# THE BOUNDARY ELEMENTS METHOD APPLIED TO INCOMPRESSIBLE VISCOUS FLUID FLOW PROBLEMS

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***Abstract.*** *Integral equations formulation for steady flow problems of a viscous fluid is presented based on the boundary elements Method (BEM). The Continuity, Navier Stokes and Energy equations are used for calculation of the flow field. The governing differential equations, in terms of primitive variables, are derived using velocity-pressure-temperature formulation. The related 2-D fundamental solution tensors are derived. Applications to simple flow cases, such as the driven cavity, step and deep cavity are presented. Convergence difficulties are indicated, which have limited the applications to flows of low Reynolds numbers.*

***Key word***: *Boundary Elements, fundamental solution, integral equations.*

## 1.INTRODUCTION.

The need for solution of the system of partial differential equations which model the flow of a fluid in channels such as pipes, blade passages, nozzles and others, appeared the very day the fluid flow was modeled. The difficulties involved in obtaining closed solutions, even for very simple flows, required the development of clever techniques, but only with the application of numerical solutions to those equations system, some flows of practical interest were calculated. Several computational techniques have been used. Finite difference, finite element, finite volume and boundary element are among those, just to name the most known. As new algorithms were discovered and faster computers were produced, each of those methods evolved in all areas in the past years. Finite difference methods have been, and already are, implemented to solve flow problems. Finite elements gained attention in the past decades; in the seventies it was still crawling. Both are bases for commercial codes for the solution of flows of almost every kind. Computer effort has been limiting the application of the numerical methods in the sense that every new discovered method of solution claims reduction in CPU time and storage requirements, but the reasons that these methods are so CPU time and storage hungry are intrinsic. Nevertheless, the boundary element method has progressed differently depending on the areas where it has been applied - fastest in areas related to solid mechanics, slowest in the ones related to fluid mechanics. The method is similar to the finite element method. While the latter solves an algebraic equation system obtained from integrals over elements in which the volume is divided (finite element), the former solves integrals over the boundary of such elements (boundary element). Surface integrals are obtained by transformation of volume integrals using the Green-Gauss theorem. The boundary element technique may reduce the computational effort because of the problem deals with contour variables only. For the development of the computational code and the formulation of the problems, the works of Tosaka et. Al (1985), Kakuda and Tosaka (1988) and Tosaka and Fukushima were used. Despite integral methods were available many decades ago for the application to flow problems of practical interest, a comprehensive study of the formulation and application to flow problems are still being considered more recently, as they are expected to alleviate sensibly the storage and hopefully CPU time, because roughly the degrees of freedom is reduced in the boundary element method.

Although this apparent advantage, requiring less computational effort when volume integrals are transformed into surface integrals, some disadvantages arise, such as higher mathematical complexity in order to get an usable computational formulation; the need for the calculation of infinite integrals; dense matrices whose inversion is more time consuming than the band matrices in the finite difference and finite element schemes. Application of the method to the calculation of flows, using the classical problems of a) stepped channel, b) box with moving lid, and c) deep cavity flow are presented.

## 2. STATEMENT OF THE PROBLEM.

Let Ω be a domain in $R^2$ and Γ its closed boundary; $\vec{n}_i$ be the normal vector to the boundary; the fluid be a perfect gas, incompressible and viscous; (*x, y*) be a point of domain. The steady state conservation equations in cartesian co-ordinates can be written as:

Mass

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

x-momentum

$$\mu\left[\nabla^2 u+\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]-\frac{\partial P}{\partial x}=\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) \tag{2}$$

y-momentum

$$\mu\left[\nabla^2 v+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]-\frac{\partial P}{\partial y}=\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) \tag{2b}$$

Energy

$$k\nabla^2 T=\rho c_v\left[u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right] \tag{3}$$

The variables are:

| | |
|---|---|
| $u$, $v$ | flow relative velocities in the x and y directions |
| ρ, $P$, $T$ | static density, pressure and temperature |
| μ, ν | absolute and cinematic viscosities |
| $k$, $c_v, c_p$ | thermal conductivity and specific heats at constant volume and pressure |

