AN APPROXIMATE ANALYTICAL SOLUTION FOR THE UPSETTING OF A POROUS DISK USING DRUCKER-PRAGER YIELD CONDITION

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Abstract. *This paper analyzes the compaction process of porous materials. A well-known metal forming technique, named Slab Method, is used to perform an approximate analysis for the upsetting of a porous disk. The yield criteria for porous materials shall consider that hydrostatic pressure can produce plastic deformation, an example of this, is the classic Drucker-Prager yield condition which takes into consideration the influence of the hydrostatic pressure in the plastic deformation. Although, more recently, other criteria have been presented in the literature related to powder metallurgy, for the sake of simplicity, a mechanical model using Drucker-Prager yield function is presented. The analysis is restricted to problems with symmetry of revolution and an approximate analytical solution for the upsetting of a porous disk is presented.*

Keywords: *Upsetting, Drucker-Prager criterion, Yield Function, Slab Method.*

1. INTRODUCTION

This paper analyzes the compaction process of porous materials. In order to develop an approximate analysis, some well-known techniques, such as those applied to the analysis of metal forming, are used. Some of these methods, as for example, the Slab Method, are able to obtain closed form solutions, which despite being approximate, can help in the description of the problem as a whole (Wagoner and Chenot, 1997). The yield criterion for these materials is still being studied and it should consider that porous materials might have plastic deformation when subjected to hydrostatic pressure. An example of a classic criterion, which takes into account the influence of the hydrostatic pressure, is the Drucker-Prager yield criterion (Lubliner, 1990). More recently, other criteria related to powder metallurgy have been presented in the literature (Akisanya et al., 1997). Section 2 presents a mechanical model using Drucker-Prager yield function. Section 3 restricts its analysis to problems with symmetry of revolution while Section 4 presents the results for the upsetting of a porous disk.

2. MECHANICAL MODEL

2.1. Yield Function

The Drucker-Prager yield function (f) can be used to model porous materials with pressure dependent plasticity, and it can be expressed by the following equation:

$$f = A\sqrt{J_2} + BI_1 \tag{1}$$

in which A and B are parameters to be determined, I_1 is the first invariant of the stress tensor, J_2 is the second invariant of the stress deviator tensor as follows:

$$J_2 = \frac{1}{2} S \cdot S \tag{2}$$

$$I_1 = \mathrm{tr}\, T \tag{3}$$

In order to figure out the material parameters A and B, two common mechanical tests might be performed, namely: the isostatic pressure and the uniaxial compression.

Considering both tests as having a perfectly uniform stress state, the stress tensors for any representative point in the material are shown below. For isostatic pressure:

$$T = \begin{bmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{bmatrix} \tag{4}$$

and for uniaxial compression:

$$T = \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5}$$

For uniaxial compression, the stress deviator tensor and the invariants J_2 and I_1 are given by:

$$S = \begin{bmatrix} \frac{2}{3}T_{11} & 0 & 0 \\ 0 & -\frac{1}{3}T_{11} & 0 \\ 0 & 0 & -\frac{1}{3}T_{11} \end{bmatrix} \tag{6}$$

$$J_2 = \frac{1}{3} T_{11}^2 \tag{7}$$

$$I_1 = T_{11} \tag{8}$$

Considering the yield in uniaxial compression, the yield stress T_Y that depends on material density ρ, is defined as the stress at the plastic deformation onset for a density ρ.

By doing $T_{11} = T_Y$ and substituting the invariants J_2 and I_1 in the yield function one has:

$$f = \left(\frac{A}{\sqrt{3}} + B \right) T_Y \tag{9}$$

Introducing the concept of equivalent stress one might write:

$$f = A\sqrt{J_2} + BI_1 = \left(\frac{A}{\sqrt{3}} + B \right) T_Y \tag{10}$$

Defining the yield pressure P_Y as the pressure at the plastic deformation onset for a density ρ and considering $P = P_Y$, the isostatic pressure test leads to $J_2 = 0$ and $I_1 = 3P_Y$, giving rise to:

