FAST COMPUTATIONS FOR OPTIMAL DESIGN: THE USE OF REDUCED BASIS METHOD

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Abstract. *In this paper we present an strategy to obtain fast optimization solutions that is based on the use of the reduced basis techniques in the function and sensitivities computations required by the optimizers. To validate the implementation several benchmark examples are considered in this work. Trusses applications under static loads are the problems addressed here.*

1. INTRODUCTION

Optimization techniques have been extensively used in the past decades. This was mainly due to the development of reliable tools such as the method of Finite Elements (FE), sensitivity analysis schemes and mathematical programming algorithms together with computational growth. As a consequence, more realistic applications, associated with practical engineering problems, have been solved. However, practical engineering applications are involved with several aims to be tackled, large number of constraints and design variables and complex behavior implying in a difficult optimization problem to be solved. Moreover, despite of the modern computational and numerical capabilities, the use of optimization techniques for such problems could be in certain cases cost prohibitive. To overcome that, several strategies could be used, namely: the use of mathematical programming algorithms tailored for large-scale problems, the use of parallel computing and the use of approximations techniques (Barthelemy and Haftka, 1993).

Problem approximation is the focus of the present work. In this context, a model approximation based on the so called reduced basis output bound method (RBOBM) is considered here. The RBOBM is a Galerkin projection onto low order approximation spaces comprising solutions of the problem of interest at selected points in the parameter/design space. The purpose of such scheme is to get high a fidelity model information without the computational expense. A standard

Structural sizing optimization (SSO) algorithm incorporating such procedure in the structural and sensitivity analyses will be used to obtain fast optimal trusses design. The output here investigated is the structural compliance. Optimization studies will be conducted for trusses problems. Comparisons will be conducted with the traditional SSO approach.

2. SSO PROCEDURE

Optimal designs fulfilling several tasks are here obtaining through the use of an automatic SSO procedure. The algorithm incorporates different tools such as geometric definition, discretization, structural and sensitivity analyses, and optimizers. The overview of the algorithm is present in Tab. (1)

Table 1. SSO algorithm: Structural Sizing optimization procedure

SSO1. Define the shape optimization problem;
SSO2. Generates the geometry;
SSO3. Discretize the domain;
SSO4. Peform Structural FE analyses;
SSO5. Conduct the sensitivity analysis;
SSO6. Obtain a new design using a mathematical program algorithm;
SSO7. Check the shape optimization convergence criteria;
SSO7.1 stop if the new design satisfies the criteria;
SSO7.2 otherwise update the design and go to SO3.;

In the sequence some remarks referring to the adopted and proposed strategy are described. For further details see (Afonso, 1995, Macedo,2002)

REMARK SSO1 - An objective function, a set of design variables, a set of constraints and limits on the design variables must be defined to set the optimization problem. A large number of options is available in our system and some will be exploited in the applications presented in this article;

REMARK SSO3 - For trusses this is coincident to the geometry representation;

REMARK SSO4 - Static's, free vibrations and linear buckling analysis can be conduct here. Two strategies are considered in this work to perform the structural analyses. The first one is the classical approach in which the FE method is used. The second strategy uses the RBOBM;

REMARK SSO5 - Sensitivities calculations are required when gradient based mathematical programming algorithms are considered in the optimization process. Several schemes such as Finite differences (FD), Semi-analytical (SA) and Analytical (A) are implemented in the integrated systems. As in the structural analysis, the sensitivities calculations schemes are implemented in the framework of both FE (classical) and the RBOBM strategies;

REMARK SSO6 - The sequential quadratic programming (SQP) (Powell,1998) algorithm is considered here. In the mathematical formulation of the optimization problem, both inequality and equality constraints for the chosen constraint functions can be used. We also consider lower and upper limits for the set of design variables. In the context of MO solutions the SSO algorithm is conducted in a internal loop to allow the Pareto points generation.