Let the following change of variable take effect in the conservation equations:

$$t^*=\frac{tV_\infty}{L} \qquad T^*=\frac{T}{T_\infty} \qquad P^*=\frac{P-P_\infty}{\rho V_\infty^2} \qquad \rho^*=\frac{\rho}{\rho_\infty} \qquad y^*=\frac{y}{L}$$

$$u^*=\frac{u}{V_\infty} \qquad v^*=\frac{v}{V_\infty} \qquad V_\infty=\sqrt{u_\infty^2+v_\infty^2} \qquad \mu^*=\frac{\mu}{\mu_\infty} \qquad \nu^*=\frac{\nu}{\nu_\infty}$$

$$k^*=\frac{k}{k_\infty} \qquad \mathrm{Re}=\frac{\rho_\infty V_\infty L}{\mu_\infty} \qquad \mathrm{Pr}=\frac{c_p\mu_\infty}{k_\infty}$$

where ∞ refer to the far stream condition. Re and Pr are the Reynolds and Prandtl numbers, respectively.

Then, the conservation equations become:

mass

$$\frac{\partial u^*}{\partial x}+\frac{\partial v^*}{\partial y}=0 \tag{4}$$

x-momentum

$$\nabla^2 u^*+\frac{\partial}{\partial x^*}\left(\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}\right)-\mathrm{Re}\frac{\partial P^*}{\partial x^*}=\mathrm{Re}\left(u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}\right) \tag{5a}$$

y-momentum

$$\nabla^2 v^* + \frac{\partial}{\partial y^*}\left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*}\right) - \mathrm{Re}\frac{\partial P^*}{\partial y^*} = \mathrm{Re}\left(u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*}\right) \tag{5b}$$

energy

$$-\frac{1}{\mathrm{Re}}\nabla^2 T^* = \mathrm{Pr}\left(u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T}{\partial y^*}\right) \tag{6}$$

For the sake of simplicity, the asterisk will be dropped in what follows.

The independent variables of the problem are $u$, $v$, $T$ and $P$. It is possible to rewrite the conservation equations in matrix form as

$$[L]\ \{U\} = \{B\} \tag{7}$$

where $[L]$ is a linear partial differential operator, $\{U\} = \{u \quad v \quad T \quad P\}^t$ is the vector of the unknowns and $\{B\}$ the vector of nonlinear convected terms. Depending on the assumptions made, $[L]$ and $\{B\}$ can take different forms. For instance, vector $\{B\}$ can be linearised and the linear terms included in $[L]$.

Let, for the moment, all non-linear terms be included into $\{B\}$. Then

$$L_{IJ} = \begin{bmatrix} \nabla^2 + D_1 D_1 & D_1 D_2 & 0 & -\mathrm{Re}\, D_1 \\ D_2 D_1 & \nabla^2 + D_2 D_2 & 0 & -\mathrm{Re}\, D_2 \\ 0 & 0 & \frac{1}{\mathrm{Re}}\nabla^2 & 0 \\ D_1 & D_2 & 0 & 0 \end{bmatrix}; U_J = \begin{Bmatrix} u \\ v \\ T \\ p \end{Bmatrix}; B_I = \begin{bmatrix} \mathrm{Re}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) \\ \mathrm{Re}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) \\ \mathrm{Pr}\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) \\ 0 \end{bmatrix} \tag{8}$$

(I, J=1, 2, 3, 4)

For the sake of simplification, let

$$D_1 = \frac{\partial}{\partial x}; \qquad D_2 = \frac{\partial}{\partial y}; \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{9}$$

## 3. THE METHOD.

It is not known any analytical solution of equation (7). Let $\tilde{U}_J$ be an approximate solution in the sense of $L_{IJ}\tilde{U}_J - B_I = R \cong 0$, that is, $\tilde{U}_J$ differs from $U_J$ very little but it is not equal to $U_J$.

A possible solution can be obtained provided that

$$\int_\Omega (L_{IJ} U_J - B_I) W_{IK}\, d\Omega = 0 \tag{10}$$

where $W_{IK}$ is an appropriate weight function. As it will be shown later, $W_{IK}$ is chosen as the fundamental solution tensor for the adjoint of $L_{IJ}$.