$$f = 3BP_Y = \left(\frac{A}{\sqrt{3}} + B \right) T_Y \tag{11}$$

From Eq. (11) one can find a relation between A and B which must be satisfied in order to make the yield function consistent with this two mechanical tests:

$$\frac{A}{B} = \sqrt{3}\left(\frac{3P_Y - T_Y}{T_Y} \right) \tag{12}$$

It is possible to rewrite the yield function as:

$$f = A\sqrt{J_2} + BI_1 = 3BP_Y \tag{13}$$

and substituting A using Eq. (12) and rearranging terms one obtains:

$$f = \sqrt{3}\left(1 - \frac{T_Y}{3P_Y} \right)\sqrt{J_2} + \frac{T_Y}{3P_Y} I_1 = T_Y \tag{14}$$

The expression above is consistent with the physical aspect that a material from ingot metallurgy has a plastic behavior that is pressure independent. This can be checked by making P_Y tend to infinity and noting that the classical Mises type yield function is recovered.

The classical material parameters A and B can be rewritten in terms of T_Y and P_Y as:

$$A = \sqrt{3}\left(1 - \frac{T_Y}{3P_Y} \right) \tag{15}$$

$$B = \frac{T_Y}{3P_Y} \tag{16}$$

Also, these parameters definition means that:

$$A = \sqrt{3}(1 - B) \tag{17}$$

At this stage, it would be useful to introduce a relation between the density dependent uniaxial yield stress $T_Y(\rho)$ and the uniaxial yield stress for the dense material T_0 , an usual relation between this two quantities is:

$$T_Y = \delta T_0 \tag{18}$$

where δ may, for example, be a simple linear function of the relative density R:

$$\delta = \frac{R - R_0}{1 - R_0} \tag{19}$$

with the relative density R being defined by:

$$R = \frac{\text{density of the porous material}}{\text{density of the dense material}} = \frac{\text{volume of the dense material}}{\text{volume of the porous material}} \tag{20}$$

which is equal to one for the fully dense material, and is equal to R_0 for the initial relative density.

From Eq. (19) one can see that δ is equal to zero when R is equal to R_0, and is equal to one when R is equal to one. Figure 1a compares the proposed function δ with a nonlinear function previously proposed by Doraivelu (1984), in the same way Fig. 1b compares the uniaxial yield stress T_Y for copper with yield stress of 60 MPa for the dense material.

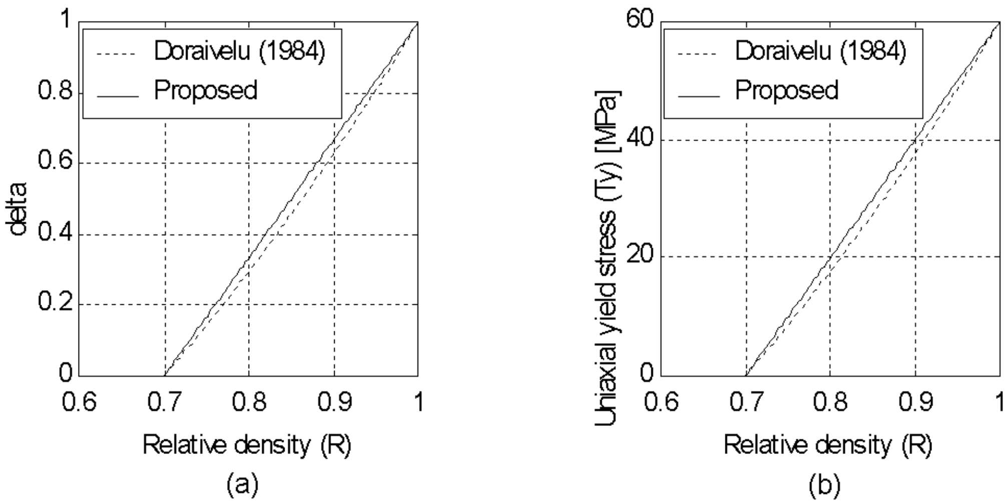

Figure 1 - Function δ and yield stress T_Y for a initial relative density R_0 = 0.7