2. STRUCTURAL AND SENSITIVITY ANALYSIS

3.1.Classical Approach

Commonly in the analysis module of a SSO procedure a discrete form of the governing equations are solved by the FE method. Under static conditions this leads to the set of equations

$$\boldsymbol{Kd} = \boldsymbol{f} \tag{1}$$

In which $\boldsymbol{K}$ is the stiffness matrix, $\boldsymbol{d}$ are the unknown displacements and $\boldsymbol{f}$ is the external load vector. If the structural compliance is the output of interest this is computed as

$$\boldsymbol{s} = \boldsymbol{d}^{\boldsymbol{T}} \boldsymbol{f} \tag{2}$$

Using the direct approach for sensitivities analyses, the displacement sensitivities w.r.t. the design variable vector $\boldsymbol{x}$ can be obtained from

$$\boldsymbol{K} \frac{\partial \boldsymbol{d}}{\partial \boldsymbol{x}} = \boldsymbol{f}^{*} \tag{3}$$

in which $\boldsymbol{f}^{*}$ is the pseudo load vector:

$$\boldsymbol{f}^{*} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} - \frac{\partial \boldsymbol{K}}{\partial \boldsymbol{x}} \boldsymbol{d} \tag{4}$$

After displacements sensitivities be computed, the derivative of the compliance is easily obtained through the following equation

$$\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{d}^{\boldsymbol{T}}}{\partial \boldsymbol{x}} \boldsymbol{f} \tag{5}$$

3.2.RBOBM Approach

3.2.1 Central idea

To construct an approximation for the displacements and consequently to any solution output (here we will focus on the compliance) satisfying efficiency and accuracy requirements.

3.2.2. Efficiency

The use of an affine decomposition to the stiffness matrix of the conventional (costly) problem is the requirement to perform inexpensive computational calculations. Thus we rewrite $\boldsymbol{K}$ as

$$\boldsymbol{K}(\mu) = \sum_{r=1}^{R} \mu_r \ \boldsymbol{K}_r \tag{6}$$

in which each parameter μ_r is a cross sectional area to be specified for different geometric regions r (r = ... *1* ..., R) of the truss. It is important to emphasize that the stiffness matrix $\boldsymbol{K_r}$ is restricted to particular geometric region r and is independent of $\boldsymbol{\mu}$.

3.2.3 Approximation

To construct the RBOM approximations, firstly a sample S^N in the design D must be chosen

$$S^N = \{(\mu_1,...,\mu_R)^1,...,(\mu_1,...,\mu_R)^N\} \tag{7}$$

where each $(\mu_1,...,\mu_R)^i$ is in *D*, this means

$$\mu^{LOW} \leq \mu_r \leq \mu^{UP} \tag{8}$$

in which μ^{LOW}, μ^{UP} are the lower and upper limits respectively of D. The associated reduced basis space is represented by

$$W^N = span\{\zeta^i, i = 1,...N\} \tag{9}$$

in which

$$\boldsymbol{\zeta^i} = \boldsymbol{d}\left((\mu_1,...,\mu_R)^{\boldsymbol{i}}\right), \quad \boldsymbol{i} = 1,...\boldsymbol{N} \tag{10}$$

Taking the above definitions, the reduced-basis approximation problem can be formulated as: for: $\boldsymbol{\mu} \in$ D find

$$\boldsymbol{s}^N(\mu) = \left(\boldsymbol{d}^N(\mu)\right)^{\boldsymbol{T}} \boldsymbol{F} \tag{11}$$

as the approximation of $s(\mu)$ and $\boldsymbol{d}^N$ in the Galerkin projection of $\boldsymbol{d}(\mu)$ onto W^N. Therefore, taking into account the observations made previously, the reduced-basis approximation $\boldsymbol{d}^N(\mu)$ is expressed as

$$d^N(\mu) = \sum_{j=1}^{N} \alpha(\mu)^j \zeta^i, \quad \alpha^j \in \Re^n \quad , \ j = 1,...,N \tag{12}$$