Hörmander´s (1965) method is used for the calculation of the weight function and fundamental solution. Although it does not provide $W_{IK}$ directly, it allows, as a first step, the combination of several partial differential operators $L_{IJ}$ into a single differential operator, from which the tensor $W_{IK}$ is calculated. The weight tensor $W_{JK}$ or the fundamental solution may be determined as a solution of steady Stokes problem with heat transfer:

$$\overline{L}_{IJ}W_{JK} + \delta_{IK}\delta(x-y) = 0 \tag{11}$$

where $\delta(x-y)$ is the Dirac delta function and $\overline{L}_{IJ}$ is the adjoint operator of $L_{IJ}$.

Hörmander's method is simultaneously applied to the Navier Stokes equations, continuity and energy for incompressible and steady flow:

$$[\overline{L}][W] = -[I]\delta(x-y) \tag{12}$$

After a left multiplication of equation (11b) by $[\overline{L}]^{-1}$ one finds $[W]$:

$$[W] = -[L]^{-1}\delta(x-y) = -[I]\delta(x-y) \tag{13}$$

where, $[L]^{-1} = \text{Adj}[L]/\text{Det}[L]$ and

$$\text{Adj}[L] = \text{Cof}^{t}[L] = \begin{bmatrix} D_2^2\nabla^2 & -D_2D_1\nabla^2 & 0 & -D_1\nabla^2\nabla^2 \\ -D_1D_2\nabla^2 & D_1^2\nabla^2 & 0 & -D_2\nabla^2\nabla^2 \\ 0 & 0 & \text{Re}\,\nabla^2\nabla^2 & 0 \\ \frac{1}{\text{Re}}D_1\nabla^2\nabla^2 & \frac{1}{\text{Re}}D_2\nabla^2\nabla^2 & 0 & \frac{2}{\text{Re}}\nabla^2\nabla^2\nabla^2 \end{bmatrix} \tag{14}$$

whose terms are $x_{ij} = (-1)^{(i+j)} m_{ij}$, where the terms $m_{ij}$ are the minors of $[L]$, and

$$\det[L] = \nabla^2\left(\nabla^2\left(\nabla^2\right)\right) \tag{15}$$

Thus,

$$[W] = [L]^{-1}\delta(x-y) = \text{Cof}^{t}[L]\,(\det L)^{-1}[I]\delta(x-y) \tag{16}$$

Let

$$\phi^* = (\text{Det}[L])^{-1}\delta(x-y) \tag{17}$$

Then

$$\text{Det}[L]\ \phi^* = \delta(x-y) \tag{18}$$

whose solution $\phi^*$ is:

$$\phi^*(x,y) = \frac{1}{128\pi} r^4 \ln r \tag{19}$$

where $r = \|x-y\|$ denotes the distance between $x$ and $y$.

Therefore, the fundamental solution tensor $W_{JK}$ can be determined explicitly from equations (13) and (16) in conjunction with (19) as follows:

$$W_{11} = \frac{1}{4\pi}\left[\ln(r) + 1 + \frac{(y-y_i)^2}{r^2}\right] \qquad W_{42} = -\frac{1}{2\pi}\left[\frac{(y-y_i)}{r^2}\right]$$

$$W_{21} = -\frac{1}{4\pi}\left[\frac{(x-x_i)(y-y_i)}{r^2}\right] \qquad W_{13} = W_{23} = W_{43}$$

$$W_{31} = 0 \qquad W_{33} = \frac{\text{Re}}{4\pi}(3 + 2\ln r)$$

$$W_{41}=\frac{1}{2\pi}\left[\frac{(x-x_i)}{r^2}\right] \qquad W_{14}=\frac{1}{2\pi\,\mathrm{Re}}\left[\frac{(x-x_i)}{r^2}\right]$$

$$W_{12}=W_{21} \qquad W_{24}=\frac{1}{2\pi\,\mathrm{Re}}\left[\frac{(y-y_i)}{r^2}\right]$$

$$W_{22}=\frac{1}{4\pi}\left[\ln(r)+1+\frac{(x-x_i)^2}{r^2}\right] \qquad W_{34}=W_{44}$$

$$W_{32}=0 \tag{20}$$

## 4. DISCRETIZATION

Let the Green-Gauss theorem be applied to equation (10) so that the domain integral is transformed into an integral over the contour $\Gamma$, divided into $n_e$ boundary elements. Then,

$$\int_{\Omega}\left(L_{IJ}U_J-B_I\right)W_{IK}\,d\Omega=0 \qquad (I,\ J,\ K=1,\ 2,\ 3,\ 4) \tag{21}$$

which tells that the system of differential equations has been transformed into a system of algebraic equations (21) that involves the values of the variables at each boundary element. If one finds the values of the variables at the elements in the boundary, the solution n the boundary is obtained.

