# AN UNSTRUCTURED EDGE-BASED FINITE VOLUME FORMULATION FOR THE SOLUTION OF THE BIPHASIC FLOW OF OIL AND WATER THROUGH AN "IMPES" PROCEDURE

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**Abstract**: Unstructured mesh based discretization techniques can offer advantages relative to standard finite difference approaches which are still largely used in petroleum reservoir simulation due to their flexibility to model complex geological features and due to their capacity to incorporate mesh adaptation techniques. In this paper we consider an unstructured edge-based implicit finite volume formulation (FVM) which is used to solve the elliptic pressure equation and the non-linear hyperbolic equation that arise in biphasic flow problems when the IMPES (IMplicit Pressure Explicit Saturation) techniques is used together with a global pressure approach. The IMPES method is a segregated type method in which the flow equations are manipulated in order to produce an elliptic pressure equation solved implicitly and a hyperbolic type saturation equation which is then solved explicitly. The numerical formulation includes the introduction of an adaptive artificial dissipative term in order to deal with the Bucklet-Leverett (saturation) equation. This finite volume formulation is very flexible and efficient, and it is equivalent to the edge-based finite element formulation (FEM) when linear triangular elements are employed. Some bidimensional model examples are solved in order to show the potential of the formulation utilized.

**Keywords:** Biphasic Flow, Buckley Leverett, IMPES, Finite Volume, Unstructured Meshes.

## 1. INTRODUCTION

Naturally occurring hydrocarbon systems found in petroleum reservoirs are mixtures of organic compounds, which exhibit multiphase behavior over wide ranges of pressures and temperatures. These hydrocarbon accumulations may occur in the gaseous, liquid and solid states or in various combinations of gas, liquid and solid phases. These differences in phase behavior, coupled with the physical properties of reservoir rock that determine the relative ease with which gas and liquid are transmitted or retained, result in many diverse types of hydrocarbon reservoirs with complex behaviors. Frequently, petroleum engineers have the task to study the behavior and characteristics of a petroleum reservoir and to determine the course of future development and production that would maximize profits. Nowadays, in order to complement and improve the accuracy of the more traditional prediction methods (e.g. experimental, analogs, etc.), numerical methods are being widely used as tools to predict the behavior of multiphase flow through reservoirs. Due to several factors (robustness, easiness of programming, etc), the finite difference method (FDM) is usually utilized in

reservoir analysis. On the other hand much effort has been recently put in methods that allow a better treatment of the complex geometries that characterize petroleum reservoirs. In this context, the adoption of methods able to deal with unstructured meshes is very attractive and highly recommended. Within such class of methods the most frequently used are the finite element method (FEM), (Zienckievicz and Morgan, 1983), and the finite volume method (FVM), (Barth, 1992). The later is particularly attractive in reservoir problems due to, among other things, its local and global conservation properties.

In this work the vertex centered finite volume formulation using median dual control volumes is implemented using an edge-based data structure that is adapted for solving two-dimensional biphasic flow problems in an IMPES (IMplicit Pressure Explicit Saturation) procedure. In this technique a sequential time stepping procedure is used to decouple the equations, which consists of basically solving one equation at a time. First the pressure equation is solved and then the velocity field is computed. This velocity field can be used as input for the saturation equation and so on. In order to account for the convective terms appearing in the saturation equation, an artificial dissipation scheme, adapted for use on unstructured meshes, is also utilized. This finite volume formulation is very flexible and efficient, and it is equivalent to the edge-based FEM when linear triangular elements are employed (Lyra, 1994 and Sorensen, 2001). The formulation is flexible to deal with any kind of unstructured meshes with elements of different types. For instance, in 2-D either triangular, quadrilateral or mixed meshes can be directly used, and the same happens when dealing with 3-D, where tetrahedral, hexahedral, pyramids, prisms and mixed meshes can be adopted. In terms of efficiency, both memory and CPU time requirements are reduced by using an edge-based implementation (Barth, 1992; Sorensen, 2001). Finally, an edge-based data structure allows for the implementation of different types of finite difference discretization in the context of 2-D and 3-D unstructured meshes (Lyra, 1994).