The above equation means that $\boldsymbol{d}^N$ can be expressed as a linear combination of the displacements solutions $\boldsymbol{\zeta^i}$. In matrix form Eq. (12) is rewritten as

$$\boldsymbol{d}^N(\mu) = \boldsymbol{Z}\alpha(\mu) \tag{13}$$

Using stationary conditions to the total potential energy and Eq. (13) to represent the displacements field , we end up in the following equation (Prud'homme et al, 2002)

$$\boldsymbol{K}^N(\mu)\alpha(\mu) = \boldsymbol{F}^N \tag{14}$$

in which

$$\boldsymbol{K}^N(\mu) = \boldsymbol{Z^T K}(\mu)\boldsymbol{Z} \quad \in \Re^{NxN}; \quad \boldsymbol{F}^N = \boldsymbol{Z^T F} \quad \in \Re^N \tag{15}$$

3.2.4 Accuracy

The accuracy checking for the method is accomplished due to consideration of a posteriori error estimator procedure. To compute the error in a computationally efficient way the following residual equation should be solved,

$$\boldsymbol{C}(\mu)\hat{\boldsymbol{e}}(\mu) = \boldsymbol{R}(\mu) \tag{16}$$

in which $\boldsymbol{C}(\mu)$ is a symmetric operator defined such that

$$\boldsymbol{e}^T(\mu)\boldsymbol{K}(\mu)\boldsymbol{e}(\mu) \le \hat{\boldsymbol{e}}^T(\mu)\boldsymbol{C}(\mu)\hat{\boldsymbol{e}}(\mu) \tag{17}$$

In the above equations $\boldsymbol{e}(\mu)$ is the exact error and $\hat{\boldsymbol{e}}(\mu)$ is the approximated error and $\boldsymbol{R}$ is the residuo. To compute $\boldsymbol{C}$ we consider the so-called point conditioner strategy (Prud´homme et al, 2002):

$$C(\mu) = g(\mu)\,\hat{K} \tag{18}$$

in which $\boldsymbol{g}(\mu) = \min\{\mu_r\}, \quad \boldsymbol{r} = 1,\ldots,\boldsymbol{R}$ and

$$\hat{\boldsymbol{K}} = \sum_{r=1}^{R} \boldsymbol{K}_r \tag{19}$$

Finally, an estimative of the error for the structural compliance can be written as

$$\Delta^N(\mu) = \hat{\boldsymbol{e}}^T(\mu)\, g(\mu)\, \hat{\boldsymbol{K}}\, \hat{\boldsymbol{e}}(\mu) \tag{20}$$

It is important here to emphasize that the solution and error computations are very fast as it take the advantage of using several precomputed quantities (μ independent). This is possible due the Eqs. (14,15) together with the decomposed form of our stiffness matrix, defined in Eq. (6).

3.2.5 Compliance Sensitivities

Differentiating Eq. (11) w.r.t. a design variable x_k and using Eq. (13), we obtain

$$\boldsymbol{s}^N(\mu)_{,x_k} = \boldsymbol{\alpha}_{x_k}^T\ \boldsymbol{F}^N \tag{21}$$

Similarly, from Eq. (14) it follows that α_{x_k} is the solution of

$$\alpha(\mu)_{,x_k} = -\boldsymbol{K}^{N^{-1}}(\mu)\boldsymbol{F}^{N*} \tag{22}$$

in which $\boldsymbol{K}^{N^{-1}}$ is the inverse of $\boldsymbol{K}^N$ and $\boldsymbol{F}^{N*}$ is the pseudo reduced-basis load force

$$F^{N*} = -K^{N}(\mu)_{,x_k}\,\alpha(\mu). \tag{23}$$

Equations (21,22) are solved for each design variable in turn. In the present context these quantities are selected from the parameters $\mu = (\mu_1,\ldots,\mu_R)$ described in previous section and the total number of design variables is n_{dv}.

