MODAL ANALYSIS OF BEAMS: AN EXPERIMENT FOR TEACHING

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Abstract. *It is presented in this paper some basic concepts of modal analysis aimed to serve as a reference for teaching this subject to undergrad students. It is described an experimental apparatus and the associated theory which allows to obtain the natural frequencies and modes of vibration of a cantilever beam. The results are attractive in the sense that the students get more involved with the subject by experimenting directly with the set up here described.*

Keywords: *modal analysis, experiments, beam vibration*

1. INTRODUCTION

The subject of structural vibration is part of the minimum curriculum in many engineering courses. Usually, a basic vibration course comprises the analysis of a mass connected to a frame with a spring and a dashpot. This simple, one degree of freedom, system is used to present the major features of the vibration motion, including natural frequency, forced vibration, damping, etc...

Single degree of freedom system can be quite versatile in the sense that many real structures can be modelled using this concept, at least in a preliminary design phase. However, although sometimes neglected in an introductory vibration course, continuous systems are more realistic, despite of being more complex to analyse. The simplest continuous system, a rod, can be used to introduce the concept of natural modes of vibration, their associated natural frequencies, and the application of Fourier series to analyse forced motion.

Nevertheless, the beams are really the first continuous system that has a wider use in the real world but it is sometimes neglected as a subject in a typical one semester vibration course. Experimental vibration is a more complex topic for teaching since it is necessary to have facilities like benches, shakers, acelerometers, amplifiers, PC data acquisition cards, osciloscopes, etc... Nevertheless, no doubt the students become more motivated and so learn much more when they can actually see a real structural vibration problem.

Bearing this in mind, the authors prepared an experiment aiming at teaching modal analysis of beams for students who have attended an introductory vibration course. This paper aims to describe it in details the basic theoretical and experimental procedure adopted in order to awake the students curiosity to the important problem of modal analysis.

2. MODAL ANALYSIS OF A CANTILEVER

An experimental set up was deviced to study the vibration of beams. It was decided to use a very simple beam arrangement which could suit well experimental facilities restraints. Accordingly, a cantilever was chosen since it is very easy to fix its clamped support to a shaker. The analysis is also somewhat simple, as described next.

Consider a cantilever beam in line with the axis x with length l, breadth b, height h, made of a material with density ρ and elastic modulus E. The moment of inertia in relation to the axis z is I(x) and the cross-section area is A(x).

Figure 1 presents the beam under analysis with a free body digramm where equilibrium of transverse force gives

$$\left(V(x,t)+\frac{\partial V(x,t)}{\partial x}dx\right)-V(x,t)+f(x,t)dx=\rho\cdot A(x)dx\frac{\partial^2 w(x,t)}{\partial t^2} \tag{1}$$

Equlibrium of bending moments actind on the infinitesimal element of width dx leads to

$$\left[M(x,t)+\frac{\partial M(x,t)}{\partial x}dx\right]-M(x,t)+\left[V(x,t)+\frac{\partial V(x,t)}{\partial x}dx\right]dx+\left[f(x,t)dx\right]\frac{dx}{2}=0 \tag{2}$$

