An inverse analysis applied to a viscoelastic constitutive model

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Abstract

The present work is aimed at modeling and characterizing viscoelastic materials. A constitutive equation for viscoelastic materials, in time domain, is proposed based on the concepts of internal variables and the thermodynamics of irreversible processes. The proposed constitutive equation is capable of dealing with common viscoelastic behavior such as creep and relaxation phenomena. Once one has chosen the parsimony of the model, a finite element model of the system, which is parameterized by a set of constitutive parameters, is built. The performance and features of the proposed constitutive modeling is assessed by means of an inverse formulation, which also provides an adequate framework to obtain the constitutive parameters and eventually propose optimal designs involving damped structures. A number of specific analysis is carried out and their results are compared to experiments.

Keywords: viscoelasticity, constitutive equation, Levenberg-Marquardt

1 Introduction

Modeling plays a crucial role in controlling and optimizing engineering applications by providing means of better understanding the involved phenomena and on improving the capacity of predicting the dynamic response. An interesting example is provided by vibrating devices, which often use viscoelastic components to enhance damping levels. Optimal design of such vibration control systems requires accurate modeling, specially in what concerns the viscoelastic dynamic behavior.

The dissipation mechanisms inherent to those materials are tied to chemical micro-structure and, therefore, a viscoelastic constitutive equation could be derived from a multiscale perspective. The main drawback of this approach would be the computational effort that a multiscale computation takes, which can lead to prohibitive costs for analyzing real applications. Taking this in consideration several phenomenological models, which leads to an interesting balance between accuracy and computational cost, have been proposed in order describe the viscoelastic behavior in terms of macro variables [1, 2].

The present work is aimed at modeling and characterizing the dynamic behavior of viscoelastic materials. A constitutive equation in the time domain is proposed based on the concepts of internal

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variables and the Thermodynamics of Irreversible Processes [1]. The proposed constitutive equation is capable of dealing with standard time viscoelastic response such as creep and relaxation. More complex phenomena can also be reproduced but their relationship with the model parameters is not straightforward. This represents an important drawback for optimal design and operation tuning.

Indeed, exploring the model dynamics and the role of its parameters represent crucial steps towards the use of simulation as an effective engineering tool. In the present work this is accomplished trough inverse formulations [3], which constitute a means of obtaining the parameters values from experimental data and provides a rational framework for understanding the connection between parameters and dynamic response as well.

The constitutive parameters required to describe the dynamic behavior of the viscoelastic material are estimated by means of the solution of the associated inverse problem which was formulated in frequency domain. The inverse problem has been solved by means of the Levenberg-Marquardt technique [4]. The effectiveness of the proposed approach has been evaluated through experimental data obtained out of a viscoelastic sandwich beam.

2 Modeling

The dynamic response of a viscoelastic body submitted to a level of excitation corresponding to small deformations is considered. Therefore, the main aspect of the mechanical modeling relies upon the constitutive equations described bellow.

Aiming at proposing a constitutive equation for viscoelastic materials the Thermodynamics of Irreversible Processes have been considered along with the concept of internal variables [5] and [1]. Considering small strain thermomechanical process, the free energy function ψ and the pseudo-potential of dissipation φ were chosen as follows

$$\psi(\boldsymbol{\varepsilon}(\mathbf{x},t),\boldsymbol{\xi}_1(\mathbf{x},t),\ldots,\boldsymbol{\xi}_I(\mathbf{x},t)) = \frac{1}{2\rho}\,[E\,\boldsymbol{\varepsilon}(\mathbf{x},t)\cdot\boldsymbol{\varepsilon}(\mathbf{x},t) + \sum_{r=1}^{I} E_r\,(\boldsymbol{\varepsilon}(\mathbf{x},t)-\boldsymbol{\xi}_r(\mathbf{x},t))\cdot(\boldsymbol{\varepsilon}(\mathbf{x},t)-\boldsymbol{\xi}_r(\mathbf{x},t))] \tag{1}$$

