VIBRATION ANALYSIS OF TIMOSHENKO ROTATING BEAMS

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Abstract: *The aim of this work is to study the vibration of Timoshenko rotating beams. An analytical solution and a Finite Element solution were proposed. The equations of motion were obtained from Lagrange equations. The gyroscopic effect according Timoshenko has been considered. In order to obtain the analytical solution and the mass, gyroscopic effect and stiffness matrices, Mathematica^{â} software was used. A computer software written in Fortran^{â} assembles the beam's global matrices and simulates the response, according to the givens dimensions, material, angular velocity and unbalancing. Results are presented, discussed and compared.*

Key-Words: *Mechanical Vibrations, Timoshenko Beams, Finite Elements, Gyroscopic Effect*

1. INTRODUCTION

Design of rotating equipments that operate under various conditions of load and speed has been a challenge to designers. Every rotating equipment has always some unbalancing, so there will always be a harmonic load and the designers have to obtain the critical speed of system, and compare with the operating range, to make sure the motion is stable at the operating condition. Timoshenko theory was first developed to study non rotating beams, and it has been extended in this work, to take into account the rotations effects. This work includes the gyroscopic effect according Timoshenko at the kinetic functional. The rotating beam is simply supported at both ends.

1.1 - Problem Formulation

Considering the beam in bending, Fig. 1.1, where $y(x,t)$ is the transverse displacement, $f(x,t)$ is the transverse force per unit length, $m(x)$ is the mass per unit length, E is the Young modulus, $A(x)$ is the area, $I_a(x)$ is the area moment of inertia e $J_a(x)$ is the mass moment of inertia of the cross sectional area.

According to the free body diagram of a beam element of length dx, Fig. 1.2, $M(x,t)$ is the bending moment, $Q(x,t)$ is the shear transverse force, $b(x,t)$ is the deformation due the bending and $z(x,t)$ is the deformation due the transverse shear

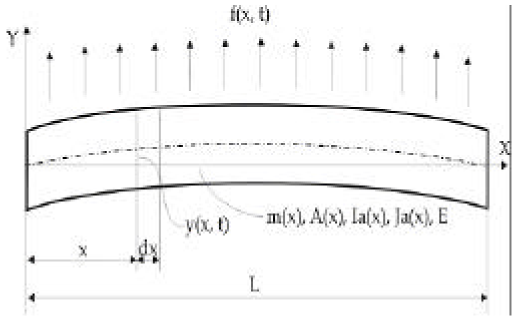

Fig. 1.1 - Beam in bending

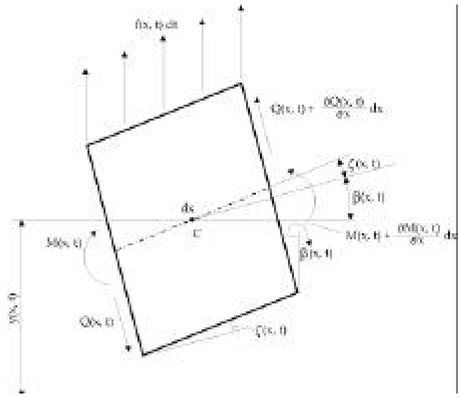
Fig. 1. 2 - Free body diagram according Timoshenko

Considering

$$\frac{\partial y(x,t)}{\partial x} = b(x,t) + z(x,t) \tag{1.1}$$

$$M(x,t) = EI_a(x)\frac{\partial b(x,t)}{\partial x} \tag{1.2}$$

$$Q(x,t) = k'GA(x)z(x,t) \tag{1.3}$$

where k' is a numeric factor depending on the shape of the cross section and G is the shear modulus.

The energy functionals are

$$T = \frac{1}{2}\cdot\int_0^L m(x)\cdot\left(\frac{\partial y(x,t)}{\partial t}\right)^2 dx + \frac{1}{2}\cdot\int_0^L J_a(x)\cdot\left(\frac{\partial b(x,t)}{\partial t}\right)^2 dx \tag{1.4}$$

$$V = \frac{1}{2}\cdot\int_0^l E\cdot I_a(x)\cdot\left(\frac{\partial b(x,t)}{\partial x}\right)^2 dx + \frac{1}{2}\cdot\int_0^l k'\cdot G\cdot A(x)\cdot\left(\frac{\partial y(x,t)}{\partial x} - b(x,t)\right)^2 dx \tag{1.5}$$

Yokoyama (1988) proposed a solution model to Timoshenko beams clamped at a rotating shaft, presenting a Finite Element that includes a stiffness matrix related to the angular velocity. According to Yokoyama (1988), without this stiffness matrix, the other matrices are the same obtained by Archer (1965).