At the present stage of the work, we are still solving the two equations separately to validate independently the formulation utilized for each equation. Though, it is our hope that very soon we will be solving the coupled problem.

## 2. MATHEMATICAL MODEL

In order to simplify our notation, but without loss of generality we present here the mathematical governing equations for immiscible biphasic flows of water and oil through rigid porous media. This model (which can be directly extend to miscible, three phase flow) is obtained combining the Darcy's Law with the mass conservation equation for each phase. The model adopted here has been successfully used by many authors (Peaceman, 1977; Ewing, 1983), though it is not commonly used in commercial reservoir simulators. Even though the approach to be used in our work seems more complex, it is far more useful when one aims for numerical accuracy and efficiency (Peaceman, 1977; Ewing, 1983; Da Silva, 2000). We are assuming that the phase velocities obey the Darcy's law, which, ignoring gravitational effects can be written for phase *i*, as:

$$\vec{v}_i = -\underset{\sim}{l}_i \nabla P \tag{1}$$

Where the phase mobility is defined in Eq. (2) as:

$$\underset{\sim}{l}_i = \underset{\sim}{K} \frac{k_i}{m_i} \tag{2}$$

Here $\underset{\sim}{K}$ denotes the absolute permeability tensor of the rock, $k_i$ is the phase relative permeability, $m_i$ is the phase viscosity and $r_i$ is the phase density. Henceforth we will also ignore the capillary pressure and will assume that $P = P_w = P_o$ where (w) and (o) stand respectively for, the wetting

(water) and the non-wetting (oil) phases. Additionally, conservation of mass for each equation can be written as in Eq. (3):

$$-\nabla\left(r_i v_i\right)+q_i=\frac{\partial\left(f r_i S_i\right)}{\partial t} \tag{3}$$

In Equation (3), $f$ is porosity, i.e. fraction of the rock which can be occupied by fluids, $q_i$ denotes sources or sinks and $S_i$ is the saturation of phase *i*, which represents the percentage of the available pore volume occupied by this phase. Due to this last definition, we can write:

$$S_o+S_w=1 \tag{4}$$

### 2.1. The Pressure Equation

Now, carrying out the differentiation in Eq. (3) yields:

$$-\nabla\left(r_i \vec{v}_i\right)=-q_i+r_i S_i\frac{\partial f}{\partial t}+f\left(S_i\frac{\partial r_i}{\partial t}+r_i\frac{\partial S_i}{\partial t}\right) \tag{5}$$

If we divide Eq. (5) by $r_i$ with *i* = o, w, and add the results we have:

$$\nabla\cdot\left(\underset{\sim}{l}\nabla P\right)-\left(\vec{v}_o c_o+\vec{v}_w c_w\right)\nabla P-f C_T\frac{\partial P}{\partial t}=-Q \tag{6}$$

Where, we have defined the following terms, the total flux, the phase compressibility and the total compressibility, Eqs. (7), (8), (9):

$$Q=\left(\frac{q_o}{r_o}+\frac{q_w}{r_w}\right) \tag{7}$$

$$c_f=\frac{1}{r_f}\frac{d r_f}{dP_f}=\frac{1}{r_f}\frac{d r_f}{dP} \tag{8}$$

$$C_T=\frac{1}{f}\frac{df}{dP_f}+S_o c_o+S_w c_w \tag{9}$$

If , additionally, we define the total velocity as in Eq. (10)

$$\vec{v}=\vec{v}_o+\vec{v}_w \tag{10}$$

We can write Eq. (6) as:

$$-\nabla\cdot\vec{v}-\left(\vec{v}_o c_o+\vec{v}_w c_w\right)\nabla P-f C_T\frac{\partial P}{\partial t}=-Q \tag{11}$$

Using Darcy's law, Eq. (1), in Eq. (11) yields:

$$\nabla \cdot \left( \underset{\sim}{l} \nabla P \right) - \left( \vec{v}_o c_o + \vec{v}_w c_w \right) \nabla P - f C_T \frac{\partial P}{\partial t} = -Q \tag{12}$$

where, $\underset{\sim}{l} = \underset{\sim}{l}_o + \underset{\sim}{l}_w$ is the total mobility.

