THE COMPOSITE ELEMENT METHOD APPLIED TO THE VIBRATIONS ANALYSIS OF TIMOSHENKO'S BEAMS.

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Abstract. Free vibrations of Timoshenko´s beams are analyzed by the Composite Element Method. This method is a new technique that combines the analytical solutions of the free vibration problem based on the Classical Theory (CT) and the shape functions of the Finite Element Method (FEM). The analytical functions must obey special boundary conditions in such a way they do not alter the nodal values determined by FEM. The addition of the analytical to the shape functions enriches the solution space and the results are more accurate. There are two types of enrichments: h-refinements, which are obtained increasing the number of elements, and c-refinements, which add analytical functions in element displacement field. The c-refinement is a kind of hierarchical method, and in the free vibration problem shows good accuracy and high convergence. Some examples are presented to compare both methods and show that the CEM is better than FEM in the case of coarse meshes and, specially, to determine higher vibrations frequencies. It is also analyzed the influence of the shear and rotatory inertia in the precision of the method.

Keywords. *Composite Elements Method, Finite Elements Method, Timoshenko´s Beam, Vibration Analysis.*

1. INTRODUCTION

Free vibration analysis of structures is an usual task on the modern engineering design and the Finite Element Method (FEM) is one of the most important tools to carry on this objective. The FEM is very accurate to determine lowers natural modes and frequencies of shallow beams but, in the case of high sections or when one intends to determine higher modes, a fine mesh is necessary and the computational effort is increased. It would be desired to obtain good precision for lowers and higher modes on the free vibration analysis even with coarse meshes. This is the advantage of the Composite Element Method (CEM). It can improve the accuracy without change the finite element mesh, but only adding analytical functions to the shape functions of the displacement field and enriching the solution space. The CEM is a method that combines the versatility of FEM and the high accuracy of closed form solutions from classical theory. The analytical solutions, which satisfy some special boundary conditions, are added to the shape functions of FEM forming a new group of interpolation functions. CEM can be improved using two types of approach: h-version and c-version. The h-version, the same of FEM, is the refinement of the element mesh increasing the number of elements. The c-version corresponds to the increase of the number of degrees of freedom related to the analytical solutions (c-DOF). The Composite Element Method was initially proposed by ZENG (1998 a, 1998b, 1998c), which applied it to trusses and framed structures (C^o formulation) and to free vibration of plates. The Composite Element Method was also developed by (Hoefel, 2002; Carvalho, 2002; Arndt ,2001; Machado, 2002 et. al), which were able to confirm the accuracy and efficiency of the CEM in many applications. Many researchers including Huang (1961, 1963), Horr and Schmidt (1995) and Yokoyama (1995), etc, have been studying the free vibration of Timoshenko's beams. Numerical methods have been used to solve these problems. In this paper, the free vibration analysis of Timoshenko's beams is developed using the Composite Element Method

2. COMPOSITE ELEMENT

The Composite Element Method is an approximation method whose displacement field is expanded by the sum of the finite element and the CT shape functions. The FEM displacement field must satisfy the nodal boundary conditions of the element. Otherwise, the analytical functions are obtained from the differential equation of the Classical Theory in the element domain, with compatible boundary conditions in order to maintain the nodal values of displacements determined by FEM. The final displacement field equation may be written as:

$$v_{CEM}(\xi) = v_{FEM}(\xi) + v_{CT}(\xi) \tag{1}$$

where v_{FEM} and v_{CT} are the two parts of the CEM displacement field. The first one is associated to the nodal coordinate system from the FEM, using the shape function vector N; the nodal displacement vector v (or the nodal degrees of freedom), and the local coordinate ξ, as the following equation:

$$v_{CT}(\xi) = \Gamma^T(\xi)c \tag{2}$$

The second part is obtained by linear combination of the analytical solutions of the Classical Theory, as the following expression:

$$v_{CT}(\xi) = \Gamma^T(\xi)c \tag{3}$$

where, Γ is the analytical shape functions vector which is obtained by CT, and c is the coefficient vector associated to the c-degrees of freedom or c-coordinates. Substituting Eq. (2) and Eq. (3) into Eq. (1) one obtains:

$$v_{CEM}(\xi) = N^T(\xi)q + \Gamma^T(\xi)c \tag{4}$$

2.1. Displacement field function by Timoshenko Theory.

The main difference between the Timoshenko's and the Euler-Bernoulli's beam theories is that the cross section, which is straight and perpendicular to the neutral axis before deformation, remains straight but not necessary perpendicular after deformation. The coupled-beam equations for the total deflection v and bending slope ψ are given by Timoshenko (1955):

$$\begin{aligned} EI\frac{\partial^2\psi}{\partial x^2} + \kappa\left(\frac{\partial v}{\partial x} - \psi\right)AG - I\rho\frac{\partial^2\psi}{\partial t^2} = 0 \\ \rho A\frac{\partial^2 v}{\partial t^2} - \kappa\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2\psi}{\partial x^2}\right)AG = 0 \end{aligned} \tag{5}$$

in which E is the elasticity modulus, G is the transversal elasticity modulus, I is the area moment of inertia of cross section, A is the cross-sectional area and κ is the shear coefficient. When the beam vibrates harmonically, one can set : $v = Ve^{ipt}$ $\psi = \Psi e^{ipt}$ and $\xi = x/L$, where V is the normal function of v, Ψ it is the normal function of ψ, $p = 2\pi\omega$ the angular frequency, L the lenght of the beam, ξ the dimensionless length, and $i = \sqrt{-1}$. Omitting the factor e^{ipt}, the Eq. (5) is reduced to

$$\begin{aligned} s^2\Psi'' - (1 - b^2r^2s^2) + V'/L = 0 \\ V + b^2s^2 - \Psi = 0 \end{aligned} \tag{6}$$

in which

$$b_i = 2\pi\sqrt{\frac{\rho A\, l^4}{EI}\,\omega_i^2} \tag{7}$$

and the primes for V, Ψ represent differentiation with respect to ξ and ω is natural frequency. Note that r^2 and s^2 describe the effects of rotatory inertia and shear deformation, respectively. Alternatively, when the shear deformation is set equal to zero and effects of rotatory inertia are neglected, the resulting model is identical to the classical Euler-Bernoulli beam theory. The solutions of Eq. (6) can be found as (Boyce and Diprima, 1988):

$$\begin{aligned} V &= c_1 \cosh b\alpha\,\xi + c_2 \sinh\, b\alpha\xi + c_3 \cos b\beta\xi + c_4\, sin\, b\beta\,\xi \\ \Psi &= \bar{c}_1 \sinh\, b\alpha\xi + \bar{c}_2 \cosh b\alpha\xi + \bar{c}_3 \sin\, b\beta\xi + \bar{c}_4 \cos b\beta\xi \end{aligned} \tag{8}$$

where

$$
\begin{aligned}
\alpha &= \frac{\sqrt{2}}{2}\{-(r^2+s^2+)+[(r^2-s^2)^2+4/b^2]^{1/2}\}^{1/2} \\
\beta &= \frac{\sqrt{2}}{2}\{(r^2+s^2+)+[(r^2-s^2)^2+4/b^2]^{1/2}\}^{1/2}
\end{aligned}
\tag{9}
$$

is assumed $\sqrt{(r^2-s^2)^2+4/b^2} > (r^2+s^2)$ and c_i are constants.

$$
\mathtt{f} = 2 - 2\cosh(\alpha\, b)\,\cos(\beta\, b) - \mathcal{D}\,\sinh(\alpha\, b)\,\sin(\beta\, b) \tag{10}
$$

where,

$$
\mathcal{D} = \frac{\nu^2-\mu^2}{\nu\,\mu}, \quad \nu = \frac{b\left(\beta^2-s^2\right)}{l\,\beta}, \qquad \mu = \frac{b\left(\alpha^2+s^2\right)}{l\,\alpha} \tag{11}
$$

The corresponding eigenfunctions, normal modes V_i and Ψ, can be obtained as:

$$
\begin{aligned}
V(\xi) &= c_4\{sen(B_i\xi) - \zeta_i\, senh\,(A_i\xi) + \mathcal{Z}_i\,[\cosh(A_i\xi) - \cos(B_i\xi)]\} \qquad (\\
\Psi(\xi) &= c_4\,\{\cos(B_i\xi)\nu_i - \cosh(A_i\xi)\nu_i + \mathcal{Z}_i\,[\,sen(\,B_i\xi)\nu_i +\ senh\,(A_i\xi)\theta_i]\}
\end{aligned}
\tag{12}
$$

where

$$
A_i = b_i\alpha_i, \quad B_i = b_i\beta_i, \quad \zeta_i = \frac{\nu_i}{\mu_i}, \quad \mathcal{Z}_i = \frac{sen\,(B_i) - \zeta_i\, senh\,(A_i)}{\cos(B_i) - \cosh(A_i)} \tag{13}
$$

