COMPUTER SIMULATION OF LAMINAR FLOWS; SOLUTION OF THE NAVIER-STOKES EQUATIONS USING THE FINITE ELEMENT MEHOD

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***Abstract**. The incompressible laminar fluid flow of three dimensional problems, are analyzed through the solution of the Navier-Stokes equations. The finite elements method, associated to the variational formulation of Galerkin, was used to obtain the numerical solution of the governing equations of the fluid motion. A computer program was written to allow solutions in both rectangular and cylindrical coordinate systems. The Poisson equation for the pressure was added to the set of Navier-Stokes equations, which brought stability and convergence to the fluid program. The mentioned equations have a characteristic nonlinear form, and for this reason, a routine has to control the iterative procedure, necessary to obtain convergence. For the discretization of the geometric part of the problem, a volume finite element, with the shape of a parallelepiped, with twenty nodes was adopted. Three velocities components and pressure where considered for each node, and each volume element had a total of 80 degrees of freedom. To solve the finite element equations, the Gauss numerical procedure was used for the integrations. Several cases, reported by the classical literature, were modeled and compared to the analytical solutions. Couette and Poseuille flows had been first simulated (both in rectangular coordinates), given the similarity between these and the flow of a bearing in which the adopted radius was considered infinite. Later, a section of a hydrodynamic bearing was simulated, with different meshes and boundary conditions. Results reveal a very satisfactory precision and validated the procedures adopted for the solution.*

Key words: Fluid Mechanics, Navier-Stokes Equations, Finite Element Method

1. INTRODUCTION

Computer simulations have been performed, based on a three dimensional fluid theory, wich considers a finite element solution for the stationary incompressible Navier-Stokes equations and Poisson equation. The subject has a broad range of applications in dealing with steady-state fluid flow problems, and provides a more realistic discription of most complex models and boundary conditions. Particularly, for the authors goals, the three dimensional solution is of great interest in solving interactions of fluids with structures and in the modeling of hydrodynamic bearings.

For the fluid domain finite elements discretization, 3D hexahedral 20-nodes elements were adopted. For the flow analysis, each node considers four degrees of freedom, three velocity components and pressure. Therefore, equal-order interpolation functions for velocity and pressure were adopted.

The objective of this work was to develop a finite element solver for the stationary incompressible Navier-Stokes equations in three dimensions, using Poisson equation to enhance stability of the original Galerkin formulation.

The stabilized finite element techniques we have developed in recent years for computational of the flow problems in various applications, least-square Galerkin formulation, streamline-upwind/Petrov-Galerkin(SUPG) and pressure-stabilizing/Petrov-Galerkin(PSPG), moving least square reproducing kernel(MLSRK) are examples of these methods. Although a mixed interpolation (the basis functions for pressure were one order lower than those for velocity) can be obtain better results, for higher-order elements this technique can be fail; besides, equal-order interpolation for velocity and pressure simplify the numerical code. Thus, we choose the Poisson equation for achieve stability owing to simplicity of the method, what preserve nature of the Galerkin formulation; besides the linearization of the Navier-Stokes equation not is necessary.

Finally, we choose a 3-D element because the fluid flows are three-dimensional in your general form (Fortuna[1])

2. NONLINEAR INCOMPRESSIBLE NAVIER-STOKES MODEL

2.1 The continuous Problem

Let Ω be a bounded domain of $\Re^3$. Let $\partial\Omega$ be its boundary. One may compute an approximate solution for the dimensionless problem:

Find u and p such that

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\nabla p - \nu\Delta\mathbf{u} = f \qquad \text{in } \Omega \tag{1a}$$

$$\nabla\cdot\boldsymbol{u} = 0 \qquad \text{in } \Omega \tag{1b}$$

where: $\boldsymbol{u}=\left(v_x, v_y, v_z\right)$ is the velocity, and p is the pressure and ν and f are the kinematics viscosity and the external force, respectively. For the numerical considerations performed in this work, the external force f was considered zero.

