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LOW DIMENSIONAL MODELS FOR VORTEX FLOW DESCRIPTIONS

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Abstract In this work we describe the application of the Proper Orthogonal Decomposition technique, combined with the Galerkin projection, to model a convected non viscous vortex immersed in an uniform flow. The obtained results show that the reduced model is adequate to describe the dynamics of this kind of flow. An important sensitivity of the long time agreement with the experimental data is significantly influenced by the space and time discretizations. The application of this technique with experimental data issued form PIV should in consequence be carefully analyzed.

Keywords: *Low dimensional models*

1- Introduction

The reduction of the Navier-Stokes equation to a system of ordinary differential equation (ODE) has been largely studied by the Computational Fluid Dynamics (CFD) community. With different tools developed in this domain, it has been possible to reproduce or predict diverse flow characteristics with a large detail.

From some years ago, different research efforts are trying to tackle the same problem but with a less ambitious goal with the so called "low dimensional dynamic models" (LODS) or "reduced models" (Rajaee 1996, Rempfer 2000, Ravindran 2000). The purpose of them is to recover just the essential characteristics of the flow (coherent structures and their dynamics) sacrificing in this attempt to gain in simplicity the detailed structure of the flow.

In contraposition to the CFD, the formulation of the ODE system with these models requires to know a set of the flow fields issued from experiments. With this data first an eduction of the coherent structures is undertaken (identification of modes with larger energy) and then a system of ODE describing its dynamics is obtained.

Some scenarios where these models are of interest are in flow control applications where it is required simple systems to delineate the strategies of actuation, in shape optimization problems where it is of interest to avoid the repetition of DNS calculations or in experimental fluid mechanics where it is required to improve the temporal resolution of flow fields measurements obtained with Particle Image Velocimetry techniques.

Eduction of coherent structures:The Proper orthogonal decomposition

The Proper Orthogonal Decomposition (POD) has been used by different authors (see for instance Berkooz 1993) as technique to obtain approximate descriptions of the large scale or coherent structures of turbulent flows. This technique is a powerful and elegant method for the analysis of data with that purpose.

If we denote by *(,)* the usual inner product of vector fields defined in $L^2(U)$ where U represents the spatial domain occupied by the flux, which we will suppose to be bounded, given a set of velocity fields $u_j(x,t)$ obtained experimentally, belonging to M discrete times, POD provides M basic functions $\hat{O}_j(x)$, which are optimal respect to the ability of represent the kinetic energy of the flux.

These functions are the eigenfunctions of the spatial correlation tensor. Because this tensor is symmetric and semi-positive definite, the eigenfunctions can be obtained mutually orthogonal, and ordered in correspondence with the magnitude of the eigenvalue $ë_j$ associated to each one of them.

Each eigenvalue $ë_j$ quantifies the occurrence of the mode $\hat{O}_j(x)$, or also, the kinetic energy that is present in it because it is verified that

$$l_j =< \left|(u_j,\Phi_j)\right|^2 >$$

where <,> denotes temporal average.

It is interesting to observe that the modes $\hat{O}_j(x)$ are built by an appropriate superposition of the velocities, the same that were considered from experimental measurements or by numerical simulations of a system, with non correlated coefficients. This is,

$$\Phi_j(x) = \sum_{k=1}^{M} a_{kj}(k) u_k(x,k)$$

where

$$< a_k, a_l >= l_k d_{kl}.$$

The variant snapshot of the POD let us to determine the coefficients of the superposition as the components of the j-eigenvector of the temporal correlation matrix *C* such that

$$C_{ij} = \frac{1}{M}(u_i, u_j)$$

It is verified that this tensor is symmetric and semi-positive definite, and also $ë_j$ is the eigenvalue associated to a_j.

The most outstanding characteristic of the POD is its optimality because it provides an efficient way to capture the dominant components of an infinite dimensional process with only a finite number of modes.

From the modal decomposition, it is possible to consider a truncated model with s modes to approximate the velocity field *u(x,t),* with . $x \in U, 1 \le t \le M, s << M$

Formulation of the low order dynamic system (LODS):POD- Galerkin projection

A method to convert a partial differential equation (PDE) system in a system of ordinary differential equation (ODE) is the Galerkin projection.