1.2 - The Gyroscopic Effect

Timoshenko (1918) showed, using the principle of angular momentum, that, under certain conditions, not only the unbalancing forces should be taking in account when obtaining the critical speeds of rotating beams. Then

$$M_y = \frac{d}{dt}\left(J_a\frac{db}{dt} - I_p\Omega g\right) \tag{1.6}$$

2. ANALYTICAL SOLUTION

Considering the rotating beam as a continuous system, Fig. 2.1.

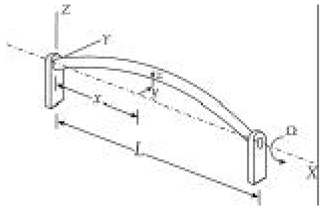

Fig. 2.1- Rotating beam in bending

2.1 – Obtaining the Equations of Motion

The equations of motion will be obtained from Lagrange Equations (Meirovitch, 1997):

$$\frac{\partial \hat{L}}{\partial q_i} - \frac{\partial}{\partial x}\left(\frac{\partial \hat{L}}{\partial q_i'}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial \hat{L}}{\partial q_i''}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}}{\partial \dot{q}_i}\right) + \frac{\partial^2}{\partial x \partial t}\left(\frac{\partial \hat{L}}{\partial \dot{q}_i'}\right) = Q_i \qquad 0 < x < L \qquad (2.1)$$

where

$$\hat{L} = \hat{T} - \hat{V} \qquad (2.2)$$

is the Lagrangian specific functional, $\hat{T}$ is the kinetic energy specific functional, $\hat{V}$ is the deformation energy specific functional, q_i are the generalized coordinates, Q_i are the generalized forces, x is the variable along the beam, and L is the rotating beam length. The over dots indicate time derivations and the primes indicate spatial derivation.

The equations of motion obtained will be for the transverse plane (YZ), Fig. 2.2.

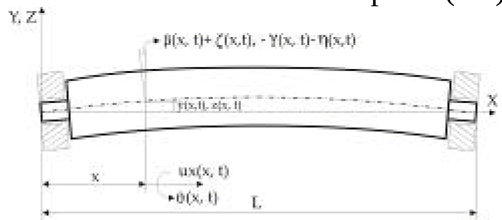

Fig 2.2 - Generalized coordinates

The transverse kinetic energy is the sum of the kinetic energy at Y and Z directions, and the kinetic energy in each direction is the sum of the kinetic energy due the translation, the rotatory inertia and the gyroscopic effect

The variation of the center of gravity position of any cross sectional can be showed at Figs. 2.3 e 2.4.

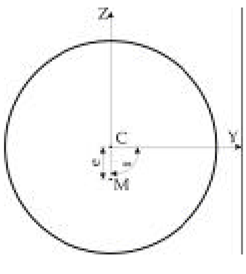

Fig. 2.3 - Cross sectional at rest

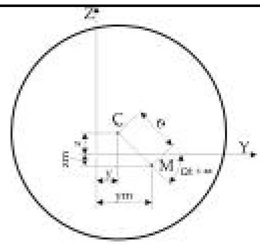

Fig. 2.4 - Cross sectional with angular velocity

where a is the angular initial position of the center of gravity M, "*e*" is the distance of the center of gravity M and the geometric center C, Ω is the angular velocity, y and z are the coordinates of the geometric center C. Only the Y direction equations will be showed

According to Meirovitch (Meirovitch, 1997):

$$T_{tyt} = \frac{1}{2}\cdot\int_0^L m(x)\cdot\left(\frac{\partial y_m(x,t)}{\partial t}\right)^2 dx \tag{2.3}$$

$$y_m(x,t) = y(x,t) + e(x)\cos(\Omega t + a(x)) \tag{2.4}$$

and $y_m(x,t)$ is the center of gravity displacements.

The kinetic energy functionals due the rotatory inertia is (RAYLEIGH, 1877):

$$T_{tyr} = \frac{1}{2}\cdot\int_0^L J_a(x)\cdot\left(\frac{\partial b(x,t)}{\partial t}\right)^2 dx \tag{2.5}$$

The kinetic energy functionals due the gyroscopic effect is (Timoshenko, 1918):

$$T_{tyg} = \frac{1}{2}\cdot\int_0^L I_p(x)\cdot\Omega\cdot(-g(x,t))\cdot\left(\frac{\partial b(x,t)}{\partial t}\right) dx \tag{2.6}$$

where $I_p(x)$ is the mass polar moment of inertia.