From Eq. (16) one can see that the parameter B relates P_Y with T_Y , and may write:

$$P_Y = \left(\frac{1}{3B}\right) T_Y \tag{21}$$

Introducing a function β , defined by:

$$\beta = \frac{1}{3B} \tag{22}$$

One has:

$$P_Y = \beta T_Y \tag{23}$$

As before, the authors try to introduce a simple form for the function β, and use the following heuristic reasoning: first, as the relative density approaches the unity the yield stress T_Y approaches T_0 and the yield pressure P_Y tends to infinity. Hence the function β should also tend to infinity, second, for the initial relative density R_0 the yield stress T_Y is assumed to be near zero. Therefore, unless β has a non-finite value, the yield pressure P_Y will be near zero too and one will assume β to be equal to a positive value k, which this work arbitrarily assumes to be equal one. Summarizing:

$$\begin{aligned} &\text{for} \quad R = 1 \quad \Rightarrow \quad P_Y = \beta T_0 \to \infty \quad \Rightarrow \quad \beta \to \infty \\ &\text{for} \quad R = R_0 \quad \Rightarrow \quad P_Y = 0 \quad \Rightarrow \quad \beta = k = 1 \end{aligned} \tag{24}$$

The following form for β is proposed:

$$\beta = \left(\frac{1 - R_0}{1 - R} \right) \tag{25}$$

which is very similar to δ and satisfies the previous conditions. Then one comes to:

$$P_Y = \beta \delta T_0 = \left(\frac{1 - R_0}{1 - R} \right) \left(\frac{R - R_0}{1 - R_0} \right) T_0 \tag{26}$$

and rearranging the terms gives (Fig. 3):

$$P_Y = \left(\frac{R - R_0}{1 - R} \right) T_0 \tag{27}$$

and one may finally write:

$$f = A\sqrt{J_2} + BI_1 = \delta T_0 \tag{28}$$

with A and B defined by:

$$A = \sqrt{3}(1 - B) \tag{29}$$

$$B = \frac{1}{3\beta} = \frac{1}{3} \left(\frac{1 - R}{1 - R_0} \right) \tag{30}$$

The authors also emphasize that B is a linear function of R. Figure 2 shows a plot of parameters A and B with the variation of the relative density R.

Doraivelu (1984) related the stress states P_Y and T_Y by equating their energy of deformation and obtaining the following expression:

$$P_Y = \frac{T_Y}{\sqrt{3}\sqrt{1-R^2}} \quad (31)$$

where R is the relative density. For $T_Y = \delta T_0$, and using Doraivelu's (1984) proposal for δ we obtain the following expression for the yield pressure in terms of the relative density R (Fig. 3):

$$P_Y = \left(\frac{R^2 - R_0^2}{1-R_0^2}\right)\frac{T_0}{\sqrt{3}\sqrt{1-R^2}} \quad (32)$$

This expression is comparable (Fig. 3) with a previous result by Ashby (Akisanya, 1997):

$$P_Y \cong 3R^2\left(\frac{R-R_0}{1-R_0}\right)T_0 \quad (33)$$

Figure 3 shows a plot of Eqs. (27), (32) and (33) along with the experimental results from (Akisanya, 1997).