According to this procedure, the functions which define the original equation are projected on a finite dimension subspace of the phase space (in this case, the subspace generated by the first *s* modes)

To proceed with the Galerkin projection, it is possible to consider either the vorticity equation or the Navier-Stokes equation. In both cases the results are systems of ODE of order 1 with quadratic terms that are solved for the corresponding initial condition, and the solutions of these systems are the temporal modes *aj(t).*

Objectives of the work

The reduced model may be obtained with "artificial" experimental data issued from analytical solutions.

By doing this a comparison of the performance of the method to recover the dynamics of the large scale structures is possible not only at the "known" discrete time of the experiment but also at any intermediate time.

Different parameters that are intrinsic to the technique (truncation criteria, solver characteristics, derivative approximation …) and other associated to the experiments itself (signal to noise ratio, temporal and spatial discretization,…) influence the ability of the model to recover the flow characteristics (Rajae 1994).

It is the objective of this work, to analyze the influence of the parameters associated with the experiments on the flow reconstruction for the case of a convected vortex.

2- Problem Description

For the description of the problems we adopt an eulerian formulation. With this formulation the flow of a convected isolated vortex is non-steady and formally the Galerkin-POD technique is not possible to be used. However if the sequence of similar experiments gives a quasi-steady character to the flow (v.g. no time-dependence of mean value of the flow field) an extension of the technique is possible (Holmes 1996).

The Convected vortex Problem

The flow of the vortex is described by the equation (Saffman 1996):

$$u(\vec{r},t) = \frac{K}{2p} \frac{\breve{k}\ x(\vec{r}-\vec{R})}{\left|(\vec{r}-\vec{R})\right|^2} + u_f(\vec{r},t)$$

with $\bar{r}$ the position vector, *K* a constant that represents the vortex intensity, $\bar{R}$ is the coordinate of the vortex axis at different times and $\bar{k}$ is the versor normal to the plane of analysis. The vortex is convected with a velocity in a fixed frame of reference

$$\frac{d\vec{R}}{dt} = u_f(\vec{r},t)$$

We analyze convection with a horizontal uniform velocity (from left to right in the figures) and the vortex appears in the domain at about the middle of the height.

The sequence of experiments consists on a set of vortex that gets inside the bidmensional domain of analysis at a coordinate that is quite close for the different vortex of the set. They are separated enough so as to neglect any interaction between them.

Physical situations

The problem corresponds to a physical situation of a vortex with a core not exceeding the size of the spatial grid (*sg*). This vortex is convected in a domain of lateral size *b* fixed so as to verify $b >> K/u_f$. The convection velocity must also satisfy:

$$u_f >> \frac{u\,b}{sg^2}$$

where õ is the cinematic coefficient of viscosity

Characteristic times of the problems are

$$Tc_1 = \frac{b}{uf}$$

The non-dimensional numbers that may be used to characterize situation with different vortex intensities and convection velocities is

$$P_1 = K/(b\ u_f)$$

3-Results

We have analyzed the case where $P_1=1$ with a characteristic time Tc_1= 0.8. The experimental data is obtained with the above cited analytical expression. To simulate experimental noise we impose a perturbation on the velocity field obtained by adding a random function on the horizontal component of the position vector. The level of noise imposed has been calculated considering a perturbation with a maximum value of 10%.

The number of experiments considered has been defined so as to assure at least 5 sequences of experiments that correspond to 5 passages of isolated vortex in the domain under study. Convergence is verified by observing that the double of the number of experiments does not modify results in more than 5%.

3.1 Reconstruction of the "experimental data" after truncation

In this section we analyze the reconstruction of the data with which we obtain the spatial modes of the POD after the truncation of the modes with less energy is performed. As we consider experiments with a poor resolution on the time domain in consequence we adopt for the proper orthogonal decomposition in the snapshot variant.

The Figure 1 shows the distribution of kinetic energy for the different modes. As we can see about 90 % of the total kinetic energy is kept with the first ten modes.