The transverse deformation energy is the sum of the deformations energy at Y e Z directions, and the deformation energy in each direction is the sum of deformations energy due the bending an transverse shear (Meirovitch, 1997):

$$V_{tyf} = \frac{1}{2}\cdot\int_0^l E\cdot I_a(x)\cdot\left(\frac{\partial b(x,t)}{\partial x}\right)^2 dx \tag{2.7}$$

$$V_{tyc} = \frac{1}{2}\cdot\int_0^l k'\cdot G\cdot A(x)\cdot\left(\frac{\partial y(x,t)}{\partial x} - b(x,t)\right)^2 dx \tag{2.8}$$

Considering an unbalancing $e(x)$, $A(x) = A$, $m(x) = m$, $I_a(x) = I_a$, $I_p(x) = I_p$, $J_a(x) = J_a$, the obtained equations of motion are

$$k'GA(y''(x,t) - b'(x,t)) - m\ddot{y}(x,t) = -m\Omega^2 e(x)\cos(\Omega t + a(x)) \tag{2.9}$$

$$k'GA(y'(x,t) - b(x,t)) - J_a\ddot{b}(x,t) + EI_a b''(x,t) + I_p\Omega\dot{g}(x,t) = 0 \tag{2.10}$$

2.2 - Solution for the Equations of Motion

Considering

$$e(x) = e\,\mathrm{sen}\left(\frac{p}{L}x\right) \tag{2.11}$$

$$a(x) = a \tag{2.12}$$

$$r = 1 \tag{2.13}$$

The solution can be expressed as:

$$y(x,t) = \mathrm{sen}\left(\frac{p}{L}x\right)\left(\sum_{i=1}^{4} C_i \cos(w_i t + f_i) + C_9 \cos(\Omega t + a)\right) \tag{2.14}$$

$$b(x,t) = \left(\frac{p}{L} + \frac{mLw^2}{k'GAp}\right)\cos\left(\frac{p}{L}x\right)\left(\sum_{i=1}^{4} C_i \cos(w_i t + f_i) + C_{10} \cos(\Omega t + a)\right) \tag{2.15}$$

$$\sqrt{\frac{mw^2}{2k'AG} + \frac{J_a w^2}{2EI_a} - \frac{I_p w\Omega}{2EI_a} \pm \frac{\sqrt{c_l}}{2k'GAEI_a}} = \frac{rp}{L} \qquad r = 1,2,... \tag{2.16}$$

$$c_l = \left(k'GAI_p w\Omega - k'GAJ_a w^2 - EI_a m w^2\right)^2 - 4k'GAEI_a\left(J_a m w^4 - k'GAmw^2 - I_p m w^3\Omega\right) \tag{2.17}$$

Equation (2.36) has been solved using Mathematica software (FALEIROS, 1998).

$$C_9 = \frac{emL^2\Omega^2\left(k'GAL^2 + EI_a p^2 + \left(I_p - J_a\right)L^2\Omega^2\right)}{-mL^2\Omega^2\left(EI_a p^2 + \left(I_p - J_a\right)L^2\Omega^2\right) + k'GA\left(EI_a p^4 - L^2\left(mL^2 + \left(J_a - I_p\right)p^2\right)\Omega^2\right)} \tag{2.18}$$

$$C_{10} = \frac{k'GAemL^3 p\Omega^2}{-mL^2\Omega^2\left(EI_a p^2 + \left(I_p - J_a\right)L^2\Omega^2\right) + k'GA\left(EI_a p^4 - L^2\left(mL^2 + \left(J_a - I_p\right)p^2\right)\Omega^2\right)} \tag{2.19}$$

One may observe that the denominator of C_9 e C_{10} are the same and will be equal to zero if

$$\Omega = \sqrt{\frac{-mL^2 EI_a p^2 - k'GAL^2\left(mL^2 + \left(J_a - I_p\right)p^2\right) + \sqrt{c_\Omega}}{2\left(I_p - J_a\right)mL^4}} \tag{2.20}$$

$$c_\Omega = L^4\left(4k'GAEI_a m p^4\left(I_p - J_a\right) + \left(mEI_a p^2 + k'GA\left(mL^2 + \left(J_a - I_p\right)p^2\right)\right)^2\right) \tag{2.21}$$

