OPTIMAL BI-IMPULSIVE ORBITAL TRANSFERS BETWEEN NON-COPLANAR ORBITS WITH TIME AND POSITION CONSTRAINTS

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Abstract. *In this work the problem of two-impulsive orbital transfers between non-coplanar orbits with minimum fuel consumption and fixed time is studied, considering fixed terminal points in the initial and the final orbits. The basis of this study are the equations presented by Eckel and Vinh (1984), which provide the transfer orbit between non-coplanar elliptical orbits with minimum fuel and fixed time of transfer, considering free terminal points. When the problem with fixed time is considered, the position of the terminal points must be taken into account. When fixed terminal points are considered, it is required to leave the initial orbit from a specific point at a specific time and to arrive to the final orbit in a specific point at a specific time. Other two cases occur when only one of the terminal points is fixed. And the last case occurs when both terminal points are free. Then, some equations considering the position constraints are added to the equations presented by Eckel and Vinh (1984), and a software for orbital maneuvers was developed. This software is available to be used in the next missions developed by INPE. The software was tested, simulating real maneuvers with success.*

Keywords: Astrodynamics, Orbital Transfer, Impulsive Maneuvers, Maneuvers Optimization

1. INTRODUCTION

The majority of the spacecrafts that have been placed in orbit around the Earth uses the basic concept of orbital transfers. During the launch, the spacecraft is placed in a parking orbit distinct from the final orbit for which the spacecraft was designed. Therefore, to reach the desired final orbit the spacecraft must perform orbital transfers. Besides that, the spacecraft orbit must be corrected periodically because there are perturbations acting on the spacecraft. Both maneuvers are usually calculated with minimum fuel consumption but without a time constraint. This time constraint imposes a new characteristic to the problem that rules out the majority of the transfer methods available in the literature: Hohmann (1925), Hoelker et al. (1959), Gobetz et al. (1969), Prado (1989), etc. Therefore, the transfer methods must be adapted to this new constraint: Prussing et al. (1986), Eckel (1982), Eckel et al. (1984), Lawden (1993) and Taur et al. (1995). In Brazil, important applications have been carried on with the launch of the Remote Sensing Satellites RSS1 and RSS2 that belongs to the Complete Brazilian Space Mission and with the launch of the China Brazil Earth Resources Satellites CBERS 1, 2, 3 and 4.

In this work the problem of two-impulsive orbital transfers between non-coplanar orbits with minimum fuel consumption and fixed time is studied, considering fixed terminal points in the initial and the final orbits (initial point and final point of the maneuver). The basis of this study are the equations presented by Eckel and Vinh (1984), which provide the transfer orbit between non-coplanar elliptical orbits with minimum fuel and fixed time of transfer; or minimum time of transfer for a prescribed fuel consumption, considering free terminal points. In this work, the problem of the fuel consumption minimization with fixed time of transfer and fixed position of the terminal points is considered. This is an extension of the work of Rocco (1997) and Rocco et al. (1999). The case of orbital transfer between non-coplanar orbits with minimum time for a prescribed fuel consumption considering free terminal points, was already studied in Rocco et al. (2000). When the problem with fixed time is considered, the position of the terminal points must be taken into account. When fixed terminal points are considered, it is required to leave the initial orbit from a specific point at a specific time and to arrive to the final orbit in a specific point at a specific time. Other two cases occur when only one of the terminal points is fixed. And the last case occurs when both terminal points are free. Then, some equations considering the position constraints are added to the equations presented by Eckel and Vinh (1984), and a software for orbital maneuvers was developed. This software is available to be used in the next missions developed by INPE. The original method, developed by Eckel and Vinh, was presented without numerical results in that paper. Thus, the modifications considering the position constraints, the implementation and the solutions using this method are contributions of this work. The software was tested, simulating real maneuvers with success.

2. DEFINITION OF THE PROBLEM

The orbital transfer of a spacecraft from an initial orbit to a desired final orbit consists (Marec 1979) in a change of state of the spacecraft, from initial conditions $\vec{r}_0$, $\vec{v}_0$ and m_0 at time t_0 to final conditions $\vec{r}_f$, $\vec{v}_f$ and m_f at time t_f ($t_f \geq t_0$). In this work we consider that the spacecraft propulsion system is able to apply an impulsive thrust. Therefore, we have the instantaneous variation of the spacecraft velocity.