After the application of the Green-Gauss theorem and integrating by parts over$\Omega$, one arrives at the following equation that holds for every boundary element as well:

$$C_{KI}(y)U_I(y)=\int_{\Gamma}\left\{u_i(x)\Sigma_{IK}(x,y)-\tau_i(x,y)W_{IK}\right\}d\Gamma(x)+$$
$$+\int_{\Gamma}\frac{1}{\mathrm{Re}}\left\{T(x)\frac{\partial W_{3K}}{\partial n(x)}(x,y)-q(x)W_{3K}(x,y)\right\}d\Gamma(x)+\int_{\Omega}B_I(x)W_{IK}(x,y)\,d\Omega(x) \tag{22}$$

where summation is implied by repeating indices.

It is worth mentioning that the second member of equation (22) comprises integrals over the boundary and over the domain, these due to the non-linear convective terms $B_I$.

In equation (22), $C_{IK}$ is the tensor coefficient dependent on the geometry of the boundary. Its value is ½, 1 or 0, provided the point *y* lies over a locally regular boundary, in the domain or outside the boundary, respectively. If *y* lies at a corner of $\Gamma$, its value is $\alpha/2\pi$, where $\alpha$ is the angle formed by the left and right tangents to $\Gamma$. Also,

$$q(x)=\frac{\partial T(x)}{\partial n} \tag{23}$$

$$\Sigma_{iK}(x,y)=(-W_{4K}\delta_{ij}+W_{iK,j}+W_{jK,i})n_j \tag{24}$$

$$\tau_i(x)=(-\mathrm{Re}\,p\delta_{ij}+u_{i,j}+u_{j,i})n_j \tag{25}$$

where the coma (,) indicates derivative with respect to the following index and summation is implied by repeating indices.

From equation (24) the values of $\Sigma_{IK}$ are calculated:

$$\Sigma_{11}=\frac{1}{\pi}\left\{\left[\frac{(x-x_i)^3}{r^4}\right]n_1+\left[\frac{(x-x_i)^2(y-y_i)}{r^4}\right]n_2\right\}$$

$$\Sigma_{21}=\frac{1}{\pi}\left\{\left[\frac{(x-x_i)^2(y-y_i)}{r^4}\right]n_1+\left[\frac{(x-x_i)(y-y_i)^2}{r^4}\right]n_2\right\}$$

$$\Sigma_{12}=\frac{1}{\pi}\left\{\left[\frac{(x-x_i)^2(y-y_i)}{r^4}\right]n_1+\left[\frac{(x-x_i)(y-y_i)^2}{r^4}\right]n_2\right\}$$

$$\Sigma_{22}=\frac{1}{\pi}\left\{\left[\frac{(x-x_i)\ (y-y_i)^2}{r^4}\right]n_1+\left[\frac{(y-y_i)^3}{r^4}\right]n_2\right\} \tag{26}$$

$$\Sigma_{14}=\frac{1}{\mathrm{Re}\,\pi}\left\{\left[-\frac{(x-x_i)^2}{r^4}+\frac{(y-y_i)^2}{r^4}\right]n_1+\left[\frac{-2\ (x-x_i)(y-y_i)}{r^4}\right]n_2\right\}$$

$$\Sigma_{24}=\frac{1}{\mathrm{Re}\,\pi}\left\{\left[-\frac{-2(x-x_i)(y-y_i)}{r^4}\right]n_1+\left[\frac{(x-x_i)^2}{r^4}-\frac{(y-y_i)}{r^4}\right]n_2\right\}$$

$$\frac{\partial W_{33}}{\partial n}=\frac{\mathrm{Re}}{2\pi}\left\{\left[\frac{x-x_i}{r^2}\right]n_1+\left[\frac{y-y_i}{r^2}\right]n_2\right\} \tag{27}$$