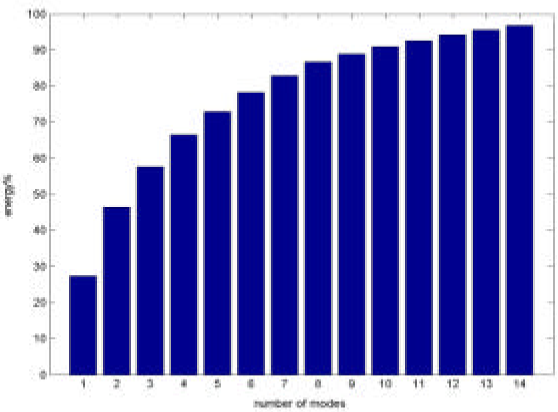

Figure 1: Typical Cumulative Kinetic energy vs. mode number: *sg=b/32, ÄT=Ä t/Tc1=0.125, M=50*

The criteria adopted for mode truncation is to keep the modes that concentrate more than 95 % of the energy. As a result we will consider only the first 14 modes.

As we can see from figure 2 the different spatial modes correspond to flow configurations quite similar to a set of aligned vortex of alternate sign. As the number of modes increases the number of vortex increases giving rise to a flow configuration for the higher modes with small velocities outside the alignment (the trajectory of the axis of the convected vortex).

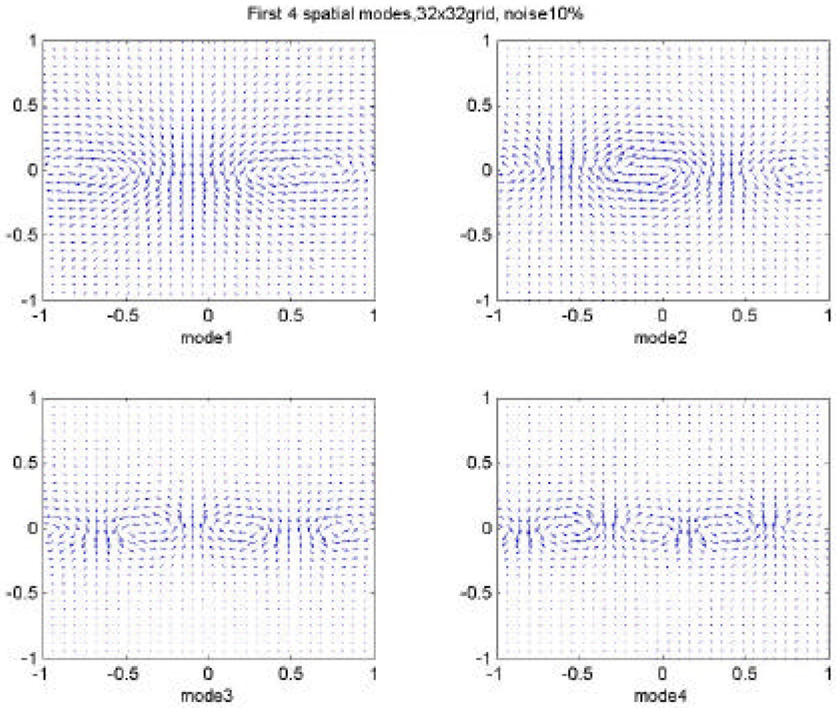

Figure 2: Topology of the spatial modes, *sg=b/32, ÄT=Ä t/Tc1=0.125 , M=50*

Figure 3 is a reconstruction at a given instant of the flow fields after a truncation of the higher modes has been performed. The “experimental data” for the same time is also superposed to compare the performance of the decomposition after truncation to recover the essential part of the flow. As we can see the reconstruction is quite satisfactory and only slight differences have been observed close to the vortex axis.

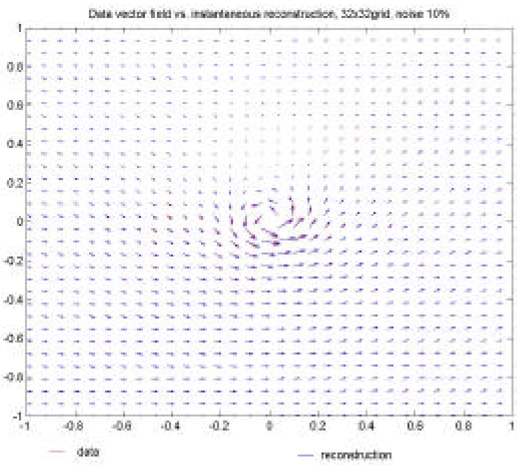

Figure 3: Typical Velocity field reconstruction at a given instant, *sg=b/32, ÄT=Ä t/Tc1=0.125 , M=50*

Influence of the grid size

The influence of the space discretization may be observed by comparing Figure 4, Figure 5 and Figure 6. The results are expressed in terms of the divisions undertaken of the lateral size of the field of view *b*. In figure 4 and figure 6 the space grid is respectively twice and the half of the one of figure 5.
As we can observe reconstruction is quite satisfactory in all cases and the influence of this parameter in the range tested is not very important.