In this case one of the natural frequencies is equal to the angular velocity and the solution of (2.21) - (2.22) can be write as:

$$y(x,t) = \mathrm{sen}\left(\frac{p}{L}x\right)\left(\sum_{i=1}^{4} C_i \cos(w_i t + f_i) + \frac{e\Omega t}{2}\mathrm{sen}(\Omega t + a)\right) \tag{2.22}$$

$$b(x,t) = \left(\frac{p}{L} + \frac{mLw^2}{k'GAp}\right)\cos\left(\frac{p}{L}x\right)\left(\sum_{i=1}^{4} C_i \cos(w_i t + f_i) + \frac{e\Omega t}{2}\mathrm{sen}(\Omega t + a)\right) \tag{2.23}$$

It is possible to see that the displacements and the rotations increase without bound as the time is increased, and the resonance occurs. So for this case of unbalancing and considering only the first vibration mode, the angular velocity from Eq. (2.20) can be considered the first critical angular velocity for this rotating beam.

3. SOLUTION USING FINITE ELEMENTS

The equations of motion will be obtained from Lagrange equations either. The kinetic and deformation energy functionals are the same used at the analytical solution.

3.1 – Obtaining the equations of motion

The behavior of any generalized coordinate inside the element is a function of the nodal displacements and the shape functions (BISMARCK, 1993):

$$q(x,t) = \sum_{i=1}^{n} f_i(x) q_i(t) \tag{3.1}$$

where $q_i(t)$ are the nodal displacements and $f_i(x)$ are the shape functions. The Lagrange equations can be simplified as (Meirovitch, 1997):

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \tag{3.2}$$

where T is the kinetic energy, V is the deformation energy and Q_i are the generalized forces.

The beam element used, Fig. 3.1, has two nodes and two degree of freedom per node, as illustrated in Fig. 3.1.

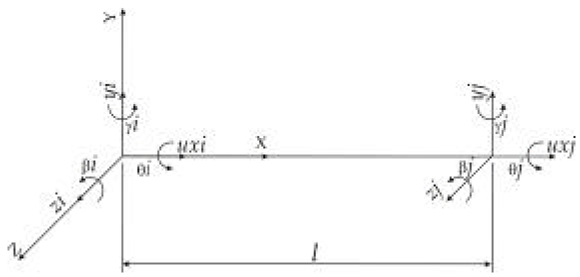

Fig. 3.1 - Finite Element Degrees of Freedom

The shape functions can be found in (Yokoyama, 1988).

The differentiations of Lagrange Equations were made using Mathematica® software (BRITO, 1999). Grouping the accelerations, velocities and displacements coefficients, the equations of motion can be written in matrix form:

$$[M]\cdot\{\ddot{q}\} + [G]\cdot\{\dot{q}\} + [K]\cdot\{q\} = \{Q\} \tag{3.3}$$

$$\{q\} = \begin{bmatrix} y_1 & b_1 & z_1 & g_1 & y_2 & b_2 & z_2 & g_2 \end{bmatrix}^T \tag{3.4}$$

$$[M]=\begin{bmatrix} My(1,1) & My(1,2) & 0 & 0 & My(1,3) & My(1,4) & 0 & 0 \\ My(2,1) & My(2,2) & 0 & 0 & My(2,3) & My(2,4) & 0 & 0 \\ 0 & 0 & Mz(1,1) & Mz(1,2) & 0 & 0 & Mz(1,3) & Mz(2,4) \\ 0 & 0 & Mz(2,1) & Mz(2,2) & 0 & 0 & Mz(2,3) & Mz(2,4) \\ My(3,1) & My(3,2) & 0 & 0 & My(3,3) & My(3,4) & 0 & 0 \\ My(4,1) & My(4,2) & 0 & 0 & My(4,3) & My(4,4) & 0 & 0 \\ 0 & 0 & Mz(3,1) & Mz(3,2) & 0 & 0 & Mz(3,3) & Mz(3,4) \\ 0 & 0 & Mz(4,1) & Mz(4,2) & 0 & 0 & Mz(4,3) & Mz(4,4) \end{bmatrix} \tag{3.5}$$