Equation (12) is the pressure equation for the immiscible biphasic flow of oil and water in porous media, ignoring gravitational and capillary effects. Finally, if we concentrate our attention on the incompressible flow (i.e. incompressible fluid and rock), we can write the pressure equation as:

$$\nabla \cdot \left( \underset{\sim}{l} \nabla P \right) = -Q \tag{13}$$

### 2.2. The Saturation Equation

With the same assumptions of the derivation of Eq. (13) we can add and subtract Eq. (2) for each phase, to find that:

$$\vec{v}_o = f_o \vec{v} \quad and \quad \vec{v}_w = f_w \vec{v} \tag{14}$$

Where $f_i = \dfrac{\underset{\sim}{l}_i}{\underset{\sim}{l}_o + \underset{\sim}{l}_w}$, is the fractional flow function for phase *i*.

Using Eq. (14) in Eq. (3) we have the saturation equation for each phase. If we write this equation for water phase we have:

$$f \frac{\partial S_w}{\partial t} + \nabla \left( \vec{F}_w ( S_w ) \right) = -q_w \tag{15}$$

Where, $\vec{F}_w = f_w \vec{v}$ is the flux function which is dependent of the phase saturation.

## 3. NUMERICAL FORMULATION

The numerical formulation used to discretize the previous equations is an edge based, vertex centered variation of the well known Finite Volume Method (FVM), (Sorensen, 2001). In this kind of formulation, most of the coefficients necessary to our computation are associated with the edges of the mesh. The edge based finite volume formulation adopted is highly computationally efficient in terms of CPU time and memory use. Besides, this kind of approach allows the utilization of 1-D "Artificial Dissipation" formulations to be extended to 2-D and 3-D unstructured meshes without excessive mathematical considerations (Lyra, 1994). This last point can be extremely useful due to the strong hyperbolic characteristic of some multiphase flow problems.

Nowadays there are many finite volume schemes that can be successfully used to solve balance equations. In this work we adopted a node centered median dual finite volume technique. In contrast to a cell centered formulation, in a node centered approach, the values of the unknowns are defined on the nodes and the control volumes (CV) are defined by the dual mesh in a way that each mesh point is associated with only one CV.

In order to discretize the bidimensional domain, we have used triangular elements, though there is, in principle, no restriction to the shape of the elements that can be used in unstructured finite volume formulations. For a triangular mesh, the control volume cells were built connecting the centroids to the middle point of the triangles that surround a specific node. The control volume created in this fashion (known as "median dual") is quite general and the Voronoi diagrams are nothing but a special case of this scheme. The structure formed by the surfaces connecting the control volumes is denominated the dual mesh. In node centered schemes, the fluxes are integrated on the dual mesh

usually through a loop over the edges (for 2-D or 3-D) and the computational cost is therefore, proportional to the number of edges of the mesh.