The displacement field function by Timoshenko's theory can be found as:

$$
v_{CT}\,(\xi,t) = \Gamma_v^T\,(\xi)\; c\,(t)\,, \qquad \psi_{CT}\,(\xi,t) = \Gamma_\psi^T\,(\xi)\; c\,(t) \tag{14}
$$

where $\Gamma_v{}^T(\xi) = [F_{v1} \quad F_{v2} \quad \dots \quad F_{vn}]$ and are the displacements fields functions or element and $c = [c_1 \quad c_2 \quad \dots \quad c_n]$ they are the constant that multiply the analytic solutions of CT.

2.2. Displacement field function of FEM.

The beam element consists of two nodes *i* and *j*, each node has degrees of freedom of lateral displacement v^e and bending rotation (or slope) ψ^e as shown in Fig. (1).

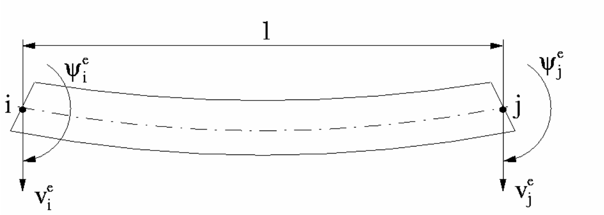

Figure 1. Timoshenko beam element.

$$\begin{aligned} v(\xi,t) &= v_i(t)N_{v1}(\xi)+\psi_i(t)N_{v2}(\xi)+v_j(t)N_{v3}(\xi)+\psi_j(t)N_{v4}(\xi) \\ \psi(\xi,t) &= v_i(t)N_{\psi1}(\xi)+\psi_i(t)N_{\psi2}(\xi)+v_j(t)N_{\psi3}(\xi)+\psi_j(t)N_{\psi4}(\xi) \end{aligned} \tag{15}$$

The displacements field function of Timoshenko's beam element can be found as:

$$v^e_{FEM}=[N_v]^T\bar{q}^e, \quad \psi^e_{FEM}=[N_\psi]^T\bar{q}^e \tag{16}$$

Here $\bar{q}^e$ is the element nodal displacement vector, and N is the shape functions (or interpolation functions):

$$\begin{aligned} N_{v1} &= \frac{[1-3\xi^2+2\xi^3+(1-\xi)]}{(1+\Phi)}, \quad N_{v2}=\frac{[\xi-2\xi^2+\xi^3+(\xi-\xi^2)\Phi/2]}{(1+\Phi)} \\ N_{v3} &= \frac{(3\xi^2-2\xi^3+\xi\Phi)}{(1+\Phi)}, \quad N_{v4}=\frac{[-\xi^2+\xi^3-(\xi-\xi^2)\Phi/2]}{(1+\Phi)} \end{aligned} \tag{17}$$

$$\begin{aligned} N_{\psi1} &= 6\frac{-\xi+\xi^2}{l(1+\Phi)}, \quad N_{\psi2}=\frac{[1-4\xi+3\xi^2+(1-\xi)\Phi]}{(1+\Phi)} \\ N_{\psi3} &= 6\frac{\xi-\xi^2}{l(1+\Phi)}, \quad N_{\psi4}=\frac{-2\xi+3\xi^2+\xi\Phi}{(1+\Phi)} \end{aligned} \tag{18}$$

where $\Phi = 12EI/\kappa GAl^2$ is the shear deformation parameter.