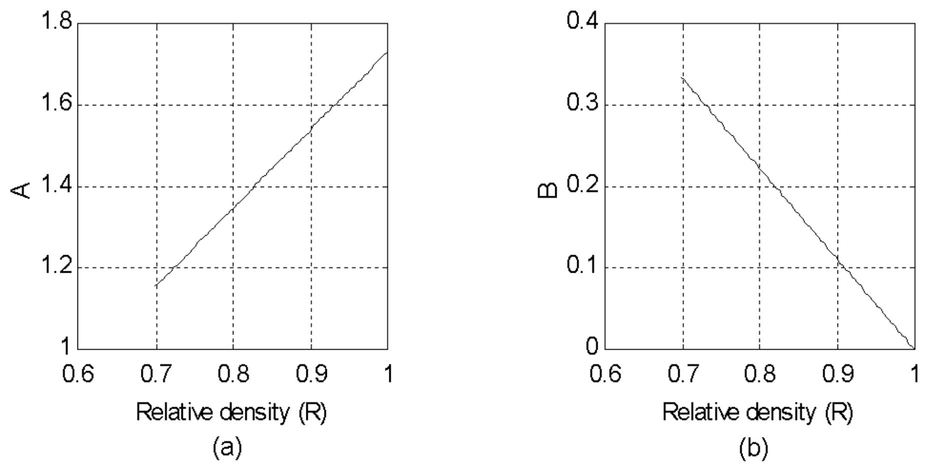

Figure 2 - Variation of parameters A and B for a initial relative density $R_0 = 0.7$

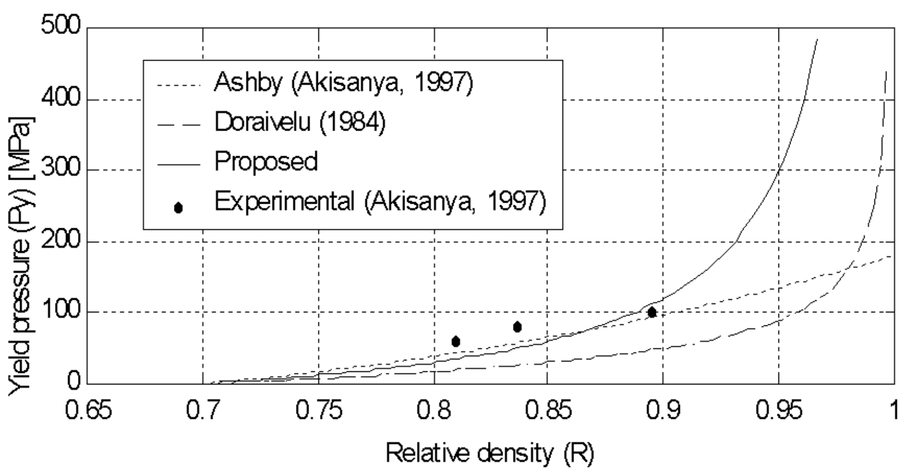

Figure 3 - Yield pressure P_Y for T_0 = 60 MPa and R_0 = 0.7

2.2. Plastic Deformation Rate

Considering the yield function, Eq. (28), as a plastic potential for the rate of deformation, i.e., assuming that normality rule applies, one has:

$$\dot{\varepsilon} = \dot{\lambda}\nabla f = \dot{\lambda}\left(A\nabla\left(\sqrt{J_2}\right) + B\nabla I_1\right) \tag{34}$$

$$\dot{\varepsilon} = \dot{\lambda}\left(\frac{A}{2}\frac{S}{\sqrt{J_2}} + B\mathbf{1}_3\right) \tag{35}$$

where $\mathbf{1}_3$ is the identity matrix.

3. YIELD FUNCTION FOR PROBLEMS WITH SYMMETRY OF REVOLUTION

In the case of the uniaxial compression of a cylindrical part one has:

$$\dot{\varepsilon}_2 = \dot{\varepsilon}_3 \tag{36}$$

with

$$\dot{\varepsilon}_2 = \dot{\lambda}\left(\frac{A}{2}\frac{S_2}{\sqrt{J_2}} + B\right) \tag{37}$$

and

$$\dot{\varepsilon}_3 = \dot{\lambda}\left(\frac{A}{2}\frac{S_3}{\sqrt{J_2}} + B\right) \tag{38}$$

then

$$S_2 = S_3 \tag{39}$$

The second invariant of the stress deviator tensor J_2 for this case is:

$$J_2 = \frac{1}{2}S \cdot S = \frac{1}{2}\left(S_1^2 + S_2^2 + S_3^2\right) = \frac{1}{2}\left(S_1^2 + 2S_2^2\right) \tag{40}$$