For constant boundary element, one has

$$\begin{cases} u_i(\Gamma_e)=u_{i_e} \\ \tau_i(\Gamma_e)=\tau_{i_e} \\ T(\Gamma_e)=T_e \\ q(\Gamma_e)=q_e \end{cases} \tag{28}$$

Substituting the indicated expressions into equation (22):

$$\begin{aligned} C_{KI}(y)U_I(y) &= \sum_{e=1}^{n_2}\int_{\Gamma}\left\{\Sigma_{iK}(x,y)u_{i_e}(x)-W_{iK}(x,y)\tau_i(x)\right\}d\Gamma(x)_e + \\ &+\sum_{e=1}^{n_2}\int_{\Gamma}\frac{1}{\mathrm{Re}}\left\{\frac{\partial W_{3K}(x,y)}{\partial n}T_e-W_{3K}(x,y)q_e\right\}d\Gamma_e+\int_{\Omega}B_I(x)W_{Ik}(x,y)\,d\Omega(x) \end{aligned} \tag{29}$$

Since the constants terms listed in equation (28), for each element, can be factored, the system of algebraic equations becomes then evident.

## 5. NUMERIC IMPLEMENTATION.

*Boundary*: Let the boundary $\Gamma$ be divided into $m$ constant elements, with the collocation points (nodes) located at mid position of each element. Application of equation (29) to m nodes gives a set of $4m$ equations with $4m$ unknowns. For the solution of this system of equations two auxiliary matrices are assembled, for each element:

$$G_{\beta\alpha}(X-Y)=\Sigma_{iK}(x,y)-\frac{1}{\mathrm{Re}}\frac{\partial W_{3K}}{\partial n}(x,y) \qquad \alpha,\ \beta=1,2,3,4 \tag{30}$$

$$H_{\beta\alpha}(X-Y)=W_{iK}(x,y)-\frac{1}{\mathrm{Re}}W_{3K}(x,y) \qquad \alpha,\ \beta=1,2,3,4 \tag{31}$$

from which

$$g^{e}_{\alpha\beta}(Y) = \int_{\Gamma_e} G_{\beta\alpha}(X - Y)\,d\Gamma_e \tag{32}$$

$$h^{e}_{\alpha\beta}(Y) = \int_{\Gamma_e} H_{\beta\alpha}(X - Y)\,d\Gamma_e \tag{33}$$

Integrals (32) and (33) are carried out numerically, using one-dimensional Gaussian quadrature, if $X \neq Y$. When $X = Y$ the integrands of (32) and (33) become infinite, requiring the calculation in the sense of Cauchy principal value. Among several techniques available to perform these calculations, in this work the method of Telles[6] was chosen.

*Domain*: To calculate the integrals over Ω, the domain is divided into *M* elements by an appropriate net. Triangular cells will be used in this work.

$$\int_{\Omega} B_I(x) W_{Ik}(x, y)\, d\Omega(x) = \sum_{e=1}^{M} \left[ \sum_{j=1}^{J} \omega_j \left( B_I W_{IK}(x, y) \right)_j \right] S_{\Omega_e} \tag{34}$$

Gauss quadrature is also used. $\omega_j$ is the Gauss weight function at point *j*, $S_{\Omega e}$ is the area of element *e*, and *J* is the number of Gauss integration points. Hammer technique, as described by Partridge et al[7], with seven integration points in each triangular cell of the sub domain, was used to determine the domain integral.

In equation (29) the values $u_i, T_i$ are known; $\tau_i$ and $q_i$ are unknown gradients of velocity and temperature.

Defining

$$\begin{aligned} \delta &= \left[ u_1\ v_1\ T_1 \ \cdots \ u_m\ v_m\ T_m \right]^{t} \\ \tau &= \left[ \tau_{11}\ \tau_{21}\ q_1 \ \cdots \ \tau_{1m}\ \tau_{2m}\ q_m \right]^{t} \end{aligned} \tag{35}$$

equation (29) is rewritten as

$$[C]\{\delta\} + [H]\{\delta\} = [G]\{\tau\} + \{D\} \tag{36}$$

Defining

$$[\overline{H}] = [C] - [H]$$

one arrives at

$$[\overline{H}]\{\delta\} = [G]\{\tau\} + \{D\} \tag{37}$$

*Boundary conditions* : The application of the boundary conditions to equation (37), it is worth noting that elements of δ and of τ have some prescribed values. Therefore it is convenient to rearrange δ and τ such that the unknowns come first and then the prescribed values, that is,

$$\begin{aligned} \delta &= \left[ \delta_u \quad \delta_p \right]^{t} \\ \tau &= \left[ \tau_u \quad \tau_p \right]^{t} \end{aligned} \tag{38}$$