2.3. Composite Element Formulation.

The displacement field of the composite element consists of two parts:

$$\begin{aligned} v^e(\xi,t) &= N_v^T(\xi)\bar{q}(t)+\Gamma_v^T(\xi)c(t) \\ \psi^e(\xi,t) &= N_\psi^T(\xi)\bar{q}(t)+\Gamma_\psi^T(\xi)c(t) \end{aligned} \tag{19}$$

or

$$v^e(\xi,t) = S_v^T(\xi)\,q(t)$$
$$\psi^e(\xi,t) = S_\psi^T(\xi)\,q(t) \tag{20}$$

The matrix function $S(\xi)$ is defined as the generalized shape function matrix of CEM which consists of both the shape function $N(\xi)$ of the conventional FEM and the mode shape $\Gamma(\xi)$ of the analytical solution of Timoshenko theory. The vector *q* is called the generalized coordinates (or DOF) of CEM which consists of both the nodal DOF $\bar{q}$ of conventional FEM and the mode coordinate *c* of the analytical solution of Timoshenko theory. The flexural strain or curvature ε_f and the shear angle θ within the element are defined as:

$$\varepsilon_f = \frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial\xi}\frac{\partial\xi}{\partial x} = [B_f]\{q\}, \quad \theta = \psi - \frac{\partial v}{\partial x} = [\bar{B}_c]\{q\} \tag{21}$$

where $B_f = \frac{1}{l}\left[\frac{\partial S_\psi}{\partial\xi}\right]$ and $\bar{B}_c = \frac{1}{l}\left[\frac{\partial S_v}{\partial\xi}\right]$. With the aid of Eq. (16) and Eq. (21), the strain energy U^e and the kinetic energy T^e can be expressed in terms of the element nodal displacement vetor q^e as:

$$U^e = \frac{1}{2}\{q^e\}^T[k_f]^e\{q\}^e + \frac{1}{2}\{q^e\}^T[k_c]^e\{q\}^e$$
$$T^e = \frac{1}{2}\{\dot{q}^e\}^T[m_t]^e\{\dot{q}\}^e + \frac{1}{2}\{q^e\}^T[m_r]^e\{q\}^e \tag{22}$$

The superposed dots denote differentiation whit respect to time *t*. By Hamilton's principle, the respective element matrices are given by:

$$[k_f] = \int_0^1 [B_f]^T EI[B_f]l\,d\xi, \quad [k_c] = \int_0^1 [B_c]^T kGA[B_c]l\,d\xi \tag{23}$$

$$[m_t] = \int_0^1 [S_v]^T \rho A[S_v]l\,d\xi, \quad [m_r] = \int_0^1 [S_\psi]^T \rho I[[S_\psi]l\,d\xi$$

Where $B_c = \left([S_\psi] - [\bar{B}_c]\right)$ and k_f is bending stiffness matrix, k_c is shear stiffness matrix, $\mathbf{m}_t$ is mass translational inertia matrix and m_r is mass rotatory inertia matrix. Note the when the shear deformation parameter Φ is set equal to zero and rotatory inertia mass matrix is neglected, the resulting model is identical to the classical Euler-Bernoulli beam.

The Hamilton´s principle (Horr and Schmidt, 1995) leads to the matrix equation governing the free vibrations of the Timoshenko beam as:

$$[M]\{\ddot{q}\} + [K]\{q\} = 0 \tag{24}$$

where

$$[M] = \sum_e([m_t]^e + [m_r]^e), \quad [K] = \sum_e([k_f]^e + [k_c]^e) \tag{25}$$

$$\{q\} = \sum_e \{q\}^e \tag{26}$$

where $[K]$ is assembled stiffness matrix, $[M]$ is assembled consistent mass matrix and $\{q\}$ is assembled nodal displacement vector. If it is assumed to be harmonic in time with circular frequency ω, Eq. (24) becomes, an eigenvalues problem of the form $(K - \omega^2 M)\{Q\} = 0$, where $\{Q\}$ is a vector of nodal displacement amplitudes of vibration (modal vector). The solution of Eq. (24) yields the natural frequencies and the corresponding mode shape.

3. NUMERICAL EXAMPLE

Example 1.

The first example is concerned for a simple supported beam. The value of the shear coefficient is 0,85, the Poisson coefficient is 0.3, the length of beam is 1 and the radius of gyration is $r_g = 0.008$. The beam is subdivided into 1, 2, 4, 8 and 10 finite elements. The FEM results are shown in Tab. (1). Considering the Composite Element Method, with just one finite element and 1, 2, 4 and 10 c-functions, the results are presented in Tab. (2). The relative errors of CEM and FEM are showed, for some eigenvalues in the Fig. (2) and relative error of the *h* and *c* refinaments of CEM are showed, for 2^{nd} eigenvalue in the Fig. (3).