such that transverse force, V (x,t), and bending moment, M(x,t), are related by

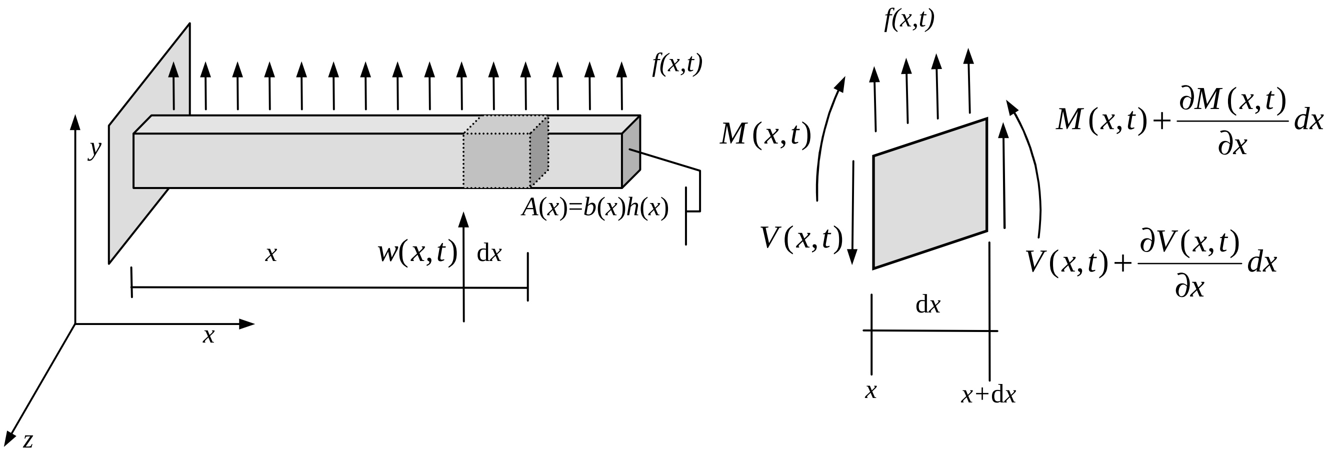

Figure 1. Cantilever beam and an infinitesimal element.

$$V(x,t) = -\frac{\partial M(x,t)}{\partial x} \tag{3}$$

when disregarding high order terms.

By invoking Hookes law and integrating it through the beam cross section, it is possible to express the bending moment with the beam transverse deflection w(x,t) using

$$M(x,t) = EI(x)\frac{\partial^2 w(x,t)}{\partial x^2} \tag{4}$$

From equations (1) to (4) and focusing on a free vibration problem, i.e. f(x,t) = 0, with A(x) e EI(x) constants, it follows that

$$\frac{\partial^2 w(x,t)}{\partial t^2} + c^2\frac{\partial^4 w(x,t)}{\partial x^4} = 0, \qquad c = \sqrt{\frac{EI}{A\rho}} \tag{5}$$

which is the equation of motion for the vibration of a beam.

Equation (5) can be solved by assuming a solution in the form w(x,t)=X(x)T(t), such that

$$c^2\frac{X^{(4)}(x)}{X(x)} = -\frac{\ddot{T}(t)}{T(t)} = \omega^2 \tag{6}$$

where the constant ω is revealed to be the natural frequency when solving

$$\ddot{T}(t) + \omega^2 T(t) = 0 \tag{7}$$

whose solution is

$$T(t) = A \cdot sin(\omega t) + B \cdot \cos(\omega t) \tag{8}$$

The constants A and B are obtained from the initial conditions of the problem.

Equation (6) also gives the spatial behaviour of a beam when solving

$$X^{(4)}(x) - \beta^4 X(x) = 0, \qquad \beta^4 = \frac{\omega^2}{c^2} = \frac{A\rho\omega^2}{EI} \tag{9}$$

whose solution is

$$X(x) = a_1 sin(\beta x) + a_2\cos(\beta x) + a_3 sinh(\beta x) + a_4\cosh(\beta x) \tag{10}$$

where β and three of the constants a_i should be determined from the boundary conditions, which are

$$M(0,t) = V(0,t) = w(l,t) = \Theta(l,t) = 0 \tag{11}$$

for the particular cantilever case here examined.

2.1 Solution

From the boundary conditions in equation (11), it is possible to show that

$$X(0) = X'(0) = EI \cdot X''(l) = EI \cdot X'''(l) = 0 \tag{12}$$

and in oder to avoid a trivial solution,

$$\cos(\beta l) \cdot \cosh(\beta l) = 1 \tag{13}$$

which gives,

$$\begin{aligned} \beta_1 \cdot l &= 1.87510407 \\ \beta_2 \cdot l &= 4.69409113 \\ \beta_3 \cdot l &= 7.85475744 \end{aligned} \tag{14}$$

The actual cantilever to be tested as described later has the following physical parameters

l = 0.5 m
E = 210 GPa
I = 2.258 x 10^{-9} m^4
A = 3.002 x 10^{-4} m^2
ρ = 7860 kg/m^3

which, when using equation (9), (13) and (14), gives the following natural frequencies

ω_1 = 199.4 rad/s .: f_1 = 31.7 Hz
ω_2 = 1249.4 rad/s .: f_2 = 198.8 Hz
ω_3 = 3498 rad/s .: f_3 = 557 Hz