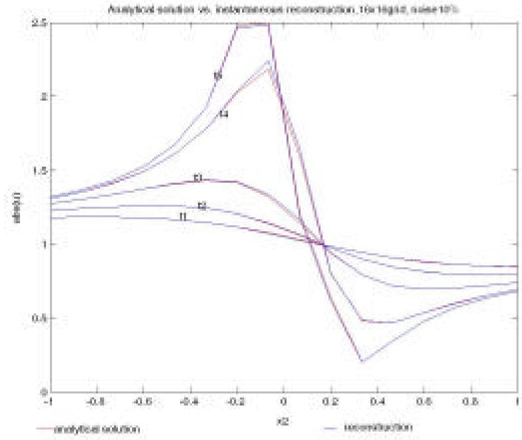

Figure 4: Velocity field reconstruction at different times. X1=-0.1615, *sg=b/16, ÄT=Ä t/Tc1=0.125 , M=50*

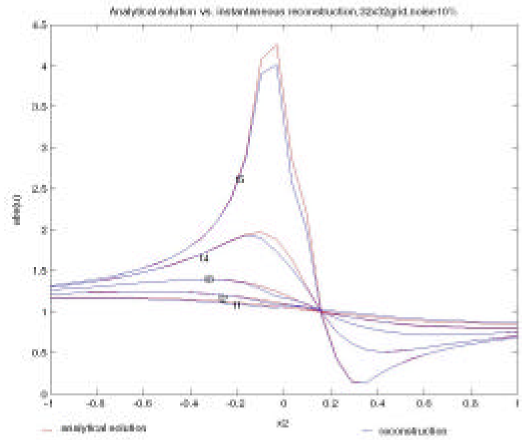

Figure 5: Velocity field reconstruction at different times. X1=-0.1615, *sg=b/32, ÄT=Ä t/Tc1=0.125, M=50*

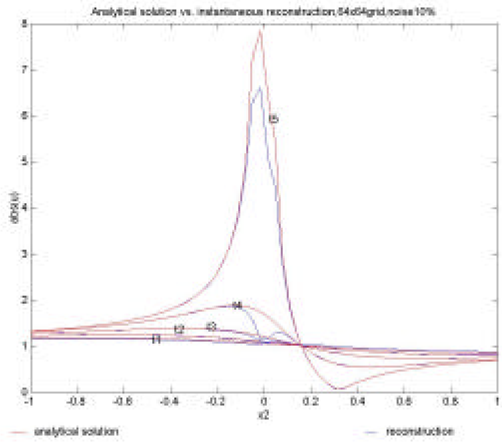

Figure 6: Velocity field reconstruction at different times. X1=-0.1615, *sg=b/64, ÄT=Ä t/Tc1=0.125 , M=50*

Influence of the time between experiments

The influence time discretization may be observed by comparing Figure 5, Figure 7 and Figure 8. In these figures the time interval of figure 5 has been shortened twice and four times respectively. Results are shown as a function of a dimensionless time by dividing this variable with the characteristic time. As we can observe the refinement of the temporal discretization does not largely improves the reconstruction or the eduction of coherent structures in the range of intervals tested.

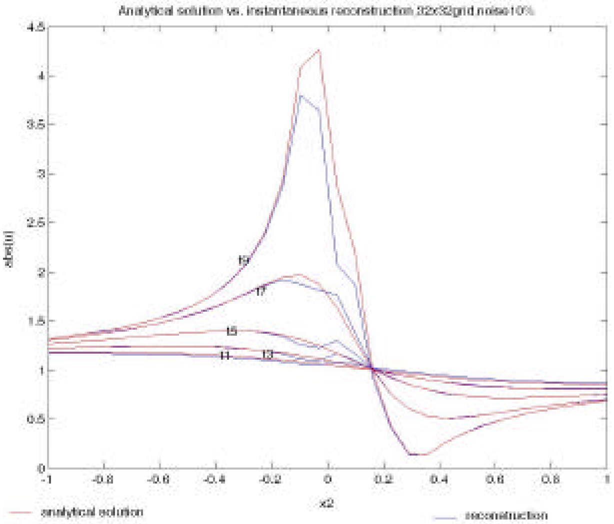

Figure 7: Velocity field reconstruction at different times. . *X1=-0.1615, sg=b/32, ÄT=Ä t/Tc1=0.062,* M=100