Table 1. Results gotten for the MEF.

Analytical	FEM (1e)	FEM (2e)	FEM (4e)	FEM (8e)	FEM (10e)
9.857	10.951	9.898	9.860	9.858	9.857
39.278	50.191	43.762	39.461	39.323	39.287
87.823		109.802	89.697	88.307	87.923
154.794		200.663	174.381	157.313	155.330
239.274			275.209	247.864	241.213
340.189			435.287	389.894	345.616
456.355			655.027	520.088	469.036
586.534			800.987	713.507	612.245

Table 2. Results gotten for the CEM.

Analytical	CEM(1e 1c)	CEM(1e 2c)	CEM(1e 4c)	CEM(1e 10c)
9.857	9.858	9.858	9.857	9.856
39.278	50.191	39.306	39.270	39.262
87.823	122.511	122.511	88.220	88.018
154.794		229.623	155.828	155.245
239.274			371.168	240.381
340.189			548.687	342.089
456.355				459.971
586.534				591.773

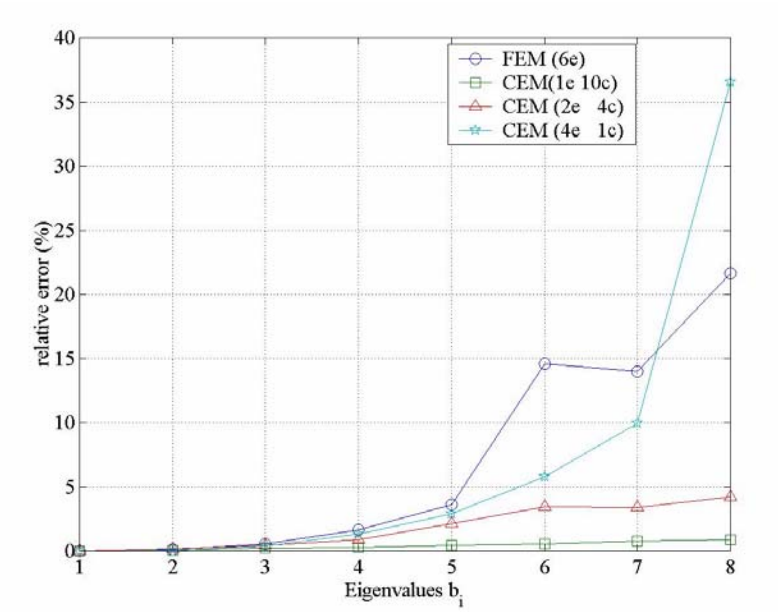

Figure 2 . Relative error – CEM and FEM.

Example 2.

To examine the influence of the height of the cross section, the same simple supported beam of the previous example is analyzed considering now the radius of gyration equal to $r_g = 0.04$. Tab. (3) shows the FEM and CEM solutions for models with the total number of dof equal to 12. In this case, the computational effort of both solutions is the same.

Table 3. Results gotten for the CEM.

Analytical	MEF (6e)	CEM (1e 10c)	CEM (2e 4c)	CEM (4e 1c)
9.571	9.576	9.519	9.591	9.581
35.359	35.620	34.830	36.685	35.814
71.657	73.760	73.920	76.319	75.142
113.845	121.948	117.055	125.948	135.893
159.136	178.970	164.238	179.00	193.746
205.992	314.134	210.928	233.663	268.205
253.582	368.293	258.320	287.815	344.070
301.458	420.616	304.063	343.221	428.277

One be noted in Tab. (3) that the FEM results are very closed to the analytical solution for lower frequencies cases, but they do not have the same accuracy for the higher ones. On the other hand, the CEM solution with coarse mesh (1e 10c) has very good approximation even for lower and higher frequencies. This behavior is not the same for finest meshes. One observes that, as the number of finite elements increase, the accuracy of CEM results decrease. As closer the mesh is to the global analytical model, better will be the CEM results. This best solution e.g., the analytical one, is “degraded” by the finite element approximation. It suggests that, the c-refinement must be used preferably in coarse meshes and may be used to determine local vibration modes specially the higher ones.