As for the vibration modes, they become

$$X_n = a_4 \left[-\cos(\beta_n x) + \cosh(\beta_n x) + \frac{\left(\cos(\beta_n l) + \cosh(\beta_n l)\right)}{sin(\beta_n l) + sinh(\beta_n l)} \cdot \left(sin(\beta_n x) - sinh(\beta_n x)\right) \right] \tag{15}$$

$n = 1,2,3...$

whose necessary parameters are all known except the constant a_4 which is an arbitrary constant. These modes of vibration are depicted in Figure 2.

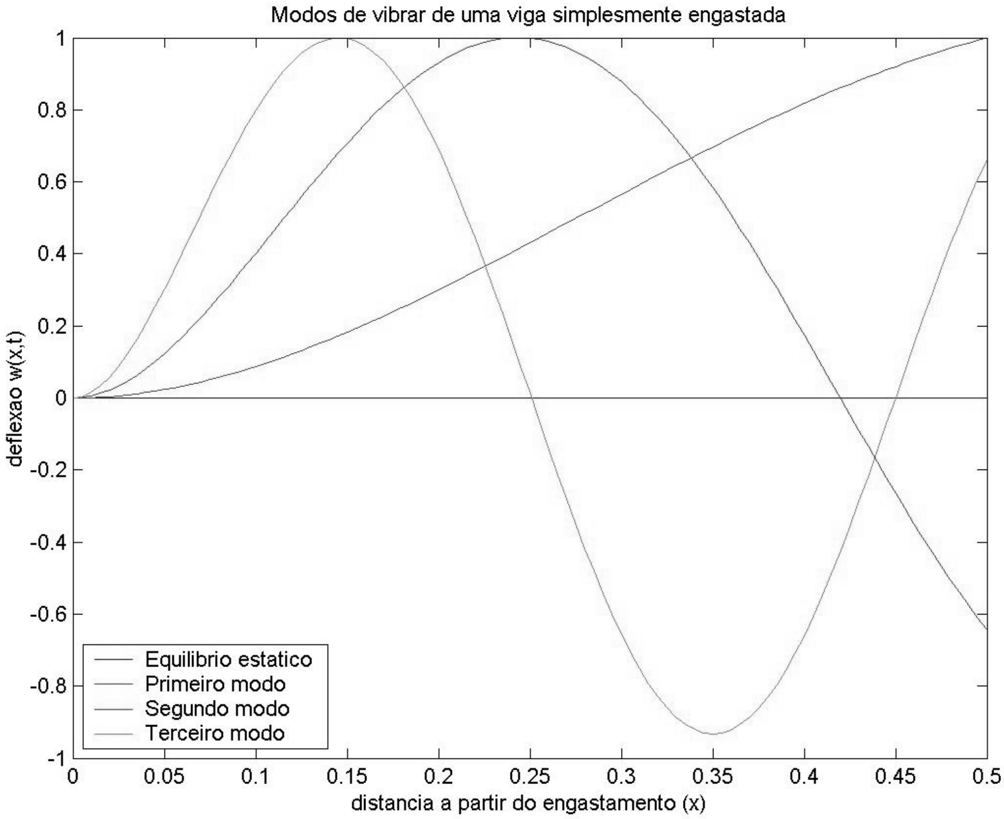

Figure 2: Theoretical modes of vibration for a cantilever beam.

3. INTRODUCING EXPERIMENTAL MODAL ANALYSIS

Modal analysis is a procedure to extract information on the natural modes of vibration and their respective natural frequencies and damping parameters. To perform an experimental modal analysis, it is necessary to excite the structure and there are a few techniques available. The excitation can come from a shaker or from the impact of a hammer. The shaker can be excited by a steady frequency sinusoidal wave or by sweeping it with a range of sinusoidal frequencies.

In the present paper, the technique adopted consists in to excite the cantilever beam with a hammer along 10 points of the beam length. Each excitation produces beam motion, detected into a single station by na accelerometer fixed at the free end of the beam. The impact produced by the hammer is capable of exciting various modes of vibration at once, which should be uncoupled by performing a spectral analysis of the response signal. Each of the 10 points, equally spaced by 5cm, was excited three times and the accelerometer signal was averaged in time before performing the spectral analysis.