2.2 The Discrete Problem

The definition of the Galerkin formulation is given by:

$$\sum_{i=1}^{20}\int_{\partial\Omega} N_i\left(equation \quad a\right)=0 \tag{2}$$

where "equation a" can be considered as Eq. (1a) or Eq. (1b), and N_i is the basis function associated to node *i*. Velocity and pressure are approximated using the appropriate polynomial basis functions, according to the following equations:

$$u^n = \sum_{j=1}^{N_0} u_j N_j\left(\xi,\eta,\zeta\right) \tag{3a}$$

$$p^n = \sum_{j=1}^{N_0} p_j N_j\left(\xi,\eta,\zeta\right) \tag{3b}$$

where N_0 is the number of the nodes of the element and ξ,η,ζ normalized coordinates. Applying the definitions (2) and (3), the corresponding discrete problem for equations (1a), (1b) is:

x component

$$\sum_{i=1}^{20}\int_V N_i\rho\left(\sum_{k=1}^{20}N_k v_{xk}\cdot\sum_{j=1}^{20}\frac{\partial N_j}{\partial x}v_{xj}\right)dV+\sum_{i=1}^{20}\int_V N_i\rho\left(\sum_{k=1}^{20}N_k v_{yk}\cdot\sum_{j=1}^{20}\frac{\partial N_j}{\partial y}v_{xj}\right)dV$$
$$+\sum_{i=1}^{20}\int_V N_i\rho\left(\sum_{k=1}^{20}N_k v_{zk}\cdot\sum_{j=1}^{20}\frac{\partial N_j}{\partial z}v_{xj}\right)dV+\sum_{i=1}^{20}\int_V \frac{\partial N_i}{\partial x}\mu\left(\sum_{j=1}^{20}\frac{\partial N_j}{\partial x}v_{xj}\right)dV$$
$$+\sum_{i=1}^{20}\int_V \frac{\partial N_i}{\partial y}\mu\left(\sum_{j=1}^{20}\frac{\partial N_j}{\partial y}v_{xj}\right)dV+\sum_{i=1}^{20}\int_V \frac{\partial N_i}{\partial z}\mu\left(\sum_{j=1}^{20}\frac{\partial N_j}{\partial z}v_{xj}\right)dV$$
$$+\sum_{i=1}^{20}\int_V N_i\left(\sum_{j=1}^{20}\frac{\partial N_j}{\partial x}p_j\right)dV-\sum_{i=1}^{20}\int_V N_i\rho g_x dV=0 \qquad (4a)$$

y component

$$\sum_{i=1}^{20}\int_V N_i\rho\left(\sum_{k=1}^{20}N_k v_{xk}\cdot\sum_{j=1}^{20}\frac{\partial N_j}{\partial x}v_{yj}\right)dV+\sum_{i=1}^{20}\int_V N_i\rho\left(\sum_{k=1}^{20}N_k v_{yk}\cdot\sum_{j=1}^{20}\frac{\partial N_j}{\partial y}v_{yj}\right)dV$$
$$+\sum_{i=1}^{20}\int_V N_i\rho\left(\sum_{k=1}^{20}N_k v_{zk}\cdot\sum_{j=1}^{20}\frac{\partial N_j}{\partial z}v_{yj}\right)dV+\sum_{i=1}^{20}\int_V \frac{\partial N_i}{\partial x}\mu\left(\sum_{j=1}^{20}\frac{\partial N_j}{\partial x}v_{yj}\right)dV$$
$$+\sum_{i=1}^{20}\int_V \frac{\partial N_i}{\partial y}\mu\left(\sum_{j=1}^{20}\frac{\partial N_j}{\partial y}v_{yj}\right)dV+\sum_{i=1}^{20}\int_V \frac{\partial N_i}{\partial z}\mu\left(\sum_{j=1}^{20}\frac{\partial N_j}{\partial z}v_{yj}\right)dV$$
$$+\sum_{i=1}^{20}\int_V N_i\left(\sum_{j=1}^{20}\frac{\partial N_j}{\partial y}p_j\right)dV-\sum_{i=1}^{20}\int_V N_i\rho g_y dV=0 \qquad (4b)$$