### 3.1. Integral Formulation and Discrete Equations

#### 3.1.1. The Implicit Pressure Equation

In order to obtain our discrete equations first we can write Eq. (13) as:

$$-\nabla \cdot \vec{v} = -Q \tag{16}$$

And the total velocity is written as:

$$\vec{v} = -\underset{\sim}{l} \nabla P \tag{17}$$

Integrating equation (17) and multiplying by (-1) yields:

$$\int_W \nabla \cdot \vec{v} \, \partial W = \int_W Q \, \partial W \tag{18}$$

Using the Gauss-Green theorem, we have:

$$\int_G \vec{v} \cdot \vec{n} \, \partial G = \int_W Q \tag{19}$$

Finally, for a node *I* of the mesh, we can write the discrete form of equation (19), as:

$$\sum_{L_{I(W)}} \left(\vec{v}_{IJ_L}\right)^{(W)T} \vec{C}_{IJ_L} + \sum_{L_{I(G)}} \left(\vec{v}_{IJ_L}\right)^{(G)T} \vec{D}_{IJ_L} = Q_I V_I \tag{20}$$

In equation (20), $V_I$ is the volume (area in 2-D) of the CV, the upper index $W$ represents approximations on the middle of every edge of the mesh which is connected to node *I*, $G$ refers only to boundary edges connected to that node, *T* represents the transpose of a vector or matrix, and the summation is performed over the edges connected to node *I*. For the 2-D case, Fig. (2) depicts the control volumes and the geometrical parameters necessary to compute coefficients $\vec{C}_{IJ_L}$ and $\vec{D}_{IJ_L}$ defined in Eq. (21).

$$\begin{aligned} \vec{C}_{IJ_L} &= A_{K-1}\vec{n}_{K-1} + A_K \vec{n}_K \\ \vec{D}_{IJ_L} &= A_L \vec{n}_L \end{aligned} \tag{21}$$

$A_{K-1}$, $A_K$ and $A_L$ are the areas of the control volume faces associated to the normals $n_{K-1}$, $n_K$ and $n_L$, respectively.

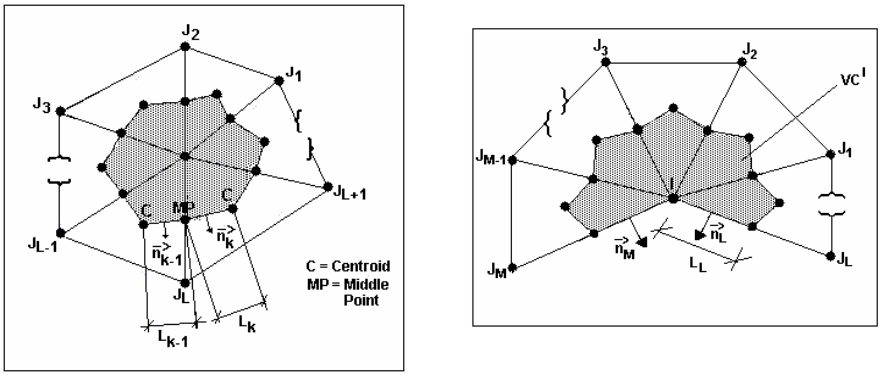


Figure 2. Internal and boundary bidimensional control volumes and their geometrical parameters

To obtain the final discrete system of equations, we must first determine the nodal velocities as functions of the discrete pressure field. Therefore we must integrate the velocity for a node *I*, Eq. (17), obtaining:

$$\int_{W_I} \vec{v}_I \, \partial W_I = -\int_{W_I} \underset{\sim}{l}_I \nabla P \, \partial W_I \tag{22}$$

Using the Gauss-Green theorem yields:

$$-\int_{\Omega_I} \underset{\sim}{l}_I \nabla P_I \, \partial \Omega_I = -\int_{\Gamma_I} \left( \underset{\sim}{l}_I P_I \right) \cdot \vec{n} \, \partial \Gamma \tag{23}$$

Assuming constant values for the left integral terms, we can write:

$$-\left( \underset{\sim}{l}_I \nabla P \right) V_I = -\left( \sum_{L_{I(\Omega)}} \left( \underset{\sim}{l}_{IJ_L} \right)^{\Omega} \left( P_{IJ_L} \right)^{\Omega} \vec{C}_{IJ_L} + \sum_{L_{I(\Gamma)}} \left( \underset{\sim}{l}_{IJ_L} \right)^{\Gamma} \left( P_{IJ_L} \right)^{\Gamma} \vec{D}_{IJ_L} \right) \tag{24}$$