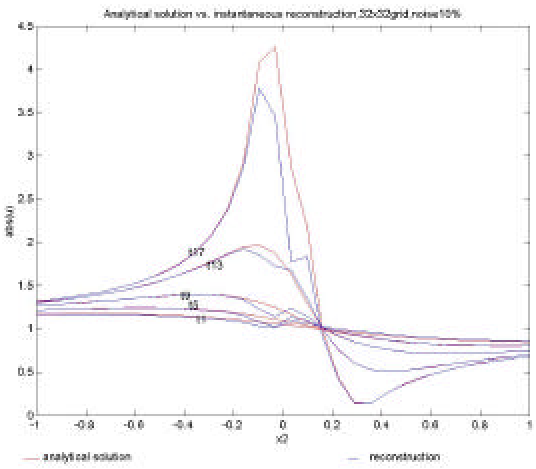

Figure 8: Velocity field reconstruction at different times. *X1=-0.1615, sg=b/32, ÄT=Ä t/Tc1=0.031,* M=165

3.2 Prediction of the LODS

In this section we analyze the performance of the reduced model to predict the behaviour of the flow at instants intermediate to those of the "experiments". The solution of the system of ODE consists on the determination of the temporal coefficients $a_j(t)$ associated with the spatial modes obtained by the POD technique. The solver considered utilizes a Runge-Kutta of 4^{th} order method.

In Figure 9 we show graphs where these coefficients are obtained by solving the ODE. In the same graph are represented the values of these coefficients corresponding to the instants where the "experimental data" has been obtained and the time domain represented is restricted to the one corresponding to the passage of only one isolated vortex through the spatial domain.

In Figure 10 we show polar graphs of some of these coefficients and the temporal coefficients of the experimental data. The experimental data corresponds to the different sequences of experiments (the whole time domain) and they appear almost superposed with those of the first sequence meanwhile the model results in these graphs correspond to only one sequence.

As we observe on both figures the predictions at times close to the initial are satisfactory but as the time increases a degradation of the model to predict data appears. The agreement of the predicted temporal coefficients with the experimental data on the polar graphs is not completely satisfactory but the shape of these curves is in some sense recovered by the model.

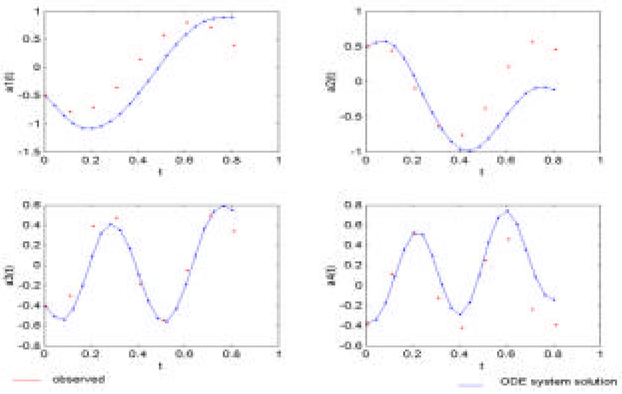

Figure 9: Prediction of Temporal coefficients as a function of time, *sg=b/32,ÄT=Ä t/Tc1=0.125, M=50*

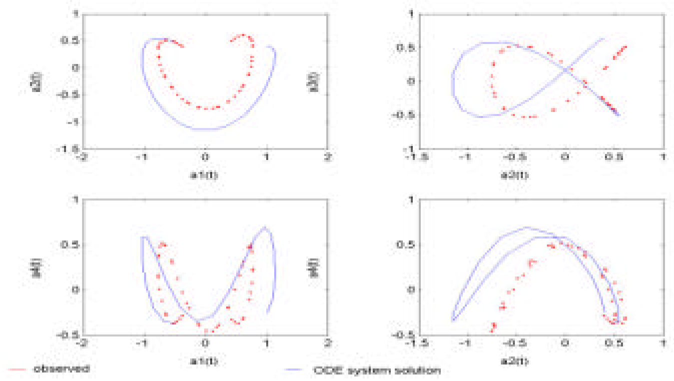

Figure 10: Polar graphs of the temporal coefficients, *sg=b/32,ÄT=Ä t/Tc1=0.125, M=50*