Writing the stress deviator components using the stress tensor components:

$$S_1 = T_1 - P = T_1 - \frac{1}{3}\left(T_1 + T_2 + T_3\right) = \frac{2}{3}T_1 - \frac{1}{3}T_2 - \frac{1}{3}T_3 \tag{41}$$

$$S_2 = T_2 - P = T_2 - \frac{1}{3}\left(T_1 + T_2 + T_3\right) = \frac{2}{3}T_2 - \frac{1}{3}T_1 - \frac{1}{3}T_3 \tag{42}$$

with

$$T_2 = T_3 \tag{43}$$

one has:

$$S_1 = \frac{2}{3}(T_1 - T_2) \tag{44}$$

and

$$S_2 = \frac{1}{3}(T_2 - T_1) \tag{45}$$

Substitution of the stress deviator components in J_2:

$$J_2 = \frac{1}{2}\left[\frac{4}{9}(T_1 - T_2)^2 + \frac{2}{9}(T_2 - T_1)^2\right] = \frac{1}{9}\left[2(T_1 - T_2)^2 + (T_1 - T_2)^2\right] = \frac{1}{3}(T_1 - T_2)^2 \tag{46}$$

then

$$\sqrt{J_2} = \frac{1}{\sqrt{3}}(T_1 - T_2) \tag{47}$$

The first invariant of the stress tensor is calculated by:

$$I_1 = T_1 + T_2 + T_3 = T_1 + 2T_2 \tag{48}$$

Substitution of the second invariant of the stress deviator tensor J_2 , Eq. (47), and of the first invariant of the stress tensor I_1 , Eq. (48), in the yield function, Eq. (14):

$$f = \left(1 - \frac{T_Y}{3P_Y}\right)(T_1 - T_2) + \frac{T_Y}{3P_Y}(T_1 + 2T_2) = T_Y \tag{49}$$

Rearranging the terms:

$$f = T_1 + \left(\frac{T_Y}{P_Y} - 1\right)T_2 = T_Y \tag{50}$$

and as:

$$B = \frac{T_Y}{3P_Y} \tag{51}$$

one can write:

$$f = T_1 + (3B - 1)T_2 = T_Y \tag{52}$$

4. APPROXIMATE SOLUTIONS - UPSETTING OF A DISK

On applying equilibrium conditions on a small element of the disk:

$$(\sigma_r + d\sigma_r)\, d\theta\, (r + dr)h - \sigma_r d\theta\, r\, h - 2\sigma_\theta dr\, h \frac{d\theta}{2} + \mu\, p\, 2\, (r\, d\theta\, dr) = 0 \tag{53}$$

Upon eliminating $d\theta$:

$$(\sigma_r + d\sigma_r)\, (r + dr)h - \sigma_r r\, h - \sigma_\theta dr\, h + \mu\, p\, 2\, (r\, dr) = 0 \tag{54}$$

Rearranging the terms and neglecting the second order differential term one obtains:

$$\sigma_r \, dr \, h + d\sigma_r \, r \, h - \sigma_\theta dr \, h + 2\mu \, p \, (r \, dr) = 0 \qquad (55)$$

For r = 0 in the differential equation of equilibrium, Eq. (55), one has:

$$\sigma_r = \sigma_\theta \qquad (56)$$

Assuming this condition to be valid for every value of r one comes to:

$$d\sigma_r \; h + 2\mu \, p \, dr = 0 \qquad (57)$$

By making $T_1 = p$ and $T_2 = \sigma_r$ in the yield function, Eq. (52):

$$p = (1 - 3B)\sigma_r + T_Y \qquad (58)$$

Upon substituting the expression p, Eq. (58), in the differential equation of equilibrium, Eq. (57), and after rearranging the terms:

$$\frac{-d\sigma_r}{(1-3B)\sigma_r + T_Y} = \frac{2\mu}{h} dr \qquad (59)$$

Knowing that in r = D/2 , $\sigma_r = 0$, one is able to perform the following integration:

$$-\int_{\sigma_r}^{0} \frac{d\sigma_r}{(1-3B)\sigma_r + T_Y} = \frac{2\mu}{h} \int_{r}^{\frac{D}{2}} dr \qquad (60)$$

and obtain:

$$\sigma_r = \left(\frac{T_Y}{1-3B}\right)\left[e^{\frac{2\mu}{h}(1-3B)\left(\frac{D}{2}-r\right)} - 1\right] \qquad (61)$$