Further we assume the following mid-edge approximations:

$$\left( P_{IJ_L} \right)^{\Omega} = \frac{P_I + P_{J_L}}{2} \tag{25}$$

$$\left( P_{IJ_L} \right)^{\Gamma} = P_I \,, \tag{26}$$

After some additional algebraic manipulation, we obtain:

$$\vec{v}_I = -\frac{1}{2V_I} \left[ P_I \left( \sum_{L_{I(\Omega)}} \left( \underset{\sim}{l}_{IJ_L} \right)^{\Omega} \vec{C}_{IJ_L} + 2 \sum_{L_{I(\Gamma)}} \left( \underset{\sim}{l}_{IJ_L} \right)^{\Gamma} \vec{D}_{IJ_L} \right) + \left( \sum_{L_{I(\Omega)}} P_{J_L} \left( \tilde{l}_{IJ_L} \right)^{\Omega} \vec{C}_{IJ_L} \right) \right] \tag{27}$$

In order to compute the velocities $\vec{v}_{IJ_L}^{P}$ on the middle of the edges we could follow the obvious choice of repeating the same strategy used in Eqs. (25) and (26) to approximate the mid-edge velocities, $\vec{v}_{IJ_L}$, of Eq. (20). Nevertheless the recursive use of the arithmetic mean to compute the interface velocity implies that the discretization of the diffusion terms in Eq. (18) involves information from two layers of points surrounding the point *I* under consideration. Furthermore, if uniform

structured quadrilateral (or hexahedral) meshes are adopted, the values computed at a given node are uncoupled from the values of those nodes directly connected to it. This fact may leads to "checker-boarding" or "odd-even" oscillations (Lyra, 1994; Sorensen, 2001). Even when computing the diffusive term in non-uniform unstructured meshes, the adoption of an extended stencil and a weak coupling with the directly connected nodes may lead to some loss of robustness and reduction of convergence rate of the resulting scheme. To overcome such weaknesses, the velocities must be computed in an alternative way. Following the procedure suggested in the literature (Sorensen, 2001) a better approach would be to use a local frame of reference in which one axis is along the edge direction (P) and another axis (N) is in the plane orthogonal to direction (P) as stated in Eq. (28).

$$\vec{v}_{IJ_L} = \vec{v}_{IJ_L}^{(N)} + \vec{v}_{IJ_L}^{(P)} \tag{28}$$

We then replace the component parallel to the edge with a local second order finite difference approximation $\vec{v}_{IJ_L}^{(P^*)}$ as shown in Eq. (30):

$$\vec{v}_{IJ_L} = \vec{v}_{IJ_L}^{(N)} + \vec{v}_{IJ_L}^{(P^*)} \tag{29}$$

Where:

$$\vec{v}_{IJ_L}^{(P^*)} = -\underset{\sim}{l}_{IJ_L} \frac{\left(P_{J_L} - P_I\right)}{\left|D_{IJ_L}\right|} \vec{L}_{IJ_L} \tag{30}$$

$\left|D_{IJ_L}\right|$ and $\vec{L}_{IJ_L}$ are, respectively, the size and the unitary vector of the edge $IJ_L$. For the definition of the midpoint values of the mobility terms, it is worthy mentioning that we use the harmonic average for the effective permeability (i.e. the product of the absolute and relative permeabilities) on the middle of the edge, due to its capacity to correctly represent the flux between two adjacent control volumes with extremely different values of permeabilities. On the other hand, the viscosity is excluded from the averaging process because it is constant under the assumption of incompressible flow. We then compute the normal component of the velocity as:

$$\vec{v}_{IJ_L}^{(N)} = \vec{v}_{IJ_L} - \vec{v}_{IJ_L}^{(P)} \tag{31}$$

Where:

$$\vec{v}_{IJ_L}^{(P)} = \vec{L}_{IJ_L} \left( \left( \vec{v}_{IJ_L} \right)^T \cdot \vec{L}_{IJ_L} \right) \tag{32}$$