3. PRESENTATION OF THE METHOD

The basis for this method are the equations presented by Eckel et al. (1984). The equations were presented in the literature but the method was neither implemented nor tested by Eckel and Vinh, and it is only valid for a specific geometry. They used the plane of the transfer orbit as the reference plane but in this work, it was decided to use the equatorial plane as the reference plane because in this way it is easy to obtain and to apply the results in real applications. Using the transfer orbit as the reference plane almost all the results obtained belong to the same specific geometry, so we change the reference system, adding the equations 1 to 6 to consider cases with more complex geometry.

Given two terminal orbits it was desired to obtain a transfer orbit which performs an orbital maneuver from the initial orbit to the final orbit with minimum velocity increment and fixed time of transfer. The orbits are specified by their orbital elements:

a → Semi-major axis
e → Eccentricity
p → Semi-latus rectum
ω → Longitude of the periapsis
i → Inclination
Ω → Longitude of the ascending node
M → Mean anomaly
E → Eccentric anomaly
λ → Angle between the planes of the initial and final orbits
β_1 → True anomaly of the point N obtained in the plane of the initial orbit
β_2 → True anomaly of the point N obtained in the plane of the final orbit
I_1 → Location of the first impulse
I_2 → Location of the second impulse
Δ → Transfer angle obtained in the plane of the transfer orbit
V_1 → Velocity increment generated by the first impulse
V_2 → Velocity increment generated by the second impulse
V → Total velocity increment
T → Time spent in the maneuver
α_1 → True anomaly of the point I_1 obtained in the plane of the initial orbit
α_2 → True anomaly of the point I_2 obtained in the plane of the final orbit
r_1 → Distance from point I_1
r_2 → Distance from point I_2
f_1 → True anomaly of the point I_1 obtained in the plane of the transfer orbit
f_2 → True anomaly of the point I_2 obtained in the plane of the transfer orbit
h_i → Horizontal component of V_i
x_1 → Radial component of the first impulse
x_2 → Radial component of the second impulse
y_1 → Transverse component of the first impulse in the plane of the initial orbit
y_2 → Transverse component of the second impulse in the plane of the transfer orbit
z_1 → Component of the first impulse orthogonal to the initial orbit
z_2 → Component of the second impulse orthogonal to the transfer orbit
N → Intersection of the orbits

From the geometry of the maneuver we obtain β_1, β_2, λ and the transfer angle Δ:

$$\beta_1 = \arctan\left[\frac{\sin(\Omega_2-\Omega_1)\tan(180^\circ - i_2)}{\sin i_1 + \tan(180^\circ - i_2)\cos i_1 \cos(\Omega_2-\Omega_1)}\right] - \omega_1 \tag{1}$$

$$\beta_2 = \arctan\left[\frac{sin(\Omega_2-\Omega_1)\tan i_1}{sini_2 + \tan i_1 \cos(180^\circ - i_2)\cos(\Omega_2-\Omega_1)}\right] - \omega_2 \tag{2}$$

$$\lambda = \arcsin\left[\frac{sin(\Omega_2-\Omega_1)sini_1}{sin(\omega_2+\beta 2)}\right] = \arcsin\left[\frac{sin(\Omega_2-\Omega_1)sini_2}{sin(\omega_1+\beta_1)}\right] \tag{3}$$

$$\cos\Delta = \cos(\beta_1-\alpha_1)\cos(\alpha_2-\beta_2) + \sin(\beta_1-\alpha_1)\sin(\alpha_2-\beta_2)\cos(180^\circ-\lambda) \tag{4}$$

$$\sin\Delta = \frac{\sin(\alpha_2-\beta_2)\sin(180^\circ-\lambda)}{\sin B} \tag{5}$$

$$B = \arctan\left[\frac{\sin(180^\circ-\lambda)}{\sin(\beta_1-\alpha_1)\cot(\alpha_2-\beta_2) - \cos(\beta_1-\alpha_1)\cos(180^\circ-\lambda)}\right] \tag{6}$$