$$[G]=\begin{bmatrix} 0 & 0 & Gy(1,1) & Gy(1,2) & 0 & 0 & Gy(1,3) & Gy(1,4) \\ 0 & 0 & Gy(2,1) & Gy(2,2) & 0 & 0 & Gy(2,3) & Gy(2,4) \\ Gz(1,1) & Gz(1,2) & 0 & 0 & Gz(1,3) & Gz(1,4) & 0 & 0 \\ Gz(2,1) & Gz(2,2) & 0 & 0 & Gz(2,3) & Gz(2,4) & 0 & 0 \\ 0 & 0 & Gy(3,1) & Gy(3,2) & 0 & 0 & Gy(3,3) & Gy(3,4) \\ 0 & 0 & Gy(4,1) & Gy(4,3) & 0 & 0 & Gy(4,3) & Gy(4,4) \\ Gz(3,1) & Gz(3,2) & 0 & 0 & Gz(3,3) & Gz(3,4) & 0 & 0 \\ Gz(4,1) & Gz(4,2) & 0 & 0 & Gz(4,3) & Gz(4,4) & 0 & 0 \end{bmatrix} \tag{3.6}$$

$$[K]=\begin{bmatrix} Ky(1,1) & Ky(1,2) & 0 & 0 & Ky(1,3) & Ky(1,4) & 0 & 0 \\ Ky(2,1) & Ky(2,2) & 0 & 0 & Ky(2,3) & Ky(2,4) & 0 & 0 \\ 0 & 0 & Kz(1,1) & Kz(1,2) & 0 & 0 & Kz(1,3) & Kz(2,4) \\ 0 & 0 & Kz(2,1) & Kz(2,2) & 0 & 0 & Kz(2,3) & Kz(2,4) \\ Ky(3,1) & Ky(3,2) & 0 & 0 & Ky(3,3) & Ky(3,4) & 0 & 0 \\ Ky(4,1) & Ky(4,2) & 0 & 0 & Ky(4,3) & Ky(4,4) & 0 & 0 \\ 0 & 0 & Kz(3,1) & Kz(3,2) & 0 & 0 & Kz(3,3) & Kz(3,4) \\ 0 & 0 & Kz(4,1) & Kz(4,2) & 0 & 0 & Kz(4,3) & Kz(4,4) \end{bmatrix} \tag{3.7}$$

The M_{yt}, M_{yr}, K_{yf}, K_{yc}, can be found at (Yokoyama, 1988)

$$[Gy]=\frac{I_p\Omega}{30l[1+\Phi]^2}\begin{bmatrix} 36 & l(3-15\Phi) & -36 & l(3-15\Phi) \\ -l(3-15\Phi) & -l^2(4+5\Phi+10\Phi^2) & l(3-15\Phi) & l^2(1+5\Phi-5\Phi^2) \\ -36 & -l(3-15\Phi) & 36 & -l(3-15\Phi) \\ -l(3-15\Phi) & l^2(1+5\Phi-5\Phi^2) & l(3-15\Phi) & -l^2(4+5\Phi+10\Phi^2) \end{bmatrix} \tag{3.8}$$

3.2 - Solution for the Equations of Motion

With the element's matrices, it is possible to obtain the global rotating beam matrices and simulate the response according initial conditions and loading. To carry out this simulation, a computer software was written in Fortran® using the Newmark Method of Direct Integration. According BATHE e WILSON (1976), an integration method is called direct when no transformation at the equations of motion is made before executing the numerical integration.

4. RESULTS

In this work, two methods for solving the dynamic response of a rotating beam according Timoshenko Theory were proposed, an analytical and a Finite Element method.

Simulations were performed for a steel cylindrical rotating beam with: Diameter = 0.025m, Length = 0.15m, k' = 0.75, Poisson = 0.3, Number of Elements = 20, e=0.00125m e a=-p/2, during 0.002 seconds.

Considering the angular velocity as 13618rad/s, the following natural frequencies for the first vibration mode are obtained (rad/s):

Table 4.1 - Natural Frequencies

Frequency [rad/s]	without gyroscopic effect	with gyroscopic effect
? 1	13618	13414+2E-10*i*
? 2		13824-2E-10*i*
? 3	459119	445914-7E-12*i*
? 4		472739+7E-12*i*

The trajectories for the middle point of rotating beam obtained by the analytical solution and by the Finite Element Method for the first vibration mode are showed. In each simulation the beam will be rotating with an angular velocity of 50%, 95%, 150% and 200% from the first natural frequency without gyroscopic effect (13618 rad/s). The trajectories with null and fist critical angular velocity are also presented. For the F.E.M an 1×10^{-5} increment of time was used.