Upon substitution of Eq. (61) in Eq. (58) one has:

$$p = T_Y \, e^{\frac{2\mu}{h}(1-3B)\left(\frac{D}{2}-r\right)} \qquad (62)$$

Assuming Coulomb friction; the shear stress at the interface is given by:

$$\tau = \mu \, p = \mu \, T_Y \, e^{\frac{2\mu}{h}(1-3B)\left(\frac{D}{2}-r\right)} \qquad (63)$$

From Eq. (48), with $T_1 = p$ and $T_2 = \sigma_r$, the hydrostatic pressure is given by:

$$P_H = \frac{1}{3} I_1 = \frac{1}{3}(p + 2\sigma_r) \qquad (64)$$

Figure 4 shows the stress results for a disk with 37.2 millimeters of diameter and 2.7 millimeters of thickness, considering Coulomb friction $\mu = 0.1$.

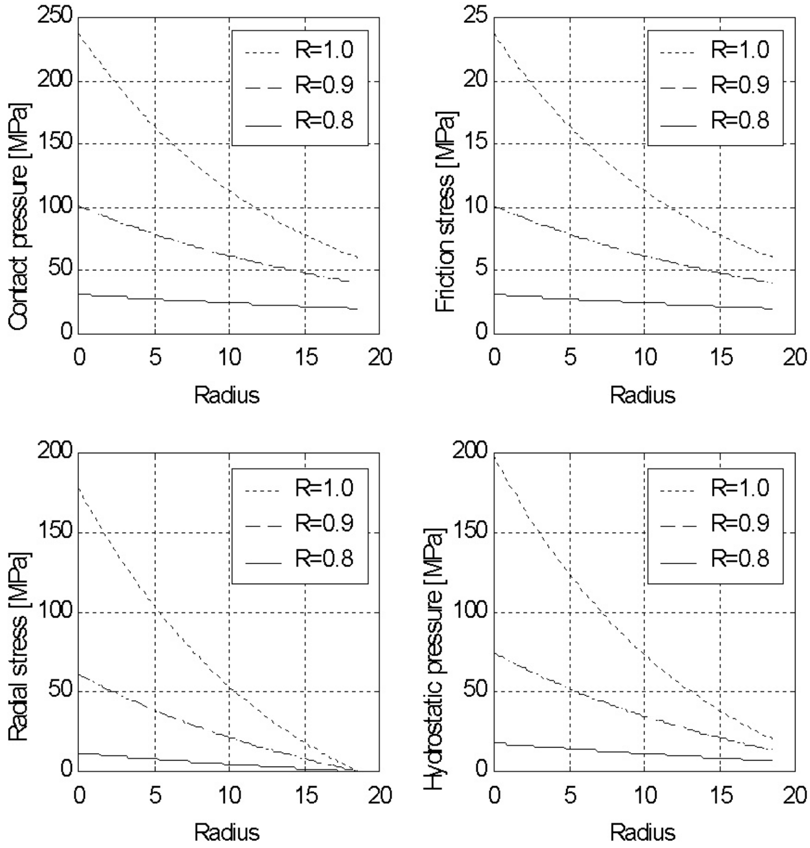

Figure 4 - Stress results for a disk with D = 37.2 mm, h = 2.7 mm, $R_0 = 0.7$ and $T_0 = 60$ MPa.

5. CONCLUSION

Using the Slab Method, an approximate analysis for the upsetting of a porous disk was presented. This analysis managed to obtain a closed form solution, which despite been approximate, can help in the description of the problem as a whole. The Drucker-Prager yield criterion was used in the analysis, which takes into account the influence of the hydrostatic pressure in the plastic deformation of the material.

6. ACKNOWLEDGEMENTS

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