The spectral analysis was done using the software MATLAB, such that the frequency response function, $H(\omega)$, was obtained for each measured station. This function presents higher amplitudes for the natural frequencies and to detect the exact frequency the Nyquist circle, formed by the real and imaginary part of the spectrum, is used. To extract and analyse the experimental data, a programm was developed which fits the Nyquist circle. From this circle, it is possible to obtain the damping coefficient for each natural frequency.

4. BASIC EQUATIONS FOR THE EXPERIMENTAL MODAL ANALYSIS

To uncouple the various recorded signals, the equation of motion for the beam can be written as

$$M\ddot{x} + C\dot{x} + Kx = fe^{j\omega_{dr}t} \tag{16}$$

where M, C and K are the mass, damping and stifness matrix. The solution of this equation for an excitation f is x(t)=**u**e $^{j\omega_{dr}t}$

$$(K - \omega_{dr}^2 M + j\omega_{dr} C)u = f \tag{17}$$

where **u** is the displacement vector whose components are associated with the motion in each degree of freedom, given by

$$u = (K - \omega_{dr}^2 M + j\omega_{dr} C)^{-1} f \tag{18}$$

or

$$\mathbf{u}=\alpha(\omega_{dr})\mathbf{f} \tag{19}$$

where $\alpha(\omega_{dr})$ is the receptance matrix, proved to be

$$\alpha(\omega_{dr})=[\mathrm{S}\,\mathrm{diag}[\omega_i^2-\omega_{dr}^2+2\zeta_i\omega_i\omega_{dr}\mathrm{j}]\mathrm{S}^{\mathrm{T}}]^{-1} = \mathrm{S}^{-\mathrm{T}}\mathrm{diag}\left[\frac{1}{\omega_i^2-\omega_{dr}^2+\zeta_i\omega_i\omega_{dr}\mathrm{j}}\right]\mathrm{S}^{-1} \tag{20}$$

with $S=M^{1/2}P$ and P a matrix formed by the eigen-vectors of $M^{-1/2}KM^{-1/2}$.

By noting that S^{-T} is formed by modes of vibration, represented by the vector $u_{i,}$ equation (20) can be further expressed as a sum of n matrix such that

$$\boxed{\alpha(\omega_{dr}) = \sum_{i=1}^{n}\left[\frac{u_i u_i^T}{(\omega_i^2 - \omega_{dr}^2) + (2\zeta_i\omega_i\omega_i)j}\right]} \tag{21}$$

which is formed again by the eigen-vectors. A given element of $\alpha(\omega_{dr})$ in the position s-r is the transfer function between the response in s, u_s, to the excitation, f_r, in r when all other excitation are fixed to zero. Hence,

$$\alpha_{sr}(\omega_{dr}) = \sum_{i=1}^{n}\frac{[u_i u_i^T]_{sr}}{\omega_i^2 - \omega_{dr}^2 + (2\zeta_i\omega_i\omega_{dr})j} \tag{22}$$

Now, by assuming that continous system exhibit modes of vibration well separated apart, the sum in equation (22) will be dominated by term associated with the natural frequency, so that, for $\omega_{dr}=\omega_i$, equation (22) becomes

$$|\alpha_{sr}(\omega_{dr})| = \frac{|u_i u_i^T|_{sr}}{|(\omega_i^2 - \omega_i^2) - 2\zeta_i \omega_i \omega_i j|} = \frac{|u_i u_i^T|_{sr}}{2\zeta_i \omega_r^2} \tag{23}$$

or

$$|u_i u_i^T|_{sr} = |2\zeta_i \omega_i^2| |H_{sr}(\omega_i)| \tag{24}$$

where $|H_{sr}(\omega_i)| = |\alpha_{sr}(\omega_i)|$ is the measured magnitude of the transfer function between the points s and r for the natural frequency i.

Equation (24) gives only the magnitude but not the signal such that it is not possible to determine the direction of the movement. However, the phase of $H(\omega_i)$ can be used to determine the signal of $|u_i u_i^T|_{sr}$.