5. CONCLUSIONS

The results obtained by the analytical solutions shows the occurrence of two natural frequencies for each vibration mode of a simply supported non-rotating beam. With the gyroscopic effect, the results present four natural frequencies for each vibration mode. The characteristic equation roots are complex numbers. One notices that the gyroscopic effect produces a deviation upwards and downwards for each natural frequency of the non-rotating beam.

The present work proposes the solution of the problem, includes a specific case of unbalancing, and a critical angular velocity is obtained. This critical angular velocity is different from the first natural frequency of a non-rotating beam, but is the same as the real part of one of the natural frequencies of the beam rotating at this angular velocity. Due to the unbalancing, if the beam is rotating with an angular velocity equal to the critical, resonance occurs. Near the critical angular velocity, the beating phenomenon occurs. The results obtained by the F.E.M reveal a good agreement in comparison with the analytical solution

In future works, numerical methods for obtaining the eigenvalues and eigenvectors can be developed, to check the higher frequencies.

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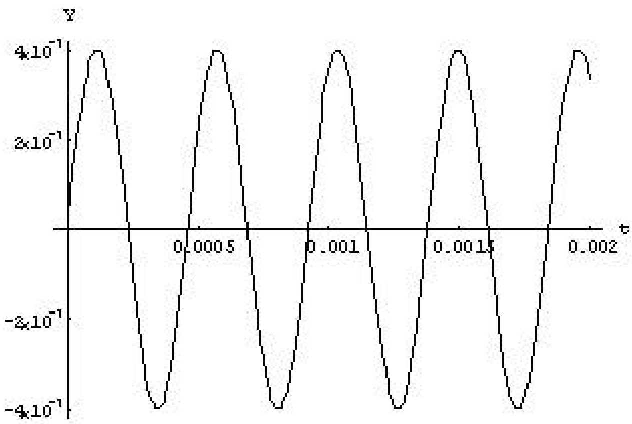

Fig. 4.1 - Y Displacement, O=0, analytical solution

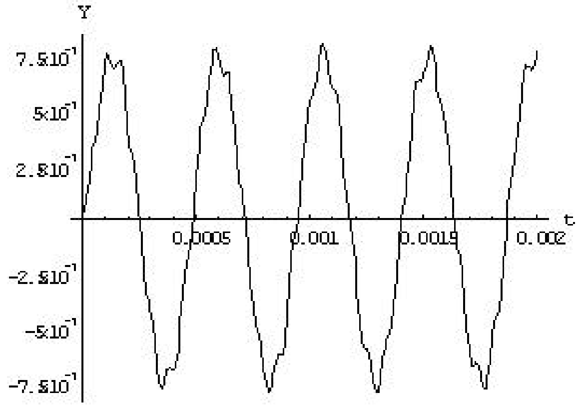

Fig. 4.2 - Y Displacement, O=0, F.E.M.

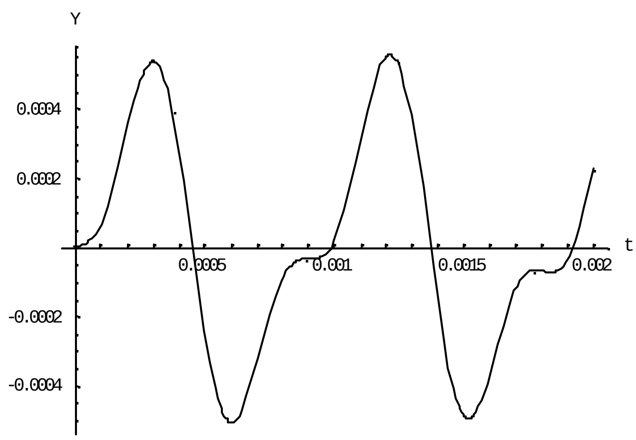

Fig. 4.3 - Y Displacement, O=50%, analytical solution

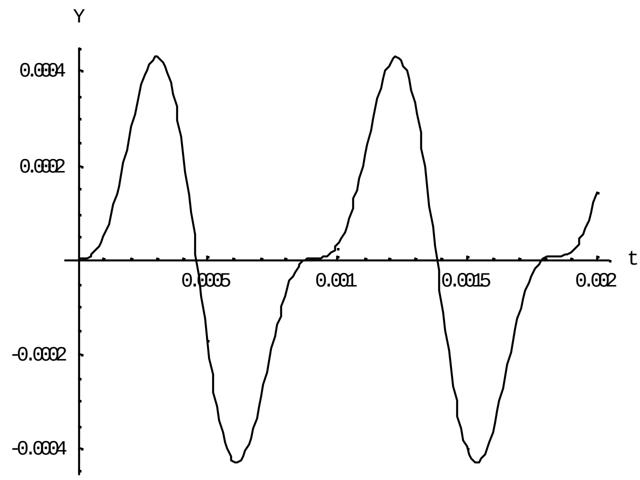

Fig. 4.4 - Y Displacement, O=50%, F.E.M.