Rearrangement of matrices $[\overline{H}]$, $[G]$ and $\{D\}$ accordingly, results in

$$\begin{bmatrix} \overline{H}_{uu} & \overline{H}_{up} \\ \overline{H}_{pu} & \overline{H}_{pp} \end{bmatrix} \begin{bmatrix} \delta_u \\ \delta_p \end{bmatrix} = \begin{bmatrix} \overline{G}_{uu} & \overline{G}_{up} \\ \overline{G}_{pu} & \overline{G}_{pp} \end{bmatrix} \begin{bmatrix} \tau_u \\ \tau_p \end{bmatrix} + \begin{bmatrix} D_u \\ D_p \end{bmatrix} \tag{39}$$

Rearrangement of equation (39) such that only the unknowns are in the left side, gives

$$[A]\{X\}=[B]\{P\}+\{D\} \quad (40)$$

Matrix [$A$] is dense so that inversion is time consuming. The inversion is carried out using the Gauss elimination algorithm. Solution of equation (40) gives the values of the unknowns on the boundary.

*Computer Program*: A modular computer program has been developed that is able to handle geometries composed of rectangles, written in FORTRAN and run in a 2.0 GHz personal computer. The computer program implementation was carried out with the following steps:

- Definition of the geometry by a combination of rectangles.
- Boundary discretization using elements of same size.
- Domain grid generation using triangular elements.
- Numbering elements counterclockwise.
- Imposition of the boundary conditions and initialization of domain variables (velocity, temperature and pressure) using reasonable guesses according to the problem being solved.
- Assembly of matrices $g^{e}_{\alpha\beta}$ and $h^{e}_{\alpha\beta}$ for each element *e.*
- Assembly of matrices [$G$] and [$H$] for all elements on the contour.
- Numeric evaluation of domain integrals equation (34).
- Solution of equation (40) for the determination of the variables on the boundary.
- Solution of equation (22) at internal nodes, with $C_{KI}=1$.

## 6. APPLICATION.

For the demonstration of the method applicability, three problems were chosen: a) the recirculating flow in a square cavity driven by a lid sliding at uniform velocity, b) the flow facing a forward step and c) the flow over a deep cavity. In the situation (b) and (c) are consider heat transfer with forced convection for low Reynolds number.

Driven cavity flow: The flow in the box is depicted using streamlines, as shown in Fig. 1. The boundary conditions are the no-slip in the box boundaries, that is, zero at the non moving surfaces and the velocity of the moving slid at the upper surface. Constant temperature was set on the boundary. Grid for Fig. 1a is 40×40 and for Fig. 1b is 30×30. Criteria of convergence were based on the difference between the previous and the current calculated values for velocities, pressure and temperature. Convergence was achieved up to Reynolds number of 400. Recirculation was detected at the bottom-right of the cavity.

Flow in a stepped channel: Streamlines of the flow in a forward facing step is shown in Fig. 2a. Boundary conditions are: parabolic distribution of velocities at inlet, no-slip condition on the walls and constant wall temperature. The results shown are for a grid of 26×30. The predicted reattachment point is in agreement with other predicted numerical methods such as the volumes finite methods, see Fig 2.b.(De Lemos, Rocamora )

In figures 3 and 4 the temperature contour is show when the upper surface is heated, and other are kept constant. According to this, the results of the temperature field are immediately obtained, due do to the consideration the fundamental tensors and the fundamental solutions coupled energy equation.

Deep cavity flow: Figure 4a shows the streamlines for Re=10, grid 40× 40 and $\varepsilon = 0.0001$, where one can see satisfactory results but with a mesh moderately refined with a grid of 20× 20 and $\varepsilon = 0.00001$, it is also possible to obtain good results in short processing time, (Figure 4.b). For methodology validation, the finite volume method was used with refined grid in the corner regions (Figure 4c). The computational processing time was also compared and it was concluded that the method of boundary elements is faster when compared to method finite volumes. Finally, in Fig. 5 one can see the comparison with the different velocity distributions obtained from BEM and FVM.