z component

$$\sum_{i=1}^{20}\int_V N_i\rho\left(\sum_{k=1}^{20}N_k v_{xk}\cdot\sum_{j=1}^{20}\frac{\partial N_j}{\partial x}v_{zj}\right)dV+\sum_{i=1}^{20}\int_V N_i\rho\left(\sum_{k=1}^{20}N_k v_{yk}\cdot\sum_{j=1}^{20}\frac{\partial N_j}{\partial y}v_{zj}\right)dV$$
$$+\sum_{i=1}^{20}\int_V N_i\rho\left(\sum_{k=1}^{20}N_k v_{zk}\cdot\sum_{j=1}^{20}\frac{\partial N_j}{\partial z}v_{zj}\right)dV+\sum_{i=1}^{20}\int_V \frac{\partial N_i}{\partial x}\mu\left(\sum_{j=1}^{20}\frac{\partial N_j}{\partial x}v_{zj}\right)dV$$
$$+\sum_{i=1}^{20}\int_V \frac{\partial N_i}{\partial y}\mu\left(\sum_{j=1}^{20}\frac{\partial N_j}{\partial y}v_{zj}\right)dV+\sum_{i=1}^{20}\int_V \frac{\partial N_i}{\partial z}\mu\left(\sum_{j=1}^{20}\frac{\partial N_j}{\partial z}v_{zj}\right)dV$$
$$+\sum_{i=1}^{20}\int_V N_i\left(\sum_{j=1}^{20}\frac{\partial N_j}{\partial z}p_j\right)dV-\sum_{i=1}^{20}\int_V N_i\rho g_z dV=0 \qquad (4c)$$

continuity equation

$$\sum_{i=1}^{20}\int_V N_i\left(\sum_{j=1}^{20}\frac{\partial N_j}{\partial x}v_{xj}\right)dV+\sum_{i=1}^{20}\int_V N_i\left(\sum_{j=1}^{20}\frac{\partial N_j}{\partial y}v_{yj}\right)dV+\sum_{i=1}^{20}\int_V N_i\left(\sum_{j=1}^{20}\frac{\partial N_j}{\partial z}v_{zj}\right)dV=0 \qquad (4d)$$

In deriving the above equations, the Green theorem has been used so reduce the second-order terms in the momentum equations. In order to discretize the domain Ω, a 20-node hexahedral element has been adopted. The normalized coordinates for the element are (ξ,η,ζ).The element, its local numbering and local coordinates are shown in Fig. 2.

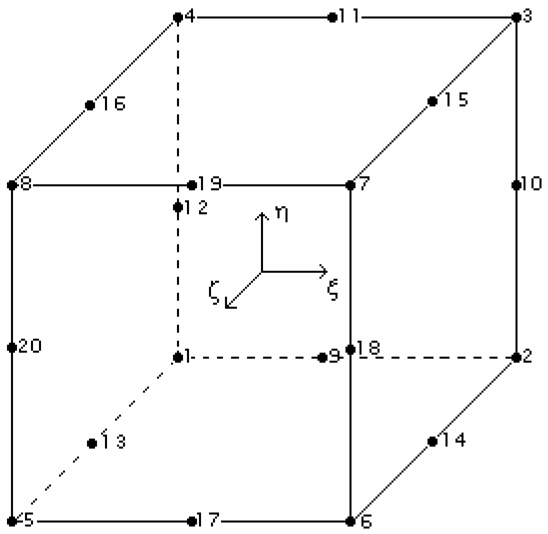

Figure 1. Hexahedral 20-noded element.

For this element, the appropriate basis functions (Serendipity family) are defined by (Hirsch, 1998),

For corner nodes:

$$N_j = \frac{1}{8}(1+\xi_0)\cdot(1+\eta_0)\cdot(1+\zeta_0)\cdot(\xi_0+\eta_0+\zeta_0-2) \qquad (5a)$$

For mid-side nodes:

$$N_j = \frac{1}{4}(1-\xi^2)\cdot(1+\eta_0)\cdot(1+\zeta_0) \quad \text{(which } \xi_j = 0, \eta_j = \pm 1, \zeta_j = \pm 1\text{)} \qquad (5b)$$

where the variables $\xi_0 = \xi \cdot \xi_j$, $\eta_0 = \eta \cdot \eta_j$, $\zeta_0 = \zeta \cdot \zeta_j$

3. POISSON EQUATION FOR PRESSURE

The stabilization of pressure terms in primitive variables CFD methods is a very old problem, especially when generated via the Galerkin Finite Element Method. According to Gresho (Gresho et al., 1981), it was discovered that equal-order interpolation, wherein the same basis functions are used for representing velocity and pressure, causes difficulty in the pressure solution. The Poisson equation is useful for this problem because provide velocity values what satisfy the continuity equation; therefore, the Poisson equation is a connection between the momentum and the continuity equations(Fortuna, 2000).