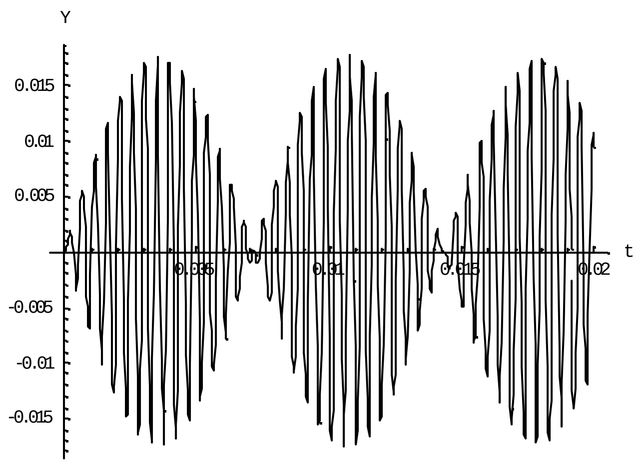

Fig. 4.5 - Y Displacement, O=95%, analytical solution for a simulation time of 0,02 secs.

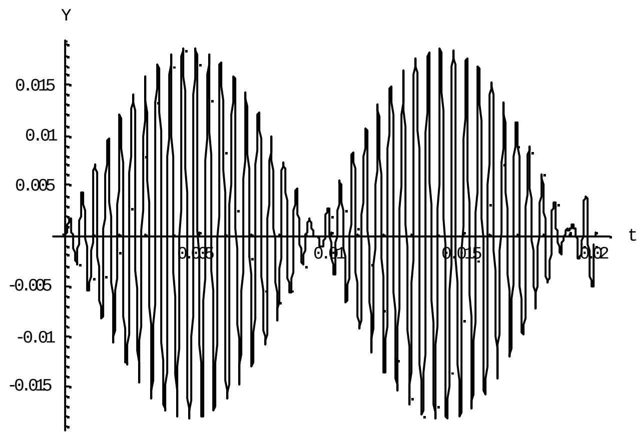

Fig. 4.6 - Y Displacement, O=95%, F.E.M. , for a simulation time of 0,02 secs

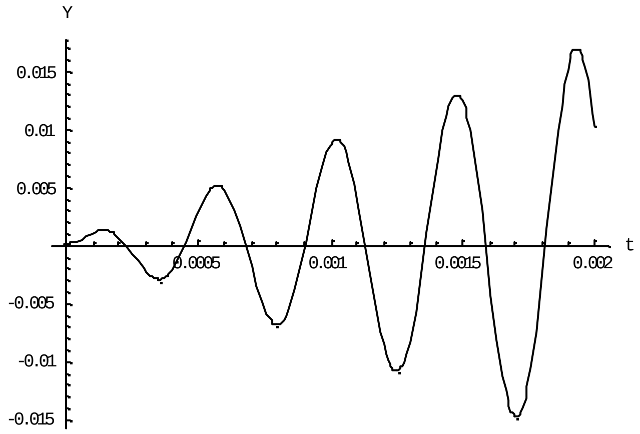

Fig. 4.7 - Y Displacement, O=critical, analytical solution

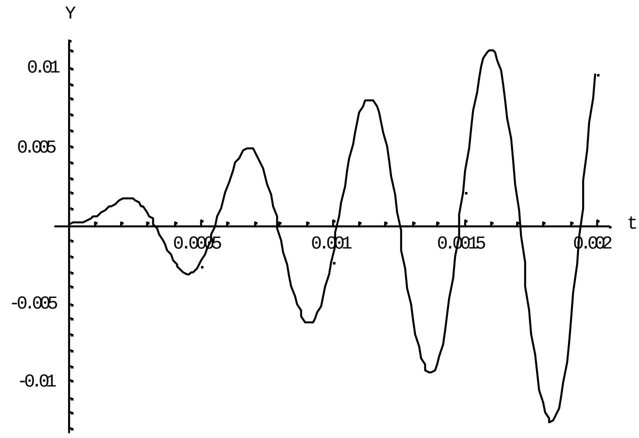

Fig. 4 8 - Y Displacement, O=critical, F.E.M.