As observed, to calculate the output sensitivities we need only to calculate $K^{N}(\mu)_{,x_k}$. Due to the affine decomposition used for $K(\mu)$ it is verified that these quantities are in fact precomputed as

$$K^{N}(\mu)_{,x_k} = K_k^{N} = Z^{T} K_k Z \tag{24}$$

Upon this observation it is easily verified that only an extra on line step have be implemented to completely evaluate the sensitivity of desired output. This is a very important feature as in the conventional (costly) approach, the sensitivity analysis can take 50% to 90% (Afonso, 1995) of the computational effort required to solve the whole optimization problem.

3.2.6 Computational Implementation

As shown in previous sections, the equation to obtain the output and its derivative and conduct error estimation involves two classes of terms respectively parameter/no parameter dependent. Therefore an off-line/on-line strategy is used for the computational implementation of the method.. As a consequence, a parameter independent quantity is calculated only once at the so-called off-line stage. Subsequently, the on-line stage access the precomputed information to provide real time response to the new parameters μ. The algorithm to compute the output and its derivative is presented in Tab. (2).

Table 2. Algorithm RBOBM: off-line/on-line stages

OFF-LINE – independent of μ:
1. Choose sample: $S^{N} = \{(\mu_1,\ldots,\mu_R)^1,\ldots,(\mu_1,\ldots,\mu_R)^N\}$
2. Construct matrix of FE solutions: $\mathbf{Z} = [\zeta^1,\ldots,\zeta^N]$;
3. Construct the reduced basis matrix: $K^{N}(\mu) = Z^{T} K(\mu) Z$;
ON-LINE – for a new vector μ:
1. From the reduced basis matrix: $K^{N}(\mu) = \sum_{r=1}^{R} \mu_r\, K_r^{N}$
2. Solves: $K^{N}(\mu)\alpha(\mu) = F^{N}$;
3. Evaluate: $s^{N}(\mu) = \alpha^{T} F^{N}$
4. Solves: $C(\mu)\hat{e}(\mu) = R(\mu)$
5. Compute the output error: $\Delta^{N}(\mu) = \hat{e}^{T}(\mu)\, g(\mu)\, \hat{K}\, \hat{e}(\mu)$
6. Compute the sensitivities: $s^{N}(\mu)_{,x_k} = \alpha_{x_k}^{T}\; F^{N}$

4. OPTIMIZATION PROBLEM

The mathematical formulation is written as (Haftka and Gurdal, 1993)

$$minimize: \quad f\left(\boldsymbol{x}\right) \tag{25}$$

Subject to the following conditions:

$$\begin{aligned} & g_i\left(\boldsymbol{x}\right) \leq 0 \quad i = 1,...m \\ & h_i\left(\boldsymbol{x}\right) \leq 0 \quad j = 1,...l \\ & x_k^l \leq x_k \leq x_k^u \quad k = 1,...n_{dv} \end{aligned} \tag{26}$$

In which **x** is the design variable vector $f\left(\boldsymbol{x}\right)$ is the objective function which is to be minimized or maximized, *nobj* is the number of objective functions, $g_i\left(\boldsymbol{x}\right)$ is an inequality constraint, $h_i\left(\boldsymbol{x}\right)$is an equality constraint and x_k^l, x_k^u are respectively the lower an upper limits on a typical design variable x_k. In a MO optimization context, the mathematical statement uses a vector $F\ (\boldsymbol{x}) = (\boldsymbol{f}_1, \boldsymbol{f}_2, ..., \boldsymbol{f}_{\boldsymbol{nobj}})$, which contains the set of objective functions instead of a unique objective as indicated in (25) (Afonso et al, 2002).

5. EXAMPLES

The approaches presented before are tested for some truss applications. In this work some benchmark examples are analyzed. The examples highlights the advantage of using the RBOBM over the conventional approach as problem complexities increases.

5.1.Problems Definitions

Three different benchmark trusses are considered. Ten bar truss, sixty-four-bar truss and two hundred-bar truss. Their geometry and loading definition are indicated in figs. (1-3). The property Elastic Modulus E is given in Tab. (3) for each truss.