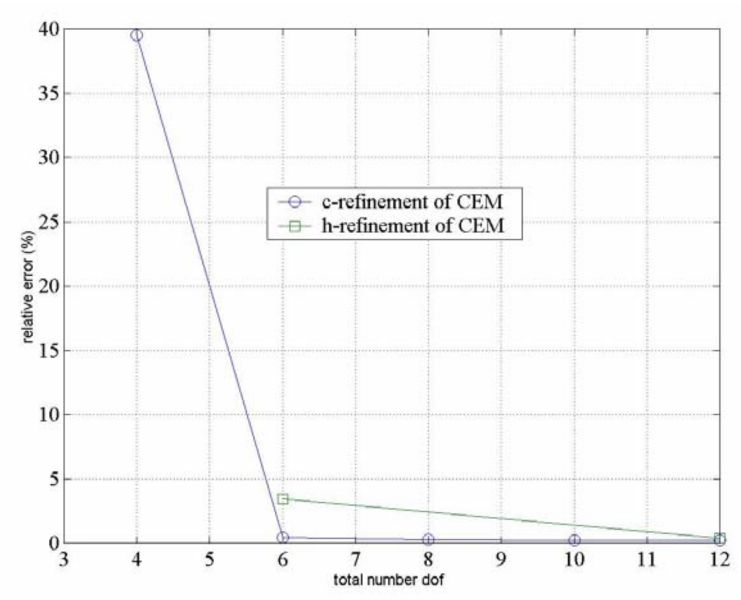

Figure 3. c-refinament and h-refinament of CEM.

It must be considered that, as higher is the height of the cross section, most important is the shear and the rotatory inertia effects. The FEM is strongly affected by this condition and its results have less accuracy compared with shallow beams. As the CEM approximation is based on the finite element displacement field then, if the finite element solution is not accurate, the CEM solution is not so accurate as one would be wished. The h-refinement of CEM is affected by the bad performance of FEM. Despite of this, for higher modes the CEM results are better than FEM ones.

The Fig. (4) shows the graph of the relative error (%) of the FEM and CEM analyses for many eigenvalues (frequencies) and considering different meshes. The next Fig. (5) and Fig (6), shows the relative error (%) for the 1st beam eigenvalues considering different radius of gyration.

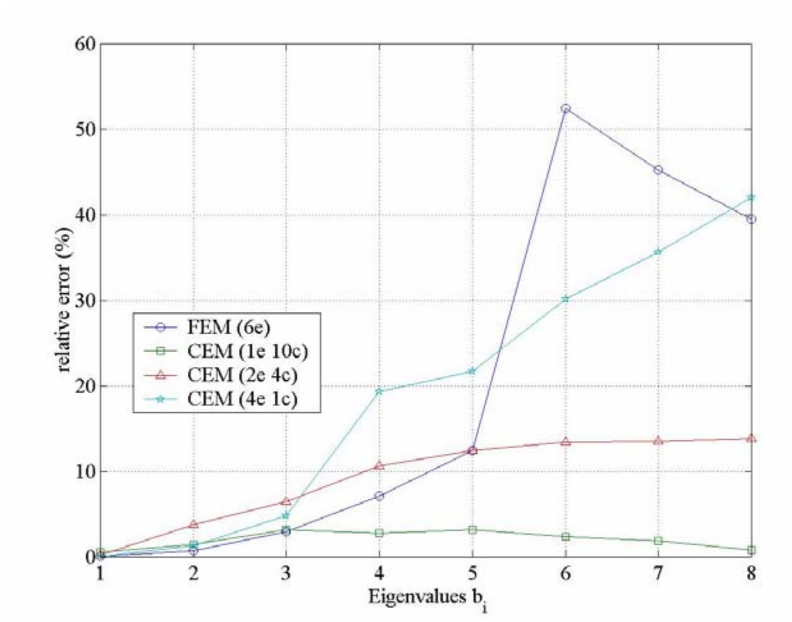

Figure 4. Results gotten for the CEM.