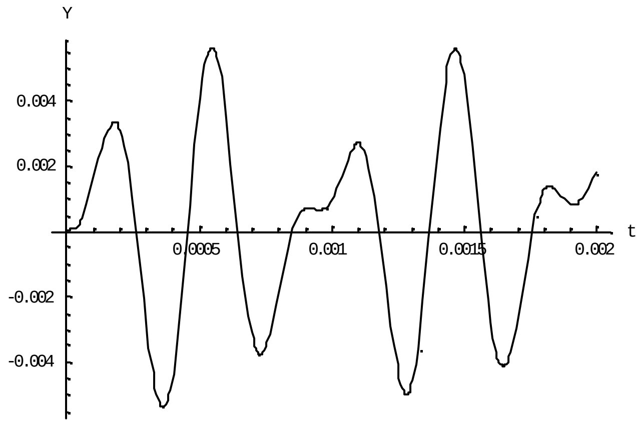

Fig. 4.9 - Y Displacement, O=150%, analytical solution

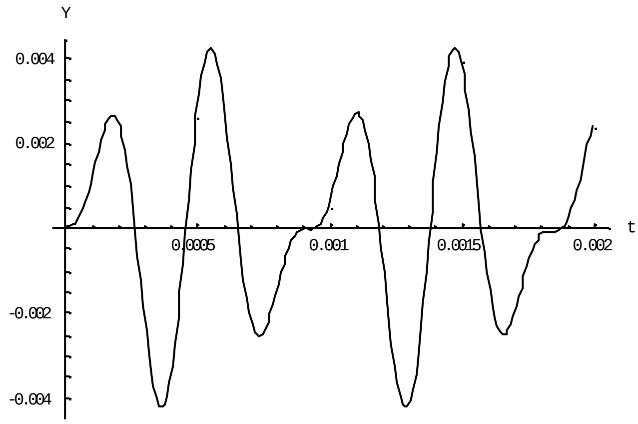

Fig. 4.10 - Y Displacement, O=150%, F.E.M.

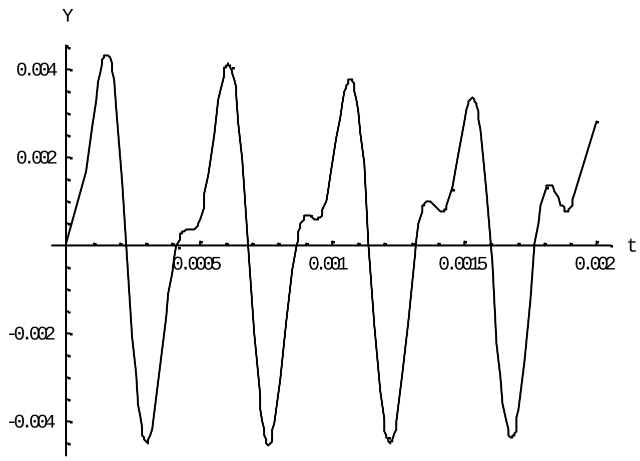

Fig. 4.11 - Y Displacement, O=200%, analytical solution

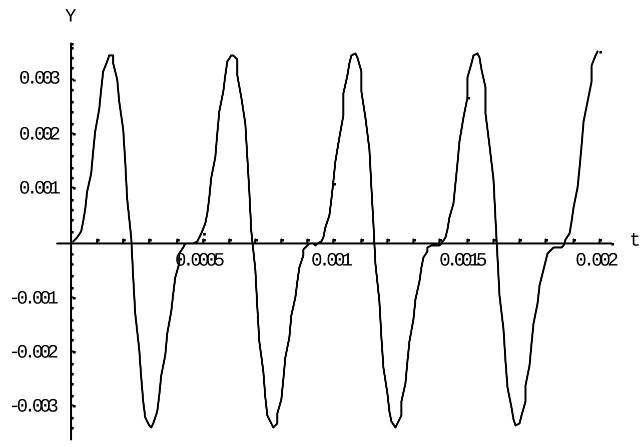

Fig. 4.12 - Y Displacement, O=200%, F.E.M.