In this work, we present the Poisson equation for pressure to add stability and accuracy for the pressure terms.

The Poisson equation, is defined as:

$$\nabla^2\left(\frac{p}{\rho}\right) = -(\nabla u):(\nabla u) \qquad (6)$$

(with $A:B=\sum_i \sum_j A_{ij}B_{ji}$)

Using the definitions of the Galerkin procedure, equation (2), and the finite element approximation, equations (3a) and (3b), the Poisson equations can be written:

$$\begin{aligned}
&\sum_{i=1}^{20}\int_V\left(N_i\sum_{k=1}^{20}\frac{\partial N_k}{\partial x}v_{xk}\cdot\sum_{j=1}^{20}\frac{\partial N_j}{\partial x}v_{xj}\right)dV+2\sum_{i=1}^{20}\int_V\left(N_i\sum_{k=1}^{20}\frac{\partial N_k}{\partial x}v_{yk}\cdot\sum_{j=1}^{20}\frac{\partial N_j}{\partial y}v_{xj}\right)dV\\
&+2\sum_{i=1}^{20}\int_V\left(N_i\sum_{k=1}^{20}\frac{\partial N_k}{\partial x}v_{zk}\cdot\sum_{j=1}^{20}\frac{\partial N_j}{\partial z}v_{xj}\right)dV+\sum_{i=1}^{20}\int_V\left(N_i\sum_{k=1}^{20}\frac{\partial N_k}{\partial y}v_{yk}\cdot\sum_{j=1}^{20}\frac{\partial N_j}{\partial y}v_{yj}\right)dV\\
&+2\sum_{i=1}^{20}\int_V\left(N_i\sum_{k=1}^{20}\frac{\partial N_k}{\partial y}v_{zk}\cdot\sum_{j=1}^{20}\frac{\partial N_j}{\partial z}v_{yj}\right)dV+\sum_{i=1}^{20}\int_V\left(N_i\sum_{k=1}^{20}\frac{\partial N_k}{\partial z}v_{zk}\cdot\sum_{j=1}^{20}\frac{\partial N_j}{\partial z}v_{zj}\right)dV\\
&-\sum_{i=1}^{20}\int_V\left(\frac{\partial N_i}{\partial x}\sum_{j=1}^{20}\frac{\partial N_j}{\partial x}p_j\right)dV-\sum_{i=1}^{20}\int_V\left(\frac{\partial N_i}{\partial y}\sum_{j=1}^{20}\frac{\partial N_j}{\partial y}p_j\right)dV\\
&-\sum_{i=1}^{20}\int_V\left(\frac{\partial N_i}{\partial z}\sum_{j=1}^{20}\frac{\partial N_j}{\partial z}p_j\right)dV=0 \qquad (7)
\end{aligned}$$

This version of the Poisson equation(incomplete Poisson equation) is obtained by assuming $\nabla\cdot u=0$ in the original equation. Based on this, the system defined by

$$\begin{cases}
x\quad component(4a)\\
y\quad componen(4b)\\
z\quad component(4c)\\
continuity\quad equation(4d)\\
complet\quad Poisson\quad equation
\end{cases}$$

is replaced by the equivalent system:

$$\begin{cases}
x\quad component(4a)\\
y\quad componen(4b)\\
z\quad component(4c)\\
Poisson\quad equation\quad for\quad pressure(7)
\end{cases}$$

For this element, the appropriate basis functions employed were (Hirsch, 1988):

For corner nodes:

$$N_j=\frac{1}{8}(1+\xi_0)\cdot(1+\eta_0)\cdot(1+\zeta_0)\cdot(\xi_0+\eta_0+\zeta_0-2) \qquad (5a)$$

For a typical mid-side node:

$$N_j=\frac{1}{4}(1-\xi^2)\cdot(1+\eta_0)\cdot(1+\zeta_0)\quad \text{(for which } \xi_j=0,\eta_j=\pm1,\zeta_j=\pm1) \qquad (5b)$$

where the variables $\xi_0 = \xi \cdot \xi_j$, $\eta_0 = \eta \cdot \eta_j$, $\zeta_0 = \zeta \cdot \zeta_j$.

