ON THE EFFECT OF NOISE FOR CONTROLLING CHAOS IN A NONLINEAR PENDULUM

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ABSTRACT

Chaotic behavior of dynamical systems offers a rich variety of orbits, which can be controlled by small perturbations in either a specific parameter of the system or a dynamical variable. Chaos control usually involves two steps. In the first, unstable periodic orbits (UPOs) that are embedded in the chaotic set are identified. After that, a control technique is employed in order to stabilize a desirable orbit. This contribution analyzes the effect of noise related to chaos control. Basically, the close-return method is employed to identify UPOs and a semi-continuous control method, which is built up on the OGY method, is used to stabilize some desirable UPO. As an application to a mechanical system, a nonlinear pendulum is considered using signals generated by numerical integration of the mathematical model. Results show situations where these techniques may be used to control chaos in mechanical systems.

Keywords: Chaos, Control, Nonlinear Pendulum.

1. INTRODUCTION

Chaos control is based on the richness of responses of chaotic behavior. A chaotic attractor has a dense set of unstable periodic orbits (UPOs) and the system often visits the neighborhood of each one of them. Moreover, chaotic response has sensitive dependence to initial condition, which implies that the system's evolution may be altered by small perturbations. Therefore, chaos control may be understood as the use of tiny perturbations for the stabilization of an UPO embedded in a chaotic attractor, which makes this kind of behavior to be desirable in a variety of applications, since one of these UPO can provide better performance than others in a particular situation. It should be pointed out that it is not necessary to have a mathematical model to achieve the control goal since all control parameters may be resolved from time series analysis.

Since noise contamination is unavoidable in cases of experimental data acquisition, it is important to evaluate its effect on chaos control procedures. Noise reduction schemes for chaotic noisy time series is an alternative (Shin *et al.*, 1999; Davies, 1994; Schreiber & Grassberger, 1991; Sauer, 1992; Enge *et al.*, 1993; Kostelich & Schreiber, 1993; Schreiber & Richter, 1999, Bröcker *et al.*, 2002) or Kalman filtering (Leung *et al.*, 2000; Lefebvre *et al.*, 2001; Wan & van der Merwe, 2001) but these subjects are beyond the scope of this paper. On the other hand, it is possible to employ robust control procedures to avoid filtering processes. Ott *et al.* (1990) say that the efficiency of the OGY method is close related to the noise level. Spano *et al.* (1991) also study the effect of noise in OGY method, confirming the previous conclusion.

This contribution evaluates noise sensitivity of chaos control techniques, previously discussed by Pereira-Pinto *et al.* (2003a,b, 2004). Basically, chaos control is applied to a nonlinear pendulum that is based on the experimental apparatus previously analyzed by Franca & Savi (2001) and Pinto & Savi (2003). This pendulum has both torsional stiffness and damping. All signals are generated numerically by the integration of the equations of the mathematical model proposed, which uses experimentally identified parameters. The close-return (CR) method (Auerbach *et al.*, 1987) is employed to determine the UPO embedded in the attractor. A variation of the OGY technique called semi-continuous control (SCC) method, proposed by Hübinger *et al.* (1994) and extended by Korte *et al.* (1995), is considered to stabilize the desirable orbit. The control analysis considers that all state variables are available. The effect of noise in the controlling procedures is analyzed, defining some limitations of these techniques. Results confirm the possibility of the use of this approach to deal with mechanical systems. Noise suppression is not considered and it is beyond the scope of this contribution the influence of noise on the determination of UPOs. For this analysis, see references: Pierson & Moss (1995), So *et al.* (1996), Pei *et al.* (1998), Dolan *et al.* (1999) and Dolan (2001).

2. DETERMINATION OF UNSTABLE PERIODIC ORBITS

The control of chaos can be treated as a two-stage process. The first stage is composed by the identification of UPO and is named as "learning stage". Since UPO are system invariants, they can be analyzed from phase space reconstructed from a scalar time series (Gunaratne *et al.*, 1989).

This article considers the close-return (CR) method (Auerbach *et al.*, 1987) for the detection of UPO embedded in the attractor. The basic idea is to search for a period–P UPO in the time series represented by vectors $\{u_i\}_{i=1}^{N}$. This state vector may be obtained by state space reconstruction from a scalar time series s_n ($n = 1, \dots N$) using delay coordinates, for example (Takens, 1981). The identification of a period–P UPO is based on a search for pairs of points in the time series that satisfy the condition $\left|u_i - u_{i+P}\right|_{i=1}^{(N-P)} \leq r_1$ where r_1 is the tolerance value for distinguishing return points. After this analysis, all points that belong to a period-P cycle are grouped together. During the search, the vicinity of a UPO may be visited many times, and it is necessary to distinguish each orbit, remove any cycle permutation and to average them in order to improve estimations as shown by Otani & Jones (1997).

After the identification of a UPO, one can proceed to the next stage of the control process that is the stabilization of the desired orbit. In the following section, it is presented one of the procedures used for this aim: the SCC control method.