The velocities were taken from the middle of the cavity. It is important to notice that the Boundary Elements Method with a grid of 20×20, gives good result when compared to the Method of Finite Volumes. Applications such as the flow field in turbines and compressors, or in turbomachinery in general, cannot be studied with this formulation because of low Reynolds number restriction, but are being studied with a new formulation using linearization of the terms B of Eq. (10). This may became very attractive if one considers the low computational coast due to the variables which would be analyzed under boundary regardless the use of refined grids.

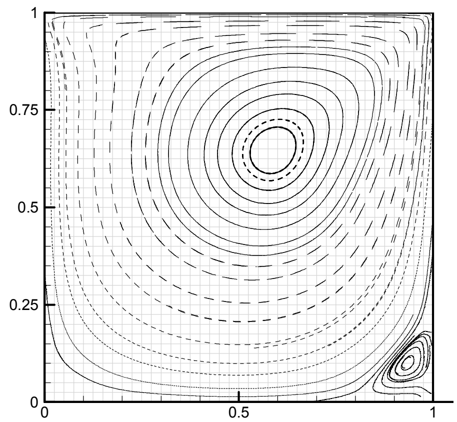

Figure 1a: Streamlines in the driven cavity flow
a) Re = 300, grid 40×40, ∈=0.0001

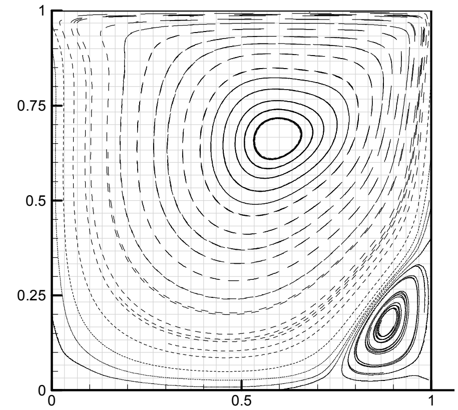

Figure 1a: Streamlines in the driven cavity flow
b) Re = 400, grid 30×30, ∈=0.001.