Considering that the spacecraft propulsion system is able to apply an impulsive thrust, and that maneuver is bi-impulsive, the total velocity increment is:

$$V = V_1 + V_2 = F(X) \tag{7}$$

The time of the transfer maneuver, when it is not considered any position constraint, is:

$$T = G(X) \tag{8}$$

Therefore, the problem is the minimization of V for a prescribed T. If the time of transfer is prescribed, being equal to a value T_0, we have the constrained relation:

$$T - T_0 = 0 \tag{9}$$

Thus, we have the performance index:

$$J = V + k(T - T_0) \tag{10}$$

From Eckel et al. (1984) we know that the solution of the problem depends on three variables: the semi-latus rectum p of the transfer orbit and the true anomaly α_1 and α_2 that define the position of the impulses in the initial and final orbits. Therefore, we have the necessary conditions:

$$\frac{\partial V}{\partial p} + k\frac{\partial T}{\partial p} = 0 \quad ; \quad \frac{\partial V}{\partial \alpha_1} + k\frac{\partial T}{\partial \alpha_1} = 0 \quad ; \quad \frac{\partial V}{\partial \alpha_2} + k\frac{\partial T}{\partial \alpha_2} = 0 \tag{11}$$

By eliminating the Lagrange's multiplier k from Equations (11) we have the set of two equations:

$$\frac{\partial V}{\partial p}\frac{\partial T}{\partial \alpha_1} - \frac{\partial V}{\partial \alpha_1}\frac{\partial T}{\partial p} = 0 \quad ; \quad \frac{\partial V}{\partial p}\frac{\partial T}{\partial \alpha_2} - \frac{\partial V}{\partial \alpha_2}\frac{\partial T}{\partial p} = 0 \tag{12}$$

Evaluating the partial derivatives in these equations and doing some simplifications we have the final optimal conditions:

$$(X_1 + YZe\sin f_2)(S_1 q_1 - T_1 e \sin f_1) + S_1 T_1 + W_1\left(\frac{W_1 - W_2}{\sin\Delta} q_2 - W_1 \tan\frac{\Delta}{2}\right) - \frac{W_1 Zer_1 e_1 \sin\alpha_1}{q_1 p_1 \sin f_1 \sin\gamma_1} = 0 \tag{13}$$

$$(X_2 + YZe\sin f_1)(S_2 q_2 - T_2 e \sin f_2) + S_2 T_2 - W_2\left(\frac{W_2 - W_1}{\sin\Delta} q_1 - W_2 \tan\frac{\Delta}{2}\right) + \frac{W_2 Zer_2 e_2 \sin a_2}{q_2 p_2 \sin f_2 \sin\gamma_2} = 0 \tag{14}$$

which utilize the relations shown in appendix A.

Considering position constraints, the time spent in the maneuver can be calculated by three different ways.

Case 1: when we consider fixed terminal points we want to leave the initial orbit from a specific point at a specific time and to arrive in the final orbit in a specific point at a specific time. In this case the time spent in the maneuver can be calculated by:

$$T = \Delta t_1 + \Delta t + \Delta t_2 \tag{15}$$

where Δt_1 is the time spent in the initial orbit; Δt is the time spent in the transfer orbit and Δt_2 is the time spent in the final orbit to reach the final terminal point.

Another two cases occur when only one of the terminal points is fixed. In these cases the problem has one constraint less than the anterior case.