And:

$$\left( \vec{v}_{(IJ_L)} \right)^{(W)} = \frac{\vec{v}_I + \vec{v}_{J_L}}{2} \tag{33}$$

$$\left( \vec{v}_{(IJ_L)} \right)^{(G)} = \vec{v}_I \tag{34}$$

Again, $W$ stands for all edges of the domain and $G$ refers only to boundary edges.
Inserting Eq. (32) in Eq. (31) yields:

$$\vec{v}_{IJ_L}^{(N)} = \left(\vec{v}_{IJ_L}\right) - \vec{L}_{IJ_L}\left(\left(\vec{v}_{IJ_L}\right)^T \cdot \vec{L}_{IJ_L}\right) \tag{35}$$

The nodal values $\vec{v}_I$ and $\vec{v}_{J_L}$ are computed using the approximation given by Eq. (27). The final discrete equation for a node *I* can be written as:

$$\sum_{L_{I(w)}}\left[\frac{\left(\vec{v}_I + \vec{v}_{J_L}\right)}{2} - \vec{L}_{IJ_L}\left(\frac{\left(\vec{v}_I + \vec{v}_{J_L}\right)^T}{2}\cdot \vec{L}_{IJ_L}\right) - \tilde{l}_{IJ_L}\frac{\left(P_{J_L} - P_I\right)}{\left|D_{IJ_L}\right|}\vec{L}_{IJ_L}\right]^T \vec{C}_{IJ_L} + \sum_{L_{I(G)}}\left(\vec{v}_I\right)^T \vec{D}_{IJ_L} = Q_I V_I \tag{36}$$

### 3.1.2. The Explicit Saturation Equation

It is well known that central difference type methods such as the Galerkin method produce unstable numerical schemes when used to discretize the convective terms that characterize hyperbolic equations (e.g. saturation equation). In order to overcome this difficulty we are testing different schemes to correctly treat convective terms. In this work we used a method that was originally proposed by Jameson et al (1981) with the modifications introduced by Peraire et al (1993). This method is based on the introduction of an adaptive artificial dissipative term that combines second order with fourth order diffusive terms (Lyra, 1994). The basic idea of the method is to introduce the second order terms in regions of high gradients and to use the fourth order terms only in regions of smooth gradients in order to stabilize the scheme.

A semi-discrete numerical scheme for the solution of the non-linear hyperbolic saturation equation, Eq. (15), can be written as:

$$f\frac{\partial S_w}{\partial t} = -\left(\sum_{L_{I(W)}} \vec{F}_{IJ_L(w)}^{(W)} \cdot \vec{C}_{IJ_L} + \sum_{L_{I(W)}} \vec{F}_{IJ_{L(w)}}^{(G)} \cdot \vec{D}_{IJ_L} + q_w\right) \tag{37}$$

The term $\vec{F}_{IJ_L(w)}^{(W)} \cdot \vec{C}_{IJ_L}$ is then replaced by the expression defined in Eq. (38) as:

$$\vec{F}_{IJ_L(w)}^{(W)} \cdot \vec{C}_{IJ_L} = \frac{1}{2}\left[\left(\vec{F}_{I(w)} + \vec{F}_{J_L(w)}\right)\right] \cdot \vec{C}_{IJ_L} + A.D. \tag{38}$$

Where *A.D.* stands for "Artificial Dissipation" terms which are computed as:

$$A.D. = -a_{IJ_L}\left|\vec{C}_{IJ_L}\right|\left[x_{IJ_L}^{(2)}\Delta S_{IJ_L} + x_{IJ_L}^{(4)}\left(\Delta S_{IJ_L} - \left(\left|D_{IJ_L}\right|\nabla S_{IJ_L}\cdot \vec{L}_{IJ_L}\right)\right)\right] \tag{39}$$