4. IMPLEMANTATIONAL ASPECTS

The numerical validation of procedure introduced early is performed using a 3D incompressible Navier-Stokes code for both rectangular and cylindrical coordinates. The code, based in Taylor's work (Taylor,1981), which uses the Frontal technique (Irons, 1970). solves the system as:

$$[C] \cdot \{X\} = 0 \tag{8}$$

where: $\{X^T\} = \{v_{r_1} \quad v_{\theta_1} \quad v_{z_1} \quad p_1 \quad v_{r_2} \quad v_{\theta_2} \quad v_{z_2} \quad p_2 ....v_{r_m} \quad v_{\theta_m} \quad v_{z_m} \quad p_m\}$ and m = total number of the nodes.

The integrals (4a), (4b), (4c) and (7) are numerically evaluated using four-points Gaussian quadrature. The boundary conditions are specified at nodal locations along the characteristic interfaces by imposing known velocities components and pressure values.

5. NUMERICAL TESTS

For the comparison purposes two numerical simulations were performed, and numerical results are presented together with classical analytical solutions. The plane Couette flow (plots of velocity, velocity vector and pressure for mesh with 12-element) and Poiseuille flow (plots of velocity, velocity vector and pressure for mesh with 9-element) are taken as references for preliminary confirmations of the numerical procedure. For both cases, charts with the greatest errors for each simulation are presented.

5.1. Plane Couette Flow

Problem description: The considered problem is that of a viscous fluid flow between two parallel rigid plates separated of a distance *D*. The bottom plate is at rest and the top plate is moving in its own plane with a constant velocity v_x.
Mesh: This problem has been modeled with four meshes, with 6, 8, 9 and 12 elements. The sketches of these meshes are presented in Fig. 10.
Boundary conditions: A no-slip velocity boundary condition is applied to the bottom plate and a constant velocity v_x = 3m/s to the top plate. Since parallel flow is assumed, all *y* and *z* components of velocity are constrained to zero. The fluid proprieties are those of the water (Streeter, 1961), and the Reynolds number for all simulations was set as 1644,3.

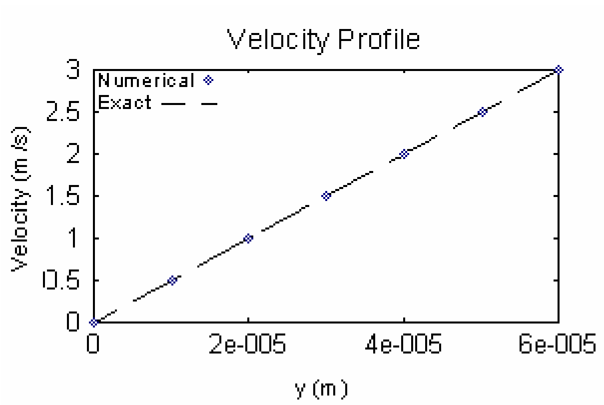

Figure 2. Re = 1644,3: Profile of horizontal velocity for 12-element mesh (Couette flow).

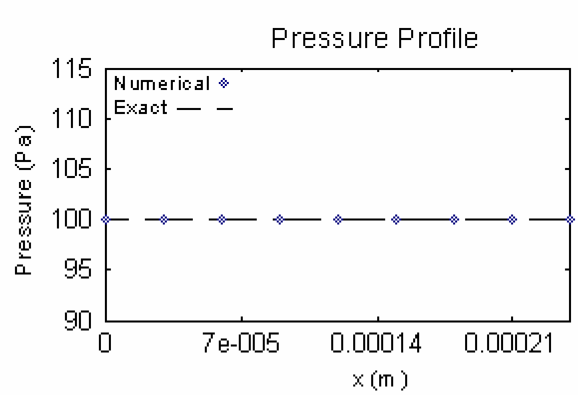

Figure 3. Re = 1644,3: Profile of pressure for 12-element mesh (Couette flow).

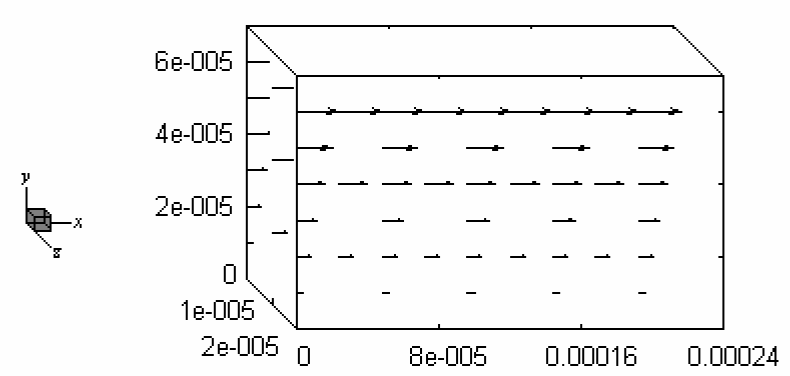

Figure 4. Re = 1644,3: Velocity vector plot for 12-element mesh (Couette flow).