5. EXPERIMENTAL RESULTS AND DISCUSSION

The beam shown in Figure 3 has at its tip an accelerometer connected to a charge amplifier, whose analogic signal is fed to a data acquisition board.

Figure 3: Cantilever beam

By processing the data generated by the beam response to the various hammer excitations, the spectrum in Figure 4[a] is obtained. It is evident in the figure the peaks in the amplitude related to the natural frequencies. This is further corroborated by plotting the real and imaginary part of this spectrum, so to obtain the Nyquist circle in Figure 4b. In Figure 4c, the phase of the transfer function is plotted, allowing the calculation of the motion direction.

The natural frequencies for the theoretical model were 31.7 Hz, 198.8 Hz and 556.7 Hz, whereas the ones obtained by the technique here described are 28.5 Hz, 184 Hz e 507 Hz.

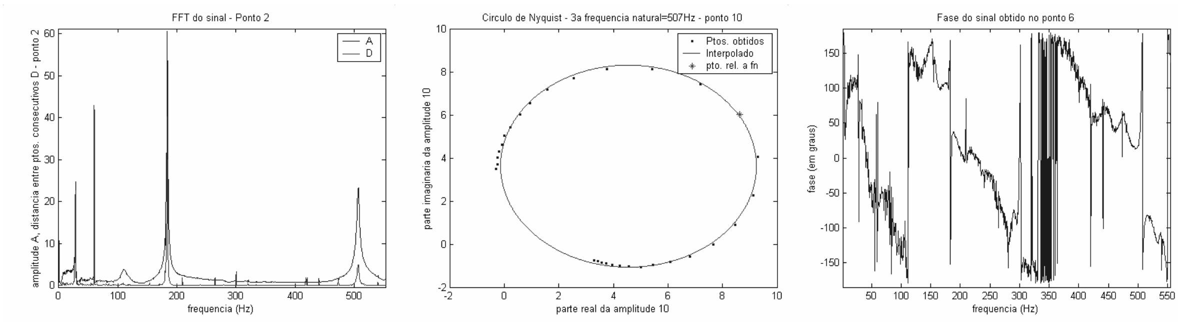

Figura 4. FFT, Nyquist circle and phase of the transfer function H(ω).

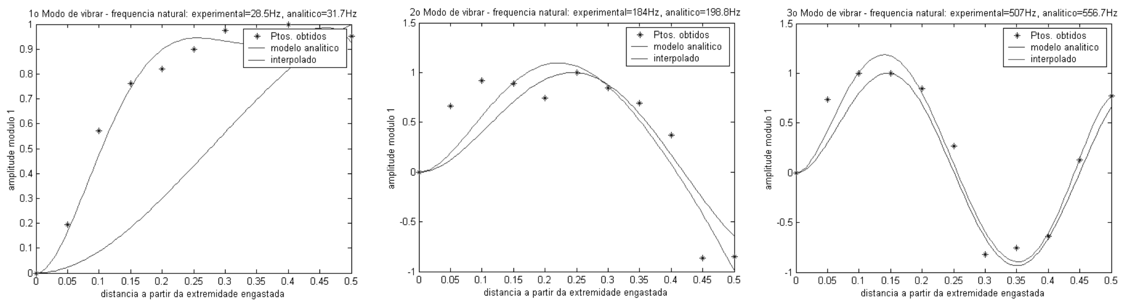

Figura 5. The first three modes of vibration for the cantilever beam. Green- analytical, red- interpolation of the experimental data, dotted in blue.

The natural frequencies were obtained within an error of no more than 10%. This is considered to be a reasonable result, when bearing in mind factors like the beam having a not perfectly rectangular cross-section and, mainly, that it is difficult to assure a fully clamped condition to the beam.

It is evident in Figure 5 that the first mode of vibration was not accurately described although higher modes were. Despite this shortcoming, the technique used here gives to the students a much more attractive way to study vibration and moda analysis. The technique used is relatively easy to implement and the experimental set up allows the determination of the natural frequencies also by merely tunning the frequency with a signal generator.

The possibility of showing to the students other techniques and to present in a clear way basic concepts like natural frequency and mode of vibration make the experiment above quite useful and attractive.

6. REFERENCES

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