$$\varphi(\dot{\boldsymbol{\varepsilon}}(\mathbf{x},t),\dot{\boldsymbol{\xi}}^1(\mathbf{x},t),\ldots,\dot{\boldsymbol{\xi}}^I(\mathbf{x},t)) = \frac{1}{2\rho}\,[\eta\,\dot{\boldsymbol{\varepsilon}}(\mathbf{x},t)\cdot\dot{\boldsymbol{\varepsilon}}(\mathbf{x},t) + \sum_{r=1}^{I}\eta_r\,\dot{\boldsymbol{\xi}}^r(\mathbf{x},t)\cdot\dot{\boldsymbol{\xi}}^r(\mathbf{x},t)] \tag{2}$$

where ρ is the specific mass, E,E_1,...,E_I and η,η_1,...,η_I are constitutive material parameters, $\boldsymbol{\varepsilon}$ is the total strain tensor and $\boldsymbol{\xi}_1$,...,$\boldsymbol{\xi}_I$ are the internal variables tensors. Once the free energy function ψ and the pseudo-potential of dissipation φ had been chosen one can obtain the constitutive equation by means of the fulfillment of the Clausius-Duhem Inequality. Therefore, the constitutive equation renders as follows

$$\boldsymbol{\sigma}(\mathbf{x},t) = E\,\boldsymbol{\varepsilon}(\mathbf{x},t) + \sum_{r=1}^{I} E_r\,(\boldsymbol{\varepsilon}(\mathbf{x},t)-\boldsymbol{\xi}_r(\mathbf{x},t)) + \eta\,\dot{\boldsymbol{\varepsilon}}(\mathbf{x},t) \tag{3}$$

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$$\dot{\boldsymbol{\xi}}_r(\mathbf{x}, t) = b_r\,(\boldsymbol{\varepsilon}(\mathbf{x}, t) - \boldsymbol{\xi}_r(\mathbf{x}, t)), \quad r = 1, \ldots, I \tag{4}$$

where $\boldsymbol{\sigma}$ is the stress tensor and the parameter b_r is defined as the inverse of the relaxation time as follows

$$b_r = \frac{E_r}{\eta_r} \tag{5}$$

It should be emphasized that the constitutive equations (3) and (4) should be able to reproduce some common dynamic behaviour of viscoelastic materials such as creep and relaxation. The ability of these constitutive equations to reproduce such phenomena can be shown through some mathematical manipulations of equations (3) and (4) [5]. The physical meaning of the constitutive parameters E, E_r and b_r, $r = 1, \ldots, I$, can be easily understood by means of the the stress relaxation response of a one-dimensional system whose constitutive equation is given by equations (3) and (4). Such a stress relaxation response is shown in equation (6)

$$\sigma(t) = E\,\epsilon_0 \left[1 + \sum_{r=1}^{I} \Delta_r\, e^{-b_r\, t}\right] \tag{6}$$

where Δ_r is defined as the ratio of E_r and E. From equation equation (6) one can conclude that Δ_r and b_r are associated to the magnitude of relaxation and to the inverse of the relaxation time of the r-th internal variable. Another aspect that should be highlighted is the fact that such constitutive equations are able to reproduce the behaviour of a viscoelastic material whose loss factor is approximately uniform over a certain frequency range.

3 Inverse analysis

The key features of the constitutive modeling are assessed within the context of an inverse analysis. This is carried out by combining experimental data, forward modeling involving Finite Elements and parameter estimation relying upon the Levenberg-Marquardt method [4]. The parameter estimation explores both time and frequency domain data [6].

The system under analysis is a sandwich beam whose core is made of a viscoelastic material and whose sketch is shown in Figure (1). The base layer and the constraining layer are made of aluminium and the core of the sandwich is a viscoelastic tape produced by $3M$. The specification of the tape is 4950.