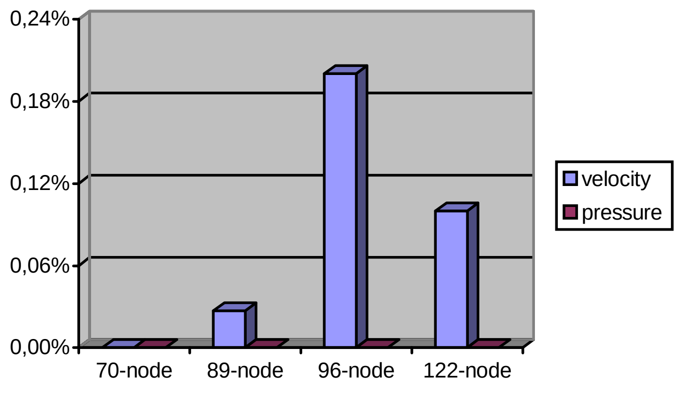

Figure 5. Greatest error, for velocity and pressure, for the simulations with 6, 8, 9 and 12 elements. (Couette flow).

5.2. Poiseuille Flow

Problem description: The problem considered here is that of a viscous flow between two parallel rigid plates separated by a distance *D*. Both plates are at rest.
Mesh: This problem is modeled for three meshes, with 4, 6 and 9 elements. The sketches of these meshes are presented in Fig. 10.
Boundary conditions: A no-slip velocity boundary condition is applied to both plates. Since parallel flow is assumed, all *y* and *z* components of velocity are constrained to zero. The values of the pressures are imposed at the inflow and outflow, having different values for each mesh. Theses values, for the inflow and outflow, respectively, are:

4-element: 1500Pa - 900Pa
6-element: 100Pa - 90Pa
9-element: 200Pa - 90Pa

The Reynolds number (calculated with values result) for each simulation, was:

4-element: 120,0
6-element: 10,0
9-element: 65,7

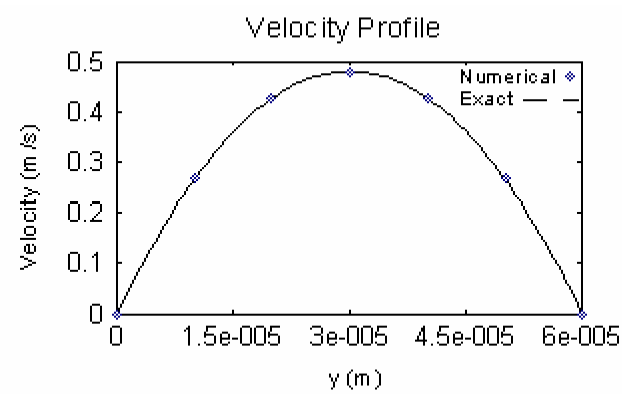

Figure 6. Re = 65,7: Profile of horizontal velocity for 9-element mesh (Poseuille flow).

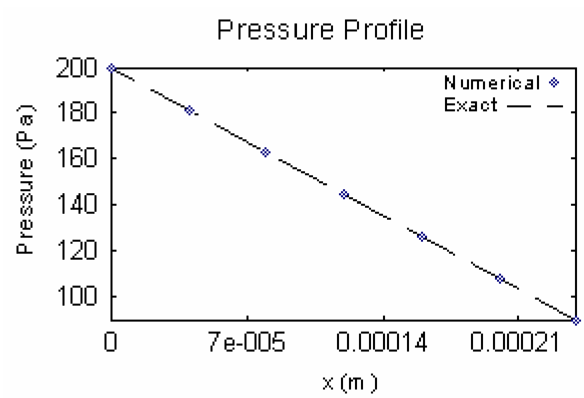

Figure 7. Re = 65,7: Profile of pressure for 9-element mesh (Pouseuille flow).