A picture of the velocity field predictions with these coefficients may be observed in Figure 11. On this figure we observe that the essential behavior of the flow is recovered by the model and that differences are significant only close the vortex axis position specially at the long time behavior.

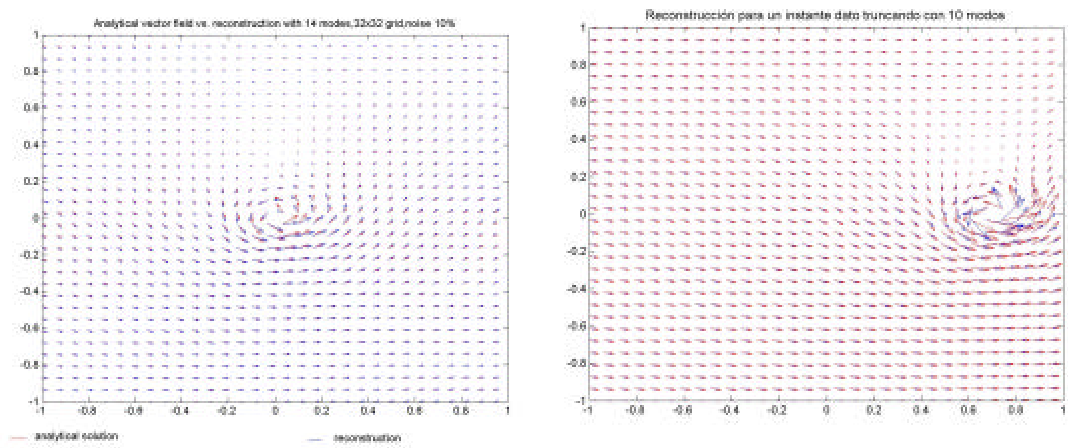

Figure 11: Velocity field reconstruction at given instants, *sg=b/32, ÄT=Ä t/Tc1=0.125, M=50*

Influence of the grid size

The grid size is determinant on the proper evaluation of the spatial derivatives used by the LODS. A comparison of figures 10, 12 and 13 enables to analyze the incidence of this variable in our problem. As it is observed a refinement of the mesh largely improves the predictions on the polar graphs restraining the prediction of these coefficients to a region closer to the experimental data. The coarser grids even that they may be useful for eduction of the coherent structures of the flow are able to capture correctly only the initial behavior.

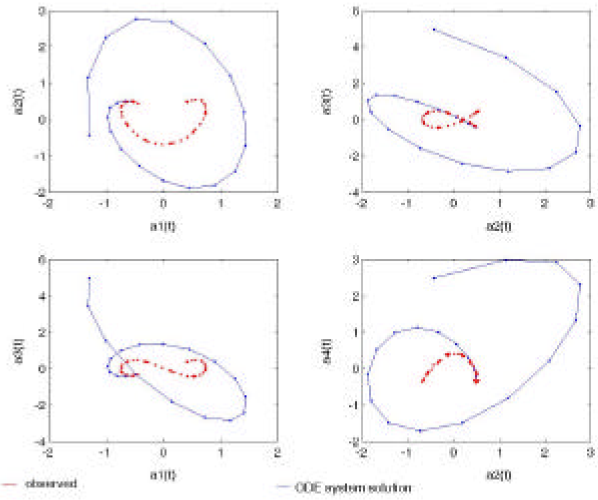

Figure 12: Polar graphs of the temporal coefficients, *sg=b/16, ÄT=Ä t/Tc1=0.125, M=50*

