 12. Nevertheless, the maximum plastic strain obtained from the linear rule is closer the obtained by FEM than the obtained from Neuber rule if the maximum plastic strain is smaller than 0.005 (0.5%). Finally, It is important to remark that the residual compressive stress and the associated plastic strain after unloading obtained from Neuber rule are always closer to the FEM predictions than the obtained from linear rule.

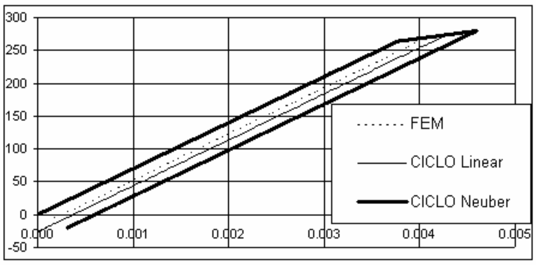

Figure 7: Stress-strain curve at the root of the notch for σ_N ranging from zero to 120MPa.

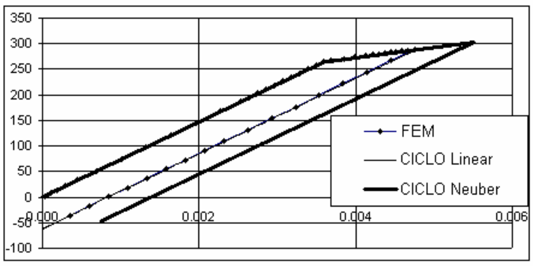

Figure 8: Stress-strain curve at the root of the notch for σ_N ranging from zero to 140MPa.

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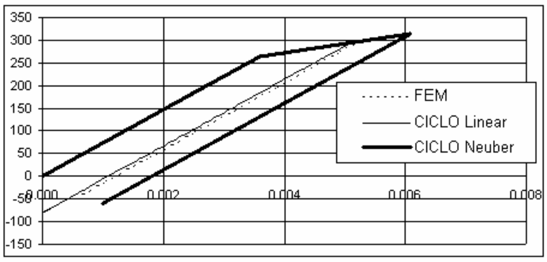

Figure 9: Stress-strain curve at the root of the notch for σ_N ranging from zero to 150MPa.

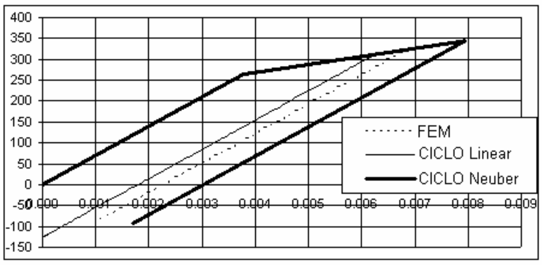

Figure 10: Stress-strain curve at the root of the notch for σ_N ranging from zero to 175 MPa.

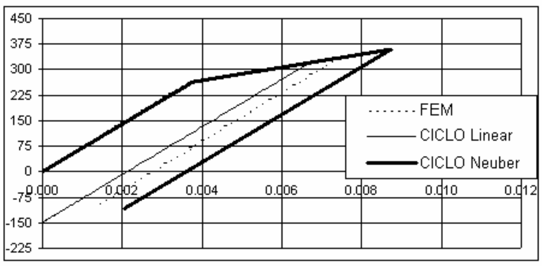

Figure 11: Stress-strain curve at the root of the notch for σ_N ranging from zero to 187.5MPa.

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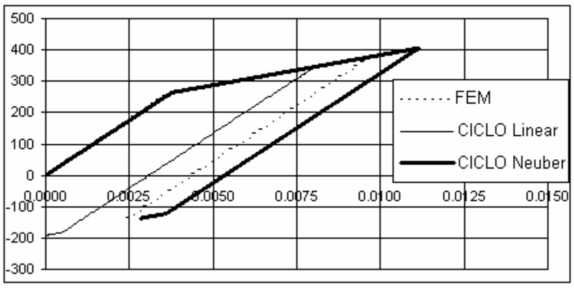

Figure 12: Stress-strain curve at the root of the notch for σ_N ranging from zero to 225 MPa.

Figure 13 shows the stress-strain curve at the root of the notch for a fully reversed cycle of the normal stress σ_N with amplitude equal to 225 MPa. Buckling was not considered in the simulation.

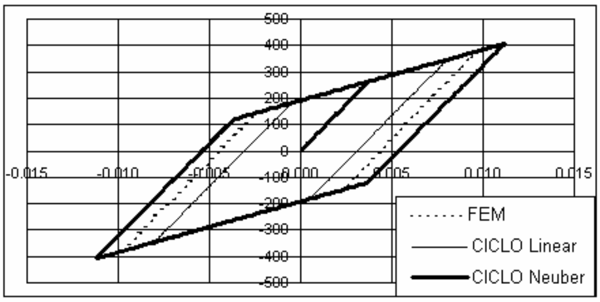

Figure 13: Stress-strain curve at the root of the notch for a fully reversed cycle of the normal stress σ_N with amplitude 225 MPa.

7 Final remarks

The methodology proposed on this paper provides a simple but adequate tool for the stress concentration analysis in thin elasto-plastic plates under tension. The use of projection techniques allows to reduce the stress concentration analysis to the simulation of a one-dimensional problem of elasto-plasticity with prescribed strain history. The linear projection technique always gives a lower bound of the maximum plastic strain at the root of the notch while Neuber rule allows to obtain an upper bound . It is important to remark that the simulation of a large number of cycles using the algorithms proposed on this paper takes few seconds (eventually much less than one second) while an accurate simulation using finite element can take a few hours in a PC. Finite element simulation should only be used if global information about the stress and plastic strain fields on plate must be obtained.

Such a methodology can be extended easily to account for elasto-viscoplastic behavior [9]. In this case it is only necessary to replace equation (10) by

$$VISCOPLASTICITY \Rightarrow \qquad \dot{\mathrm{p}} = \left\langle \frac{\Phi}{K} \right\rangle^{\mathrm{N}}; \mathrm{p}(\mathrm{t}=0) = 0$$

where <x> = max{0,x}. It can also be extended to study the stress concentration in inelastic bars and shafts under flexion [12].

Programs (CICLO-Plast, CICLO-Visc) to automatically compute the stress-strain curve at the root of the notch for elasto-plastic and elasto-viscoplastic plates using that methodology for different kinds of notches and complex loading histories can be free-downloaded in http://www.lmta.mec.uff.br. Also a program (COEFS-ciclo) to automatically identify the material parameters from one single cyclic test can be downloaded at this site.

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