3. SEMI-CONTINUOUS CONTROL METHOD

The OGY (Ott *et al.*, 1990) approach is described considering a discrete system of the form of a map $\xi_{i+1} = F(\xi_i, p)$, where $p \in \Re$ is an accessible parameter for control. This is equivalent to a parameter dependent map associated with a general surface, usually a Poincaré section. Let $\xi_F = F(\xi_F, p_0)$ denote the unstable fixed point on this section corresponding to an orbit in the chaotic attractor that one wants to stabilize. Basically, the control idea is to monitor the system dynamics until the neighborhood of this point is reached. After that, a proper small change in the parameter p causes the next state ξ_{i+1} to fall into the stable direction of the fixed point. In order to find the proper variation in the control parameter, δp, it is considered a linearized version of the dynamical system near the equilibrium point.

$$\delta\xi_{i+1} \cong A\delta\xi_i + w\delta p_i \tag{1}$$

where $\delta\xi_i = \xi_i - \xi_F$, $\delta p_i = p_i - p_0$, $A = D_\xi F(\xi_F, p_0)$, and $w = \partial F / \partial p(\xi_F, p_0)$.

The OGY method can be employed even in situations where a mathematical model is not available. Under this situation, all parameters can be extracted from time series analysis. The Jacobian A and the sensitivity vector w can be estimated from a time series using a least-square fit method as described in Auerbach *et al.* (1987) and Otani & Jones (1997).

In order to overcome some limitations of the original OGY formulation such as control of orbits with large instability, measured by unstable eigenvalues, and orbits of high period, Hübinger *et al.* (1994) introduced a semi-continuous control (SCC) method or local control method, which description is presented as follows.

The SCC method lies between the continuous and the discrete time control because one can introduce as many intermediate Poincaré sections, viewed as control stations, as it is necessary to achieve stabilization of a desirable UPO. Therefore, the SCC method is based on measuring transition maps of the system. These maps relate the state of the system in one Poincaré section to the next.

In order to use N control stations per forcing period T, one introduces N equally spaced successive Poincaré sections Σ_n, $n = 0,...,(N-1)$. Let $\xi_F^n \in \Sigma_n$ be the intersections of the UPO with Σ_n and $F^{(n,n+1)}$ be the mapping from one control station Σ_n to the next one Σ_{n+1}. Here, the superscript n is used instead of the subscript i of the OGY method, to differentiate both methods. Hence, one considers the map

$$\xi_F^{n+1} = F^{(n,n+1)}(\xi_F^n, p^n). \tag{2}$$

A linear approximation of $F^{(n,n+1)}$ around ξ_F^n and p_0 is considered as follows:

$$\delta\xi^{n+1} \cong A^n \delta\xi^n + w^n \delta p^n, \tag{3}$$

where $\delta\xi^{n+1} = \xi^{n+1} - \xi_F^{n+1}$, $\delta p^n = p^n - p_0$, $A^n = D_{\xi^n} P^{(n,n+1)}(\xi_F^n, p_0)$, and $w^n = \dfrac{\partial P^{(n,n+1)}}{\partial p^n}(\xi_F^n, p_0)$.

Hübinger *et al.* (1994) analyze the possibility of the eigenvalues of A^n be complex numbers and then they use the fact that the linear mapping A^n deforms a sphere around ξ_F^n into an ellipsoid around ξ_F^{n+1}. Therefore, a singular value decomposition (SVD),

$$A^n = U^n W^n (V^n)^T = \{u_u^n \quad u_s^n\} \begin{bmatrix} \sigma_u^n & 0 \\ 0 & \sigma_s^n \end{bmatrix} \{v_u^n \quad v_s^n\}^T , \qquad (4)$$

is employed in order to determine the directions v_u^n and v_s^n in Σ_n which are mapped onto the largest, $\sigma_u^n u_u^n$, and shortest, $\sigma_s^n u_s^n$, semi-axis of the ellipsoid in Σ_{n+1}, respectively. Here, σ_u^n and σ_s^n are the singular values of A^n.

Korte *et al.* (1995) establish the control target as being the adjustment of δp^n such that the direction v_s^{n+1} on the map *n*+1 is obtained, resulting in a maximal shrinking on map *n*+2. Therefore, it demands $\delta\xi^{n+1} = \alpha v_s^{n+1}$, where $\alpha \in \Re$. Hence, from Equation (3) one has that

$$A^n \delta\xi^n + w^n \delta p^n = \alpha v_s^{n+1} , \qquad (5)$$

which is a relation from what α and δp^n can be conveniently chosen.

4. CONTROLING A NONLINEAR PENDULUM

As a mechanical application of general chaos control procedure here presented, a nonlinear pendulum is considered. The motivation of the proposed pendulum is an experimental set up, previously analyzed by Franca & Savi (2001) and Pinto & Savi (2003). Here, a mathematical model is developed to describe the dynamical behavior of the pendulum while the corresponding parameters are obtained from the experimental apparatus. Numerical simulations of such model are employed in order to obtain time series related to the pendulum response. Finally, some unstable periodic orbits are identified with the CR method and their control simulated employing the SCC method.