The length of the beams is $1.46\,m$. The system is instrumented with four piezoelectric accelerometers ($PCB\,SN$ 13575) placed at 1/4, 2/5, 1/2 and 3/4 of the beam length and with an electromechanical shaker collocated with accelerometer number one. The first layer, called base layer, is the only one which is connected to the support as shown in Figure(1). The base layer is hinged at both ends.

As a mathematical model of the system shown in Figure(1) is required to be used in the estimation process, a finite element model of this system was built [5]. This finite element model takes the constitutive equations (3) and (4) into account and the kinematics that was adopted for this model is shown in figure (2).

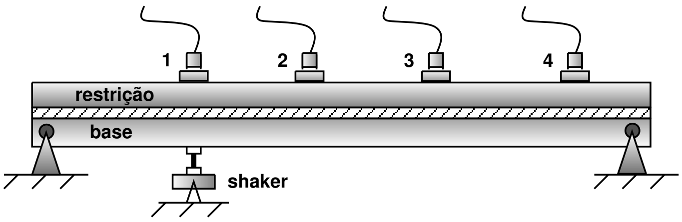

Figure 1: Sketch of the viscoelastic sandwich beam.

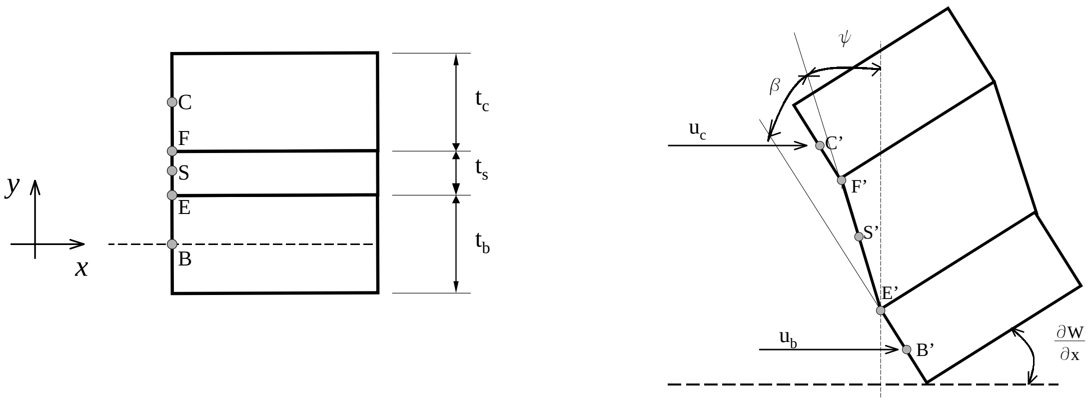

Figure 2: Finite element model.

The estimation process considers the FRF of the first and second accelerometers within the band $(0-100)Hz$, containing 200 points each. The first analysis considers the viscoelastic model containing one internal variable and the parameters have been denoted as follows:

$$G = p_1 \times 10^6 \tag{7}$$

$$G_1 = p_2 \times 10^6 \tag{8}$$

$$b_1 = p_3 \tag{9}$$

In order to guide the inverse formulation, it was considered a simple test to determine the order of magnitude of parameter G. The test that has been performed considered a sandwich beam similar to the under analysis but with an elastic core whose first three natural frequencies were evaluated for a set of values of G. It was concluded that these three natural frequencies are close to the information contained in the FRFs for values of G within (3.5,5.5) MPa. Such information was important to determine the initial guesses for G and G_r. Unfortunately the authors did not have a specific pretest for determining a suitable range of initial guesses for b_1. Three different initial guesses were tested, namely: $\mathbf{p}^{(0)} = \{5, 1, 1\}^T$, $\mathbf{p}^{(0)} = \{5, 1, 10\}^T$ and $\mathbf{p}^{(0)} = \{3, 3, 1\}^T$. All of them converged to the estimated vector $\hat{\mathbf{p}} = \{1.58, 10.82, 652.6\}^T$. Figure (3) graphs the experimental and estimated frequency response functions of the accelerometers 1 and 2 respectively.