Case 2: when the initial terminal point is fixed the method can be applied to optimal orbital transfers where the localization of the maneuver is specified, for example, when the maneuver need to be performed in visibility. The time spent in the maneuver can be calculated by:

$$T = \Delta t_1 + \Delta t \tag{16}$$

Case3: When the final terminal point is fixed the method can be applied to rendezvous maneuvers. The time spent in the maneuver can be calculated by:

$$T = \Delta t + \Delta t_2 \tag{17}$$

Case 4: The last case occurs when both terminal points are free and the only constraint is the duration of the maneuver. In this case, the interest is not where the maneuver is going to be performed, but how long the spacecraft is going to stay in the transfer orbit. Thus, the method can be applied in maneuvers of remote sensing satellites because these satellites can not work properly in the transfer orbit. The time spent in the maneuver can be calculated by:

$$T = \Delta t \tag{18}$$

The variables Δt_1, Δt and Δt_2 can be calculated by:

$$\Delta t_1 = \frac{a_1^{1.5}\left(E_1 - e_1 \sin E_1\right)}{\sqrt{\mu}} \tag{19}$$

$$\Delta t = \left(\frac{p}{1-e^2}\right)^{1.5} \frac{\left(E_f - e\sin E_f\right) - \left(E_i - e\sin E_i\right)}{\sqrt{\mu}} \tag{20}$$

$$\Delta t_2 = \frac{a_2^{1.5}\left(E_3 - e_2 \sin E_3\right) - \left(E_2 - e_2 \sin E_2\right)}{\sqrt{\mu}} \tag{21}$$

where E_i is the eccentric anomaly in the transfer orbit of the position of the first impulse; E_f is the eccentric anomaly in the transfer orbit of the position of the second impulse; E_3 is the eccentric anomaly in the final orbit of the position of the second terminal point.

Thus, we have an equation system composed by Equations (9), (13) and (14). Solving this equation system by Newton Raphson Method (cf. Press et al. 1992) or by the Least Square Method (cf. Rocco 2002), we obtain the transfer orbit which performs the maneuver between two non-coplanar terminal orbits spending a minimum fuel consumption but with a specific time of transfer.

4. RESULTS

Figures (1) to (9) present some results obtained with the software developed. Only the case 2 was considered in this example, but, the other cases were also studied and implemented. It was decided to show only case 2 because this case presented the best results until the moment. The graphs were obtained through the variation of the time spent in the maneuver. These graphs not only show the tendency of the parameters, but they quantify the evolution of the variables studied.

It was utilized as an example the correction maneuver between two elliptical non-coplanar orbits where the initial orbit have the semi-major axis of 7122.237 km, eccentricity 0.014161, longitude of the periapsis 1.72253089 rad, longitude of the ascending node 0.005 rad and inclination 0.005 rad. The final orbit shows the semi-major axis of 7148.865 km, eccentricity 0.0011, longitude of the periapsis 1.57079633 rad, longitude of the ascending node 0.01 rad and inclination 0.01 rad. We utilized in this example the initial values p = 7095 km, α_1 = 1.0 rad, and α_2 = 2.0 rad. The graphs were obtained through the variation of the time spent in the maneuver from 1950 to 2450 s.

5. CONCLUSION

In Figures 1 to 9 it can be verified the behavior of some orbital elements of the transfer orbit when the time spent in the maneuver is varied.

In Figure 1 it can be observed that the semi-major axis decreases, from the value 7097 km, when *T* is 1950 s, and it begins to increase when *T* is 2250 s. In Figures 2 and 3, that show the variation of the eccentricity and the inclination, we have a similar behavior. This happens due to the geometric arrangement of the initial and final orbits. Figures 4, 5 and 6 show respectively, the variation of the longitude of the ascending node, the variation of the longitude of the periapsis and the variation of the transfer angle ($\Delta = E_f - E_i$). Figure 7 shows the time spent by the satellite in the initial orbit Δt_1, from the longitude of the periapsis until the point of the application of the first impulse, and the time spent in the transfer orbit Δt. It can be seen that Δt_1 decreases while Δt increases with the increase of *T*. This was expected, because as it was seen in Figure 6 the transfer