For the reduced-basis solution, each truss is subdivided in three regions, regions I, II and III, indicated in Figs. (1-3), in which the struts are determined to have different cross sectional areas. This leads to R = 3 in the present study. The initial area (A_1, A_2, A_3) of each one of the three trusses are the same .It is considered N = 9 and the sample S^N for all trusses is indicated in Tab. (4). The quantity of interest to be computed is the compliance.

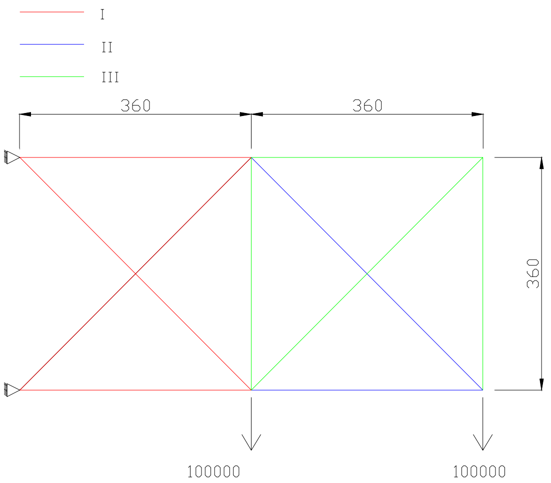

Figure 1. Ten bar truss

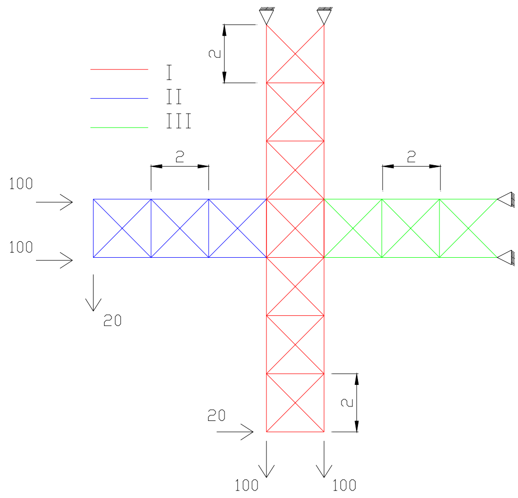

Figure 2. Sixty-four bar truss

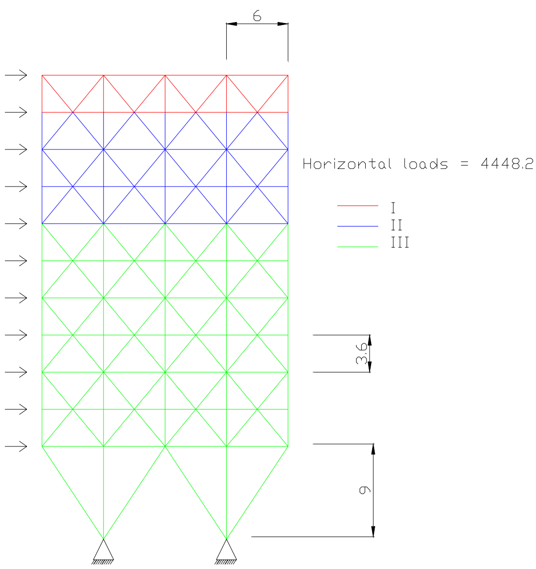

Figure 3. Two hundred bar truss

Table 3. Trusses examples material proprieties

	10 bars	64 bars	200 bars
E	$1x10^7$	$2,07x10^8$	$2.068x10^{11}$

Table 4. Trusses example – sample S^N

(μ_1,μ_2)								
(0.1,0.1)	(1,0.1)	(10,0.1)	(0.1,1)	(1.1)	(10,1)	(0.1,10)	(1,10)	(0.1,0.1)

5.2. Structural and Sensitivity Analysis Studies

The compliances (S), (S^N) obtained respectively using FE method and the RBOBM are indicated in Tab. (5). In this table the error estimative for the reduced basis approximations is also given. As observed the accuracy of the RBOBM is very good despite the small value for N.