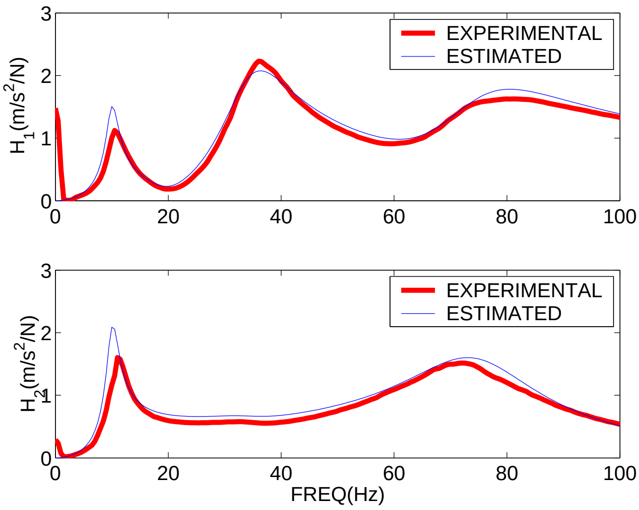

Figure 3: Experimental an estimated FRFs for accelerometers 1 and 2.

From figure (3) one may conclude that the estimated and the experimental FRFs are in agreement. The L_2 norm of the difference of the experimental and estimated FRFs of the accelerometers 1 and 2 are 12.45 and 8.88, respectively.

In order to evaluate the role played by different internal variables it is considered a different model for the viscoelastic core whose dynamics is assumed to be described by two internal variables. The vector on unknown parameters may be defined as follows

$$G = p_1 \times 10^6 \tag{10}$$

$$G_1 = p_2 \times 10^6 \tag{11}$$

$$G_2 = p_3 \times 10^6 \tag{12}$$

$$b_1 = p_4 \tag{13}$$

$$b_2 = p_5 \tag{14}$$

It was used the experimental FRFs of the accelerometers 1 and 2 and the initial guess was chosen as follows $\mathbf{p}^{(0)} = \{2, 2, 2, 1, 1\}^T$. The estimated parameter vector is $\mathbf{p}^{(0)} = \{0.469, 2.14, 12.6, 59.95, 1090\}^T$. Figure (4) show the experimental and estimated FRFs of acceleromters 1 and 2, respectively. One can clearly see from figure (4) that the level of agreement between the estimated FRFs and the experimental ones is higher than the one shown in figure (3). The L_2 norm of the difference of the experimental and estimated FRFs of the accelerometers 1 and 2 for this 2 internal variable model are 11.88 and 7.14, respectively.

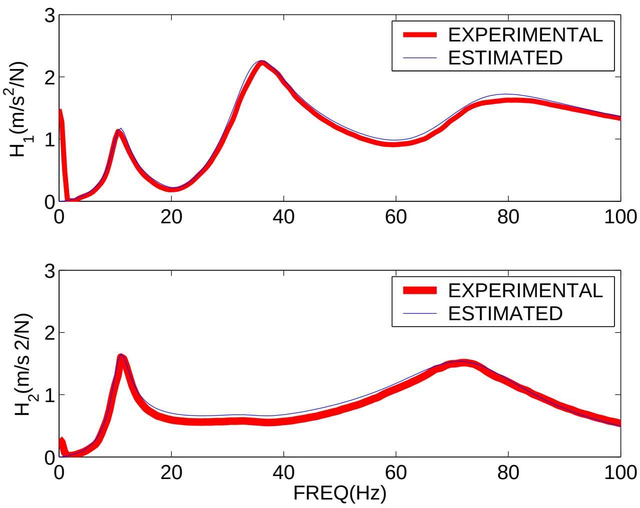

Figure 4: Experimental an estimated FRFs for accelerometers 1 and 2.