The gradient of the saturation on the middle of the edge is computed through an arithmetic average, $\nabla S_{IJ_L} = \frac{\nabla S_I + \nabla S_{J_L}}{2}$, and $\Delta S_{IJ_L} = S_I - S_{J_L}$. Parameters $x_{IJ_L}^{(2)}$ and $x_{IJ_L}^{(4)}$ are adapted to the flux flow and are defined according to:

$$\begin{aligned} x_{IJ_L}^{(2)} &= m^{(2)}\, max\left(i_I, i_{J_L}\right) \\ x_{IJ_L}^{(4)} &= max\left(0, m^{(4)} - x_{IJ_L}^{(2)}\right) \end{aligned} \tag{40}$$

With:

$$i_I = \frac{\left|S_{J_L} - 2S_I + S_{I_L}\right|}{(1-q)\left(\left|S_{J_L} - S_I\right| + \left|S_I - S_{I_L}\right|\right) + q\left(S_{J_L} + 2S_I + S_{I_L}\right) + e} \tag{41}$$

In the above equations $m^{(2)}$ and $m^{(4)}$ are user specified coefficients. The factor $i_I$ is a sensor designed to detect discontinuities. The $q$ parameter represents a weighting coefficient, $0 \le q \le 1$, and $e$ is used only to avoid the appearance of zero in the denominator. For the diffusive parameter $a_{IJ_L}$ we used the modulus of the total velocity, though other choices could be used (Lyra, 1994). $S_{I_L}$ is the value of the saturation obtained through a gradient reconstruction along the edge $IJ_L$ on a point distant $\left|D_{IJ_L}\right|$ from node *I*. For further details see (Lyra, 1994 and Peraire et al, 1993). Equation (37) can be further discretized in time to produce a system of algebraic equations. In the present work these equations were solved through a simple two-level time step explicit scheme (Euler forward).

## 4. EXAMPLES

In the first example we present a very simplified two dimensional (areal) example which resembles the well known "quarter of five spot problem" with a point source representing an injector well and a point sink representing a producer well (Ewing, 1983). The boundary conditions for this problem are null fluxes through the four lateral faces. For a non-dimensional form of the pressure equation, the fluid mobility is unitary, the source term is $q_1$= 100.0 and the sink term is $q_2$ = -100.0. Figs. (3) and (4) show respectively, the contours of pressure and the triangular mesh utilized to solve the pressure equation model problem of section 4 and its correspondent dual mesh.

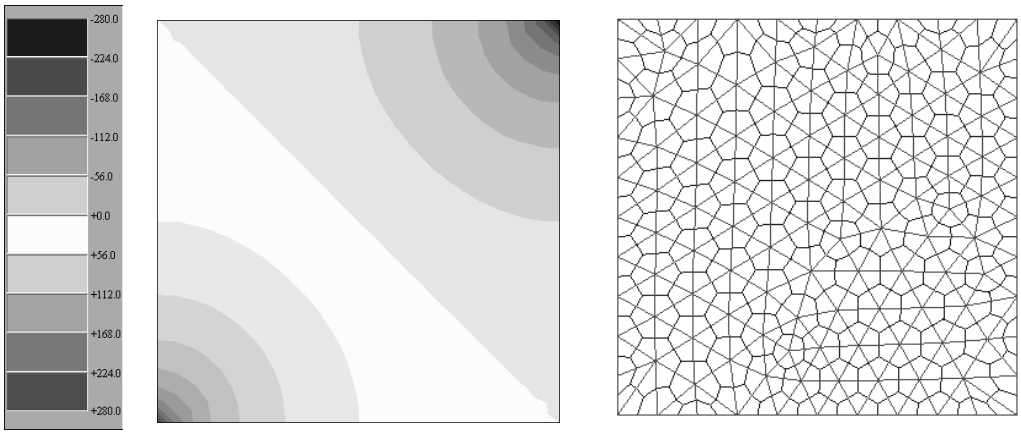


Figure 3. Contours of pressure for the problem of the "quarter of a five-spot", and unstructured triangular mesh with its correspondent dual mesh.