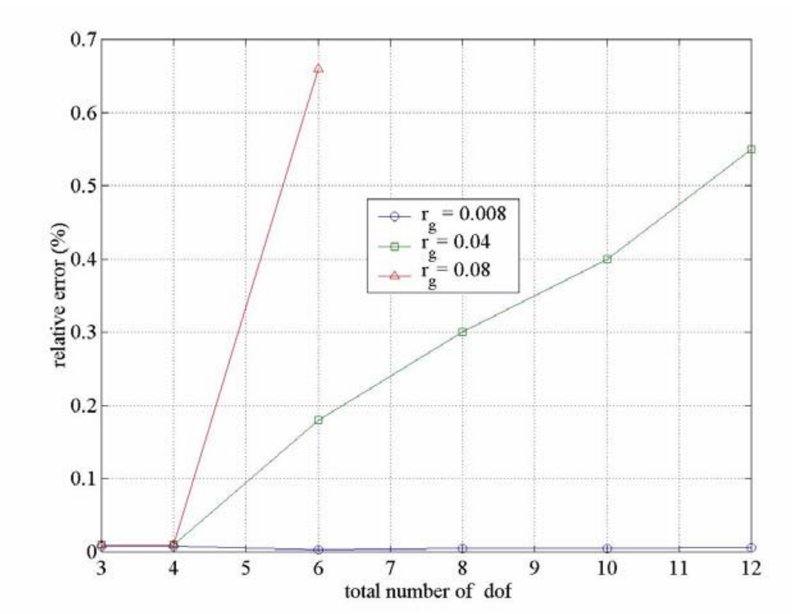

Figure 5. Relative error (%) for the 1st eigenvalue gotten for CEM.

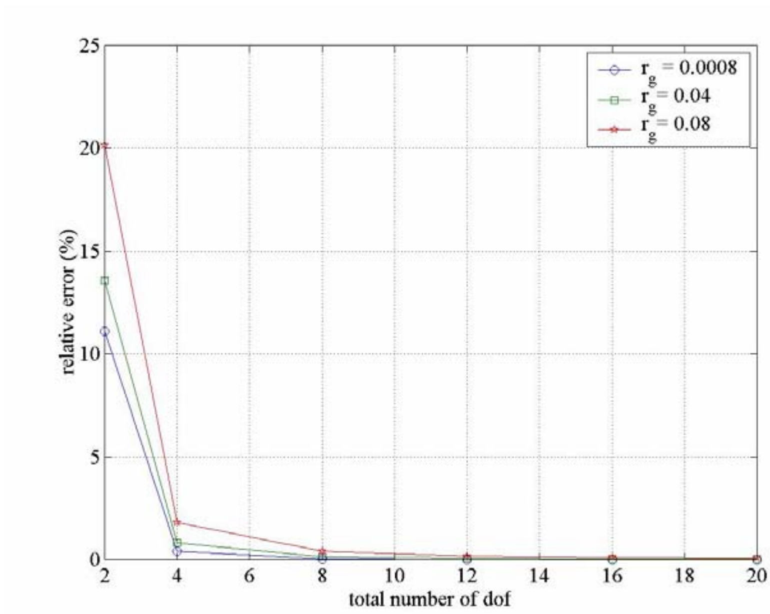

Figure 6. Relative error (%) for the 1st eigenvalue gotten for FEM.

4. CONCLUSIONS

A numerical technique has been presented for vibration analysis of Timoshenko's beams. The Composite Element Method is developed by the superposition of the conventional FEM and analytical solutions of the classical Timoshenko's theory. The CEM is more accurate numerical method than the FEM, with the same versatility and efficiency. Two types of refinements may be considered. The h refinement, like the FEM, that corresponds to increase the number of elements; and the c-refinement that corresponds to increase the number of analytical functions into the local displacement element field. This enrichment does not alter the stiffness and mass matrices, which were previously calculated, it just aggregates new terms without re-built the hole matrices. It can be inferred that the c-refinement is of a hierarchical type. The interpolation functions of the CEM are completed and consistent.

The c-refinement of CEM leads a super convergency of the results. With the same number of dof, e.g., the same computational effort, the eigenvalues associated to the CEM are more accurate than the FEM. Than, to obtain higher efficiency for the CEM, it must be adopted a minimum number of finite elements to represent the geometry and choose an appropriated number of c-dof. Considering the presented examples, one concludes hat the addition of rotational inertia and the shear effects provokes a decrease of the accuracy of the beam frequencies. The bigger the contribution of the rotational inertia and shear, e.g., the bigger the gyration ratio, the less accurate will be the solution obtained by the FEM and the CEM.

5. ACKNOWLEDGEMENTS

This research has been supported by CAPES.

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