Aiming at validating the provided results it is considered a new set of experimental data. The first validation considers time response of the first and third acceleromters when the system is excited with a sine-chirp sweeping the band (0,100) Hz. The validation for accelerometer number one is shown in figure (5) and for accelerometer number three is shown in figure (6). The responses graphed in figures (5) and (6) are in favor of the estimated parameters. Although the estimated responses provided by the one internal variable model and by the two internal variable model seems to be quite similar in figures (5) and (6) the model which best describes the system is the one which contains two internal variables. Such a conclusion can be obtained out of the comparison between figures (3) and (4) and by the H_2 norm of the differences between the experimental and estimated FRFs for these two models.

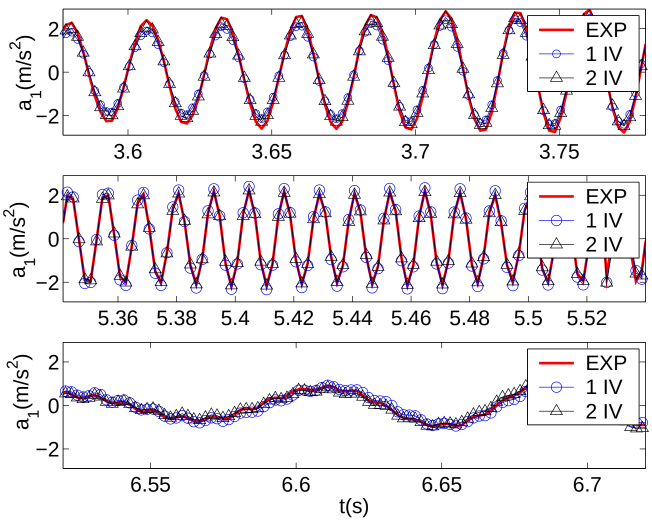

Figure 5: Experimental and estimated time responses measured by accelerometer number one for a sine chirp excitation.

4 Final remarks

The present work introduced a internal variable based constitutive equation to describe the dynamical behavior of viscoelastic materials. This constitutive equation is linear and it seems that it is able to describe common viscoelastic behavior such as creep and relaxation. The parameters that characterize the constitutive equation have been estimated by means of the classical Levenberg-Marquardt technique.

The suitability of the proposed constitutive equation has been assessed on a set of experimental data out of a sandwich beam whose core is made of viscoelastic material. The inverse problem has been formulated in frequency domain and it has been used the frequency response function of two of the accelerometers within the band (0,100) Hz. As a means of validating the estimative obtained for the parameters, it has been been used the time domain response of two accelerometers due to a sine chirp excitation. The validation step showed agreement between the experimental and the predicted response.

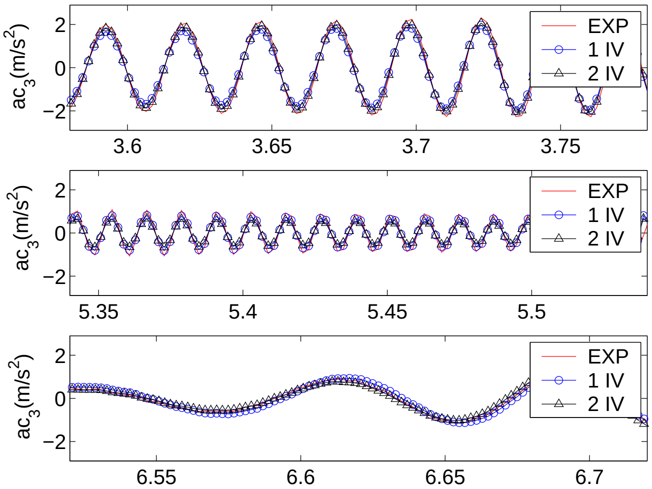

Figure 6: Experimental and estimated time responses measured by accelerometer number three for a sine chirp excitation.

The contribution of this work is to propose a constitutive equation for linear viscoelastic materials in time domain and provide a rational framework for obtaining the model parameters. As this constitutive equation is defined in time domain it is straightforward to build a time domain mathematical model of the system after the estimation of the constitutive parameters. Such a time domain model can be used to simulate the dynamical behavior of the system under different environmental conditions.

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