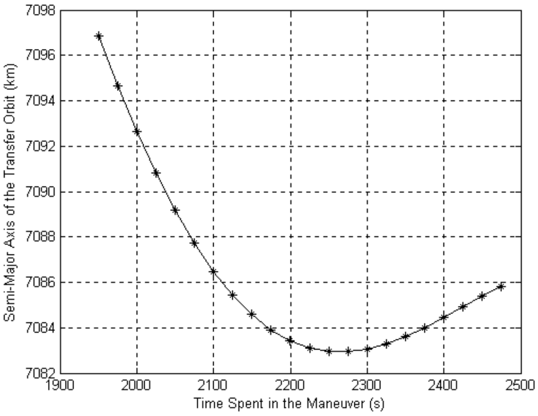

Fig. 1 – Semi-Major Axis vs. Time

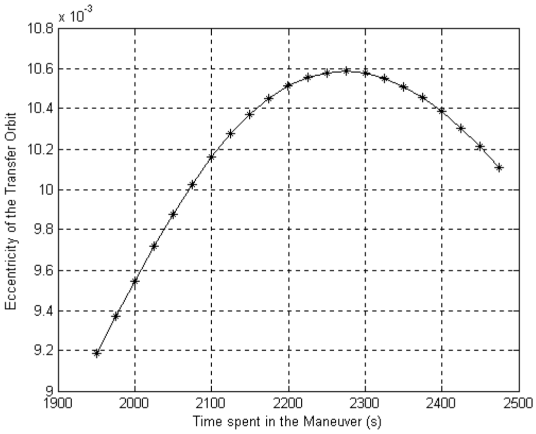

Fig.2 – Eccentricity vs. Time

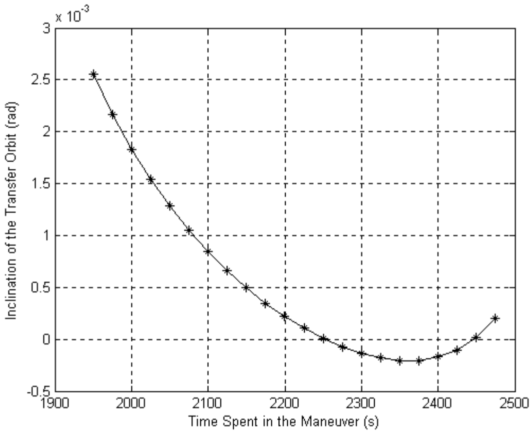

Fig. 3 – Inclination vs. Time

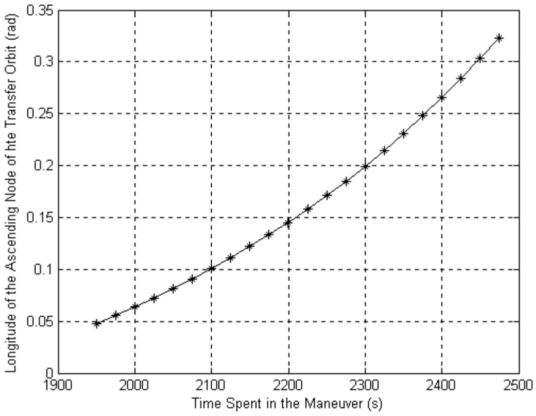

Fig. 4 – Ascending Node vs. Time

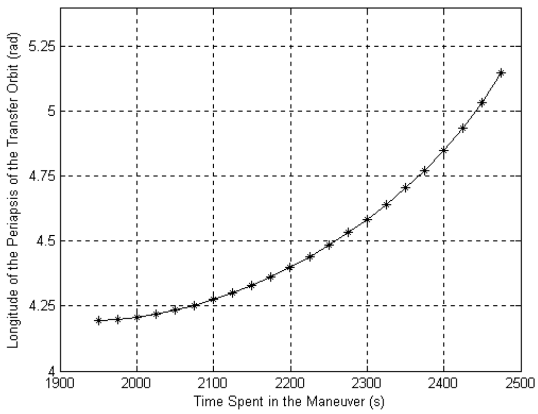

Fig.5 – Periapsis vs. Time

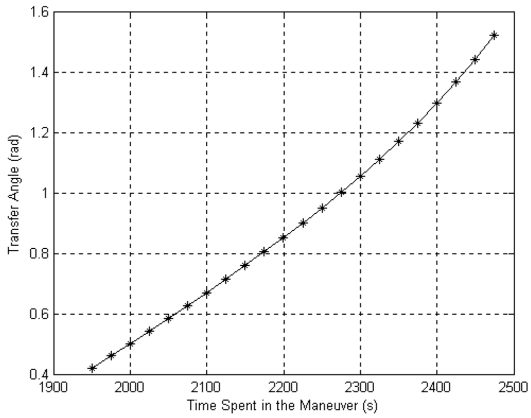

Fig. 6 – Transfer Angle vs. Time

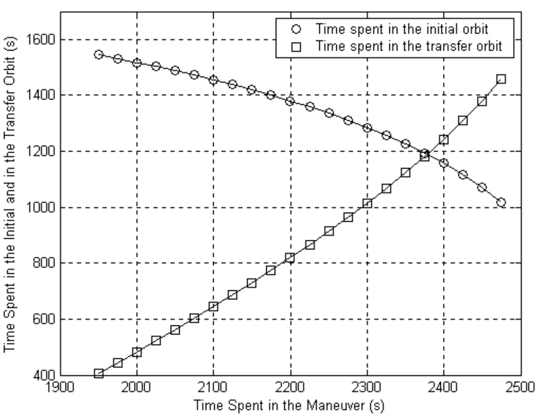

Fig. 7 – Δt_1 and Δt vs. Time

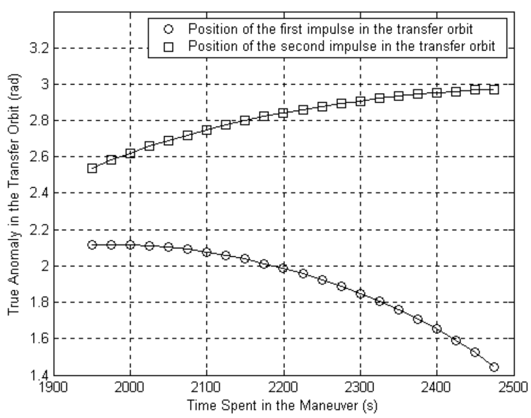

Fig. 8 –True Anomaly vs. Time

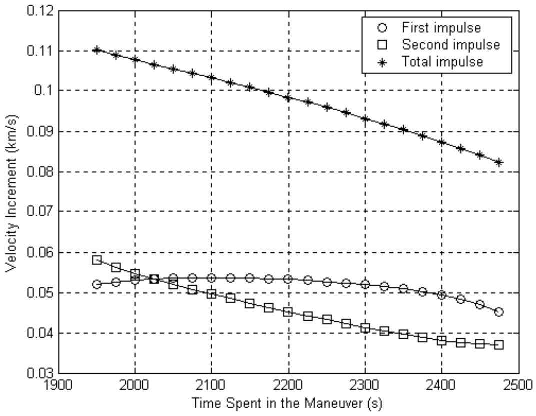

Fig. 9 – Velocity Increment vs. Time

angle Δ increases with the increase of *T*. The increase of Δ can also be verified in the Figure 8 that shows the location of the points of application of the impulses in the transfer orbit. In Figure 9 it is verified that the total velocity increment decreases with the increase of *T*, as it was expected due to the increase of Δ with the increase of *T*. When the time *T* is increased the necessary velocity increment decreases drastically because in this case the maneuver is accomplished with a larger transfer angle so that the direction of the impulses approaches the directions of the velocity vector of the satellite in the initial and final orbits. Thus the fuel consumption becomes smaller. But this is an academic study. For practical applications we have to consider the limitations of the satellite propulsion system. Besides that, we should advise that the developed program can not supply the solution for all combinations of the input parameters. For very small or very large values of the time spent in the maneuver the solution can not exist, or the numerical algorithms used in the program do not converge for the solution, because the initial values used can be too far from the solution. So, it is recommended a physical analysis of the problem, that takes into account the geometry of the maneuver, to find the range of values for the time, so that it is possible to accomplish the maneuver. It is important to notice that the software tests all the results, verifying if the maneuver obtained is just a mathematical solution or if it can really be implemented. When we use numerical methods there are some solutions, which satisfy the equations, however, in practice, they are impossible. This happened especially in cases 1 and 3, where some solutions presented negative values of the eccentricity, Δt_1 and Δt_2. Concluding, we can verify that these results are very similar to the results obtained by Rocco (1997) for the coplanar case. Therefore, the cases considering coplanar and non-coplanar maneuvers, were studied, implemented and tested with success. The simulations showed that the software developed can be used in real applications and it is capable to generate reliable results.