Table 5. Initial results for FE and RBOBM methods

Bar	Area	FE Solution	RBOBM	
		S	S^N	Δ^N
10	5	1.483x10^6	1.483x10^6	4.4703 x10^{-9}
64	1	0.0033	0.0033	1.1449x10^{-16}
200	6.452	0.1256	0.1256	4.147x10^{-15}

The sensitivities studies are conducted considering the area of each region (I, II, III) as design variable. Tab. (6) reports the compliance sensitivity calculations for the truss design variables investigated. As before both procedures (conventional and RBOBM) give the same results.

Table 6. Compliance sensitivity

Trusses	Conventional			RBOBM		
	$\partial s / \partial x_1$	$\partial s / \partial x_2$	$\partial s / \partial x_3$	$\partial s^N / \partial x_1$	$\partial s^N / \partial x_2$	$\partial s^N / \partial x_3$
10 bar	-1.969x10^5	-0.1976 x10^5	0.1301x10^5	-1.969x10^5	-0.1976 x10^5	0.1301x10^5
64 bar	-0.0020	-0.0006	-0.0008	-0.0020	-0.0006	-0.0008
200 bar	-3.913 x10^{-5}	-18.418x10^{-5}	-1924.4x10^{-5}	-3.913 x10^{-5}	-18.418x10^{-5}	-1924.4x10^{-5}

5.3. Optimization Studies

To demonstrate the capabilities of SSO procedure single optimization problems are considered first. Again, the three benchmarks trusses are considered.

The compliance is the objective function to be minimized and the initial total volume, indicated in Tabs. (7,8), is considered as a constraint. A part from that, the design space considered is D = [0.1 10] × [0.1 10] × [0.1 10]. This means that $\mu_1 = A_1, \mu_2 = A_2$ e $\mu_3 = A_3$ are the design variables and their lower and upper limits are respectively 0.1 and 10.

Tables (7) and (8) show the optimization results considering both strategies (conventional and RBOBM) investigated here. Both strategies converge to the same optimum. However the fast computation inherit to RBOBM is perceived. Also the difference in CPU time using both schemes increases as the structure get more complex. For the RBOBM the CPU time is almost constant as the number of degrees of freedom increases. For the ten-bar truss N is greater then the total number of degrees of freedom. Therefore, as the structure of $\boldsymbol{K}^N$ matrix is dense this explains the CPU time consumed for that case. This highlights the importance of the use RBOBM for the design of large structures.

Table 7 .Optimization results – conventional

	Optimum						
bars	A_1	A_2	A_3	S	V	N_{int}	CPU
10	8.7447	6.4696	0.1000	$1.483x10^6$	$2.098x10^4$	7	17.646
64	1.0001	0.9998	1.0000	0.0033	149,5391	1	4.908
200	6.449	6.445	6.455	0.1255	$5.679x10^3$	1	12.277

Table 8 .Optimization results – RBOBM

	Optimum						
bars	A_1	A_2	A_3	S	V	N_{int}	CPU
10	8.7447	6.4696	0.1000	$1.483x10^6$	$2.098x10^4$	7	5.748
64	1.0001	0.9998	1.0000	0.0033	149,5391	1	0.951
200	6.449	6.445	6.455	0.1255	$5.679x10^3$	1	0.901

6. CONCLUSIONS

Optimum designs were here obtained for classical trusses problems. The RBOBM was integrated in a SSO algorithm in order to conduct fast computations. A certify of fidelity for the reduced basis was obtained through the implementation of a posteriori error estimator.

The results were compared to the conventional SSO approach, which employs FE method. As the complexity of the FE's equation increases the advantage of using the reduced basis in the SSO algorithm was highlighted.

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