In the second example we present the classical Buckley-Leverett problem. This model problem can be basically defined as the 1-D incompressible flux of oil and water through porous media, where the gravity and the capillary effects are neglected. The basic Buckley-Leveret equation is the saturation equation (Eq. (15)). For the curves shown in Fig. (4), we have used $S_{rw} = S_{ro} = 0.1$, where $S_{ri}$ are the residual phase saturations for *i* = water and oil respectively. The boundary condition utilized was that $S_w = 1 - S_{ro} \quad x = x_0$. We also used a quadratic relative permeability saturation relationship (Pinto, 1991 ). The time step utilized was *Dt* = 0.0005 with 101 nodal points on the x direction. The courant number used was, $C_{rt} = \left|\vec{v}\right| Dt / Dx = 0.05$. As can be seen in Fig. (4), the numerical results, which are plotted against the analytical solution for instants *t* = 0.2 and *t* = 0.5, have shown

very good agreement with the analytical solutions and are comparable to the best results obtained in Pinto (1991).

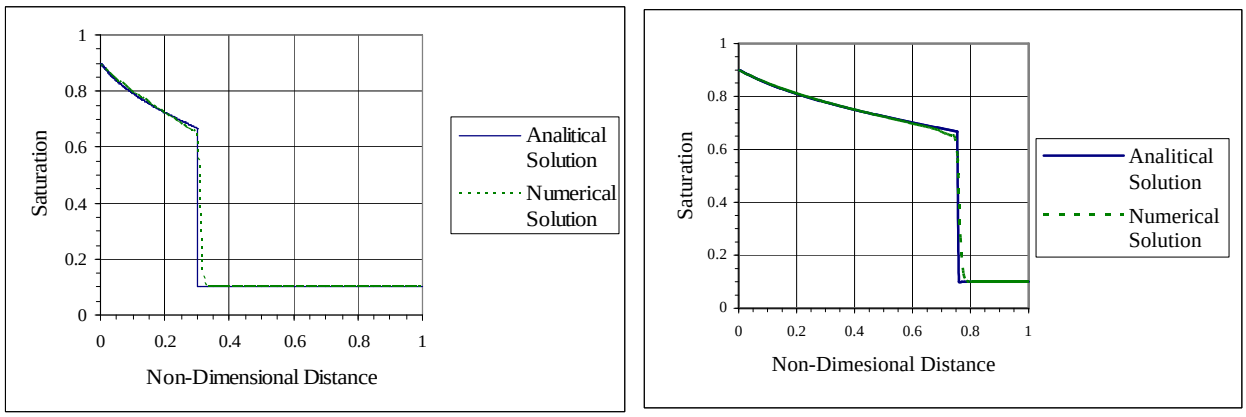


Figure 4. Saturation profile for the Buckley-Leverett problem for instants t = 0.2 (left) and t = 0.5.

## 5. CONCLUSIONS

In this paper we have described in details an edge-based finite volume discretization technique to solve the pressure and the saturation equations which arise in biphasic flow problems in the IMPES method context. Particularly, we have shown a complete description of the numerical formulation utilized to discretize both equations for a simplified model considering the incompressible flow of two immiscible fluids ignoring the gravitational and capillary effects.

The next step in our research involves the coupling between the pressure and the saturation equations in order to solve the full IMPES problem. Some care must be taken in this step since the incorrect treatment of the velocity equation, Eq. (17) may lead to poor approximations for the saturation field. Some alternatives to this problem may be devised and we are currently investigating such issues.

## 6. ACKNOWLEDGEMENTS

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