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8. APPENDIX A

$$r_i = \frac{p_i}{1 + e_i \cos\alpha_i} \tag{A.1}$$

$$f_1 = \arctan\left[\cot \varDelta - \frac{r_1(p - r_2)}{r_2(p - r_1)\sin \varDelta}\right]$$

$$f_2 = \arctan\left[\frac{r_2(p - r_1)}{r_1(p - r_2)\sin \varDelta} - \operatorname{cotg} \varDelta\right] \tag{A.2}$$

$$p = \frac{r_1 r_2(\cos f_1 - \cos f_2)}{r_1 \cos f_1 - r_2 \cos f_2} \tag{A.3}$$

$$e = \frac{r_2 - r_1}{r_1 \cos f_1 - r_2 \cos f_2} \tag{A.4}$$

$$a = \frac{p}{1 - e^2} \tag{A.5}$$

$$\gamma_i = \arcsin\left[-\frac{\sin(\beta_i - \alpha_i)}{\sin \varDelta}\sin\phi\right] \tag{A.6}$$

$$h_i = \left(y_i^2 + z_i^2\right)^{1/2} \tag{A.7}$$

$$V_i = \left(x_i^2 + h_i^2\right)^{1/2} \tag{A.8}$$

$$S_i = \frac{x_i}{V_i} \tag{A.9}$$

$$T_i = \frac{y_i}{V_i} \tag{A.10}$$

$$W_i = \frac{z_i}{V_i} \tag{A.11}$$

$$q_i = \frac{p}{r_i} \tag{A.12}$$

$$\begin{aligned} x_1 &= \sqrt{\mu}\left(\frac{e}{\sqrt{p}}\sin f_1 - \frac{e_1}{\sqrt{p_1}}\sin\alpha_1\right) \\ x_2 &= \sqrt{\mu}\left(\frac{e_2}{\sqrt{p_2}}\sin\alpha_2 - \frac{e}{\sqrt{p}}\sin f_2\right) \end{aligned} \tag{A.13}$$

$$\begin{aligned} y_1 &= \frac{\sqrt{\mu}}{r_1}\left(\sqrt{p} - \sqrt{p_1}\cos\gamma_1\right) \\ y_2 &= \frac{\sqrt{\mu}}{r_2}\left(\sqrt{p_2}\cos\gamma_2 - \sqrt{p}\right) \end{aligned} \tag{A.14}$$

$$z_i = \frac{\sqrt{\mu\, p_i}}{r_i}\sin\gamma_i \tag{A.15}$$

$$E_i = \arccos\left(\frac{e + \cos f_i}{1 + e\cos f_i}\right) \tag{A.16}$$

$$\sin E_i = \frac{\sqrt{1-e^2}\sin f_i}{1 + e\cos f_i} \tag{A.17}$$

$$M_i = E_i - e\sin E_i \tag{A.18}$$

$$T = \sqrt{\frac{a^3}{\mu}}\left(M_{I_2} - M_{I_1} + 2\pi N\right) \tag{A.19}$$

$$\begin{aligned} X_1 &= \frac{S_1\cos\Delta - S_2}{\sin\Delta} + T_1 \\ X_2 &= \frac{S_1 - S_2\cos\Delta}{\sin\Delta} + T_2 \end{aligned} \tag{A.20}$$

$$Y = \frac{1}{\left(1-e^2\right)\sin\Delta}\left[3e^2\mathrm{T}\sqrt{\frac{\mu}{p^3}} - 2e\left(\frac{1}{q_2\sin f_2} - \frac{1}{q_1\sin f_1}\right) + \cot f_2 - \cot f_1\right] \tag{A.21}$$

$$Z = \frac{q_2X_2 - q_1X_1 + \left(S_1 + S_2\right)\tan \Delta/2}{\cot f_1 - \cot f_2 + Y\left[\left(1+e^2\right)\sin\Delta + 2e\left(\sin f_2 - \sin f_1\right)\right]} \tag{A.22}$$