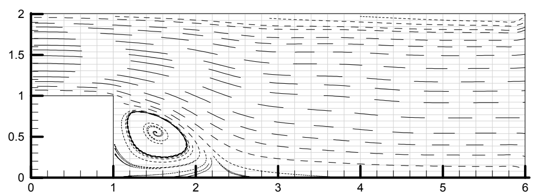

Figure 2a. BEM. Streamlines in the channel flow, Re = 30, grid 26×30, ∈=0.0001

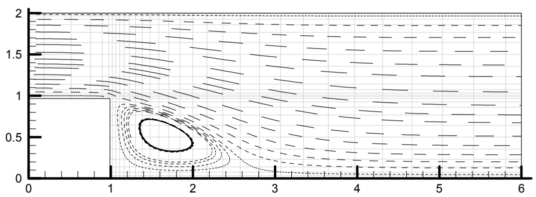

Figure 2b. MFV. Streamlines in the channel flow, Re = 30, grid 26×30,

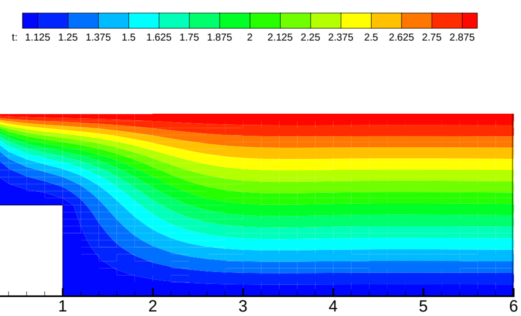


Figure 3a.- Chanel flow contour temperature

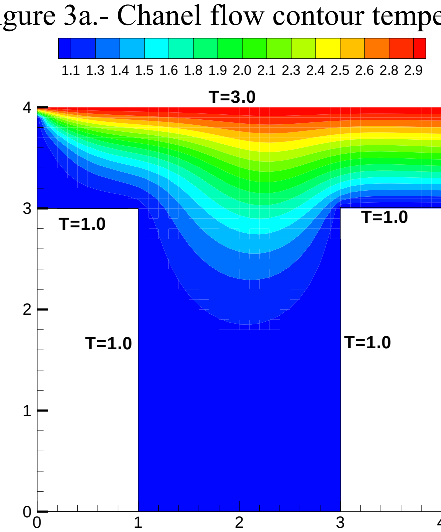


Figure 3b. Channel flow deep contour temperature

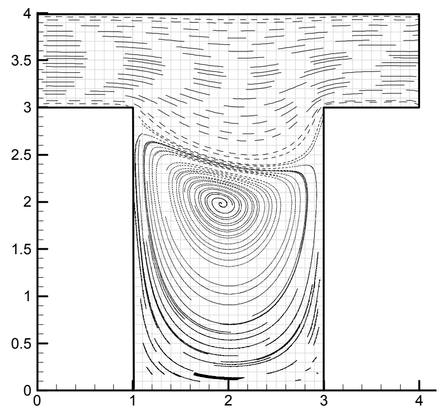


Figure 4b - Streamlines in the deep cavity flow
Re = 10, grid 40x40, ∈=0.0001

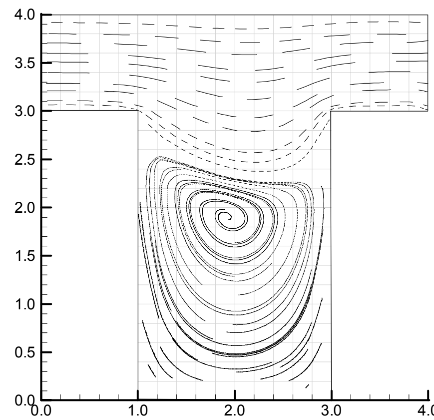


Figure 4d - Streamlines in the deep cavity flow
Re = 10, grid 20x20, ∈=0.00001

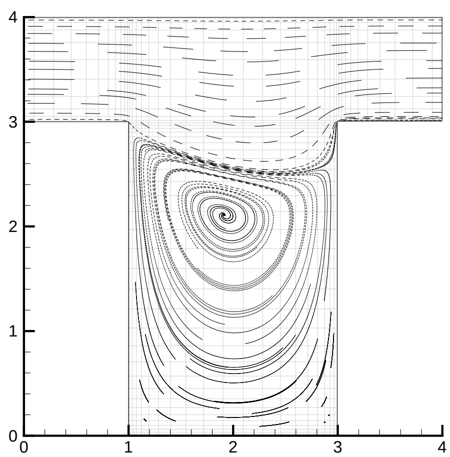


Figure 4.c.-MFV Streamline in deep cavity flow
Re=10, grid 20×20, ε= 0.0001

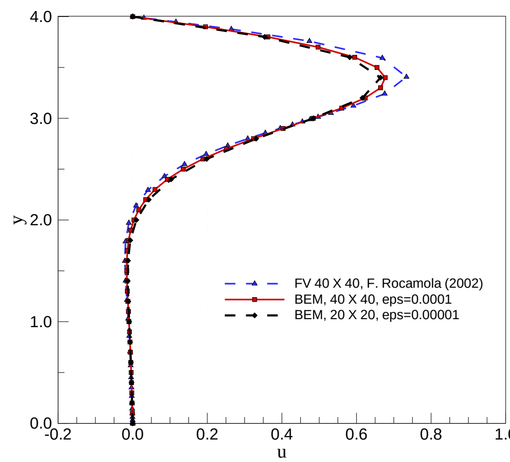


Figure 5.- Velocities distribution *u* in medium section of deep cavity flow

## 7. CONCLUSION.

The boundary element method can be applied to the calculation of incompressible viscous flows as demonstrated above. Although coarse grids were used, the results are quite satisfactory, capturing regions of reverse flows. Comparison of CPU times for the boundary element method and for a finite difference scheme indicated that the results obtained for the boundary element calculations are much faster. Consequently, it is important to investigate other applications such as the flow in blade passages and the ultimate goal for the research under way. Those problems require a method that converges for much higher Reynolds number, study that this being developed.

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