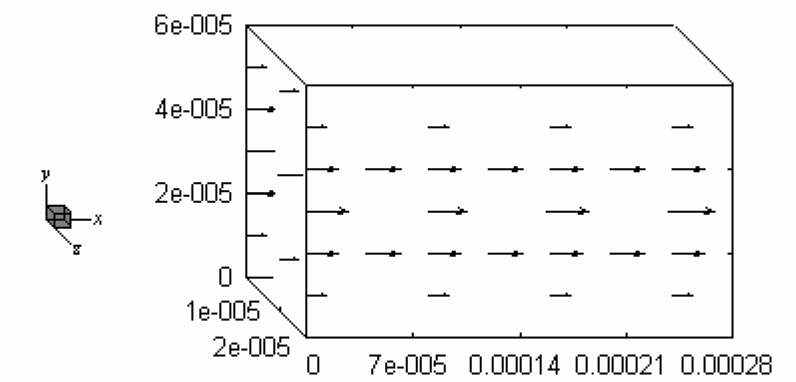

Figure 8. Re = 65,7: Velocity vector plot for 9-element mesh (Poseuille flow).

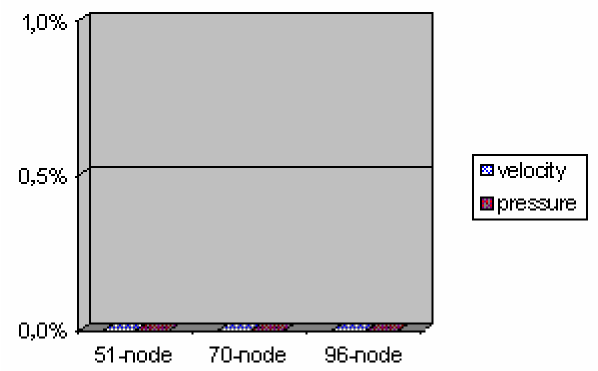

Figure 9. Greatest error, for velocity and pressure, for the simulations with 4, 6 and 9 elements. (Pouseuille flow).

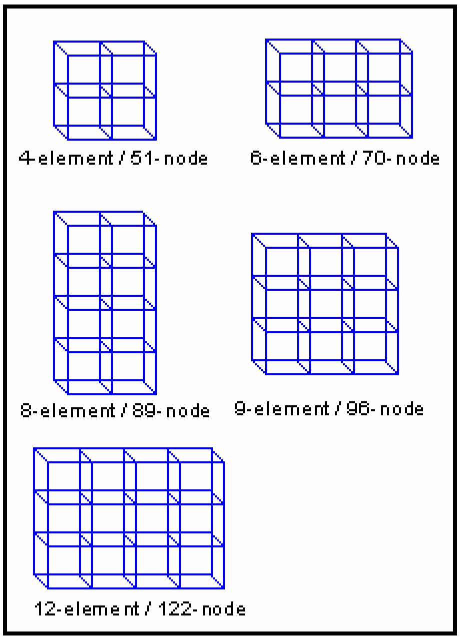

Figure 10. Sketch of meshes used in the simulations.

6. CONCLUSIONS

A finite element formulation, with equal-order-interpolation functions for the computation of the incompressible stationary Navier-Stokes equations, has been proposed. Two classical cases discussed in this paper presented satisfactory results.

For plane Couette flow modeled with four meshes, with 6, 8, 9 and 12 elements, in comparison to the exact solution, presents results which follow very closely those of the exact solution, with a maximum deviation of about 0.2% for the 96 elements mesh. Concerning pressure values, the same problem shows results with errors very close to zero. Numerical solution of Poiseuille flow, using 4, 6 and 9 discretization, evidences the highest level of accuracy for deviations approaching zero for all tested meshes, either for velocities or pressure values. Based on the preceding analyses, the proposed formulation has documented excellent accuracy for the incompressible stationary flow problem.

One may register that Poisson equation for pressure added stability for the numerical method, and accuracy to the solution. Solutions converged, for all cases, with a number of iterations between 20 and 25. Previous numerical trials performed by the authors, considering Navier-Stokes equations coupled to the continuity equation, did not present satisfactory low levels of deviations, most particularly for pressures results.

Finally, it may be emphasized that the computer code, being developed, is potentially important in solutions of fluid-structure three-dimensional problems, and those simulations concerning accurate depiction of fluid behavior of hydrodynamic bearings.

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