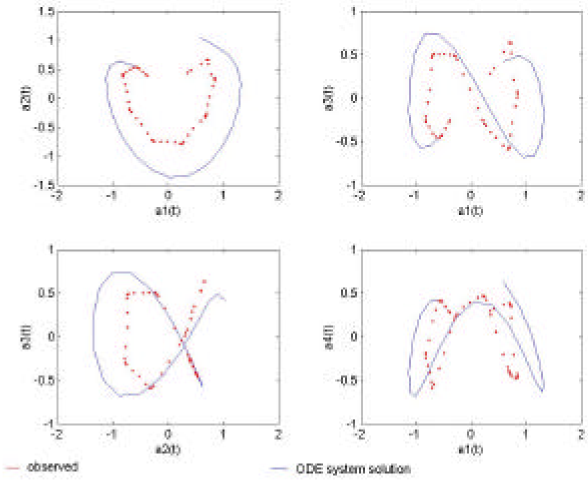

Figure 13: Polar graphs of the temporal coefficients. *sg=b/64, ÄT=Ä t/Tc1=0.125, M=50*

Influence of the time between experiments

The influence of the time between experiments is of significance on the proper evaluation of the time derivatives used by the LODS. A comparison of figures 10, 14 and 15 enables to analyze the incidence of this variable in our problem.

It is observed that the initial behavior is in general well recovered in all cases but the long time behavior is better reproduced when the temporal discretization is refined.

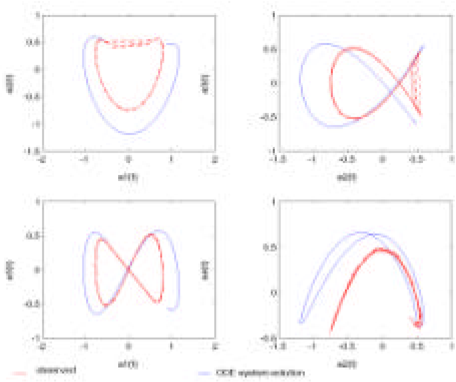

Figure 14: Polar graphs of the temporal coefficients, . *sg=b/32, ÄT=Ä t/Tc1=0.062,* M=100

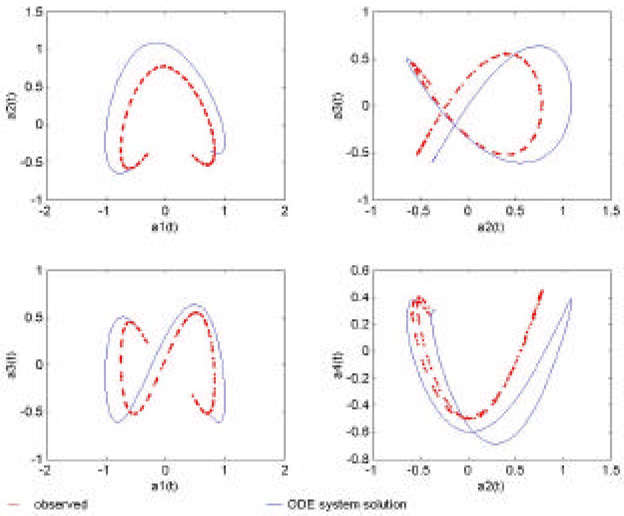

Figure 15: Polar graphs of the temporal coefficients, *sg=b/32, ÄT=Ä t/Tc1=0.031, M=165*

4 -CONCLUSIONS

In this work we analyze for the case of a convected vortex the influence of the quality of the experimental data on the ability of the reduced models obtained by POD-Galerkin technique. In general, this technique reveals as a good tool to recover the essential fluid dynamic behavior.

As a summary it is observed that in the range tested of time and space discretization the eduction of coherent structures and reconstruction of the essential flow fields is not largely affected. On the contrary it is observed that predictions with LODS at long times are significantly influenced by the refinement of the spatial mesh and of time interval.

It is expected that experimental data issued from "real experiments" as those that can be obtained by Particle Image technique in similar situation should exhibit the same behaviour. In consequence, flow field measurements with vortex convection should be carefully analyzed if this technique is desired to be applied to "complete" the experimental data between two consecutive flow fields.

Finally, the polar graphs or phase space graphs may be used to determine if the low order dynamic system obtained has a divergent, trivial or limit cycle behavior. In our case for a fixed signal to noise ratio, temporal and spatial discretization the behavior is determined by the non dimensional numbers *P1* Future work should determine the real incidence of these parameters on the system behavior.

6. Acknowledgments

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7. References

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