The considered nonlinear pendulum is shown in Figure 1. The left side presents the experimental apparatus while the right side shows a schematic picture. Basically, pendulum consists of an aluminum disc (1) with a lumped mass (2) that is connected to a rotary motion sensor (4). A magnetic device (3) provides an adjustable dissipation of energy. A string-spring device (6) provides torsional stiffness to the pendulum and an electric motor (7) excites the pendulum via the string-spring device. An actuator (5) provides the necessary perturbations to stabilize this system by properly changing the string length.

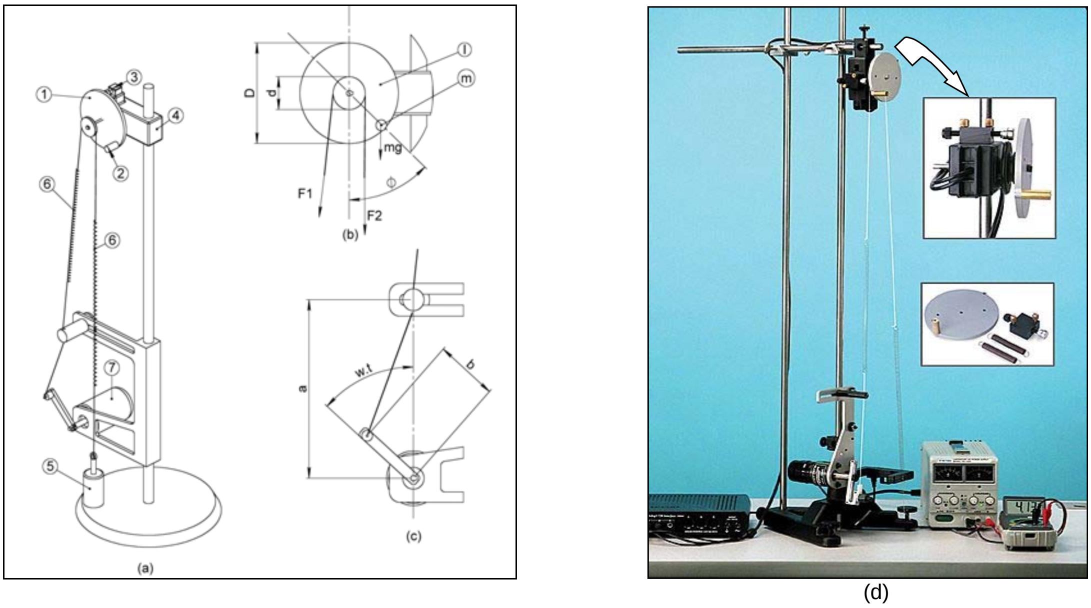


Figure 1 – Nonlinear pendulum. (a) Physical Model. (1) Metallic disc; (2) Lumped mass; (3) Magnetic damping device; (4) Rotary Motion Sensor; (5) Actuator; (6) String-spring device; (7) Electric motor. (b) Parameters and forces on the metallic disc. (c) Parameters from driving device. (d) Experimental apparatus.

In order to describe the dynamics of this pendulum, a mathematical model is proposed. Let F_1 and F_2 be the forces exerted on the rotating masses and given by:

$$F_1 = k(\sqrt{a^2 + b^2 - 2ab\cos(\varpi t)} - (a-b) - \frac{d}{2}\phi) \qquad\qquad F_2 = k(\frac{d}{2}\phi - \Delta l) \tag{6}$$

where ϖ is the forcing frequency, a defines the position of the guide of the string with respect to the motor, b is the length of the excitation arm of the motor, D is the diameter of the metallic disc and d is the diameter of the driving pulley. The Δl parameter is the length variation in the string provided by the linear actuator (5) shown in Figure 1a. This parameter is considered as the variation on the accessible parameter for control purposes. Therefore, the equation of motion is given by:

$$\begin{Bmatrix} \dot{\phi} \\ \ddot{\phi} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{kd^2}{2I} & -\frac{\zeta}{I} \end{bmatrix} \begin{Bmatrix} \phi \\ \dot{\phi} \end{Bmatrix} + \begin{Bmatrix} 0 \\ \frac{kd}{2I}[\sqrt{a^2 + b^2 - 2ab\cos(\varpi t)} - (a-b) - \Delta l] - \frac{mgD}{2I}\sin(\phi) \end{Bmatrix} \tag{7}$$

where I is the total inertia of rotating parts, m is the lumped mass and ζ is the dissipation parameter. The determination of parameters in equation of motion is done by considering the experimental setup of Franca & Savi (2001). Table 1 shows the parameters that are evaluated from the experimental setup.

Table 1 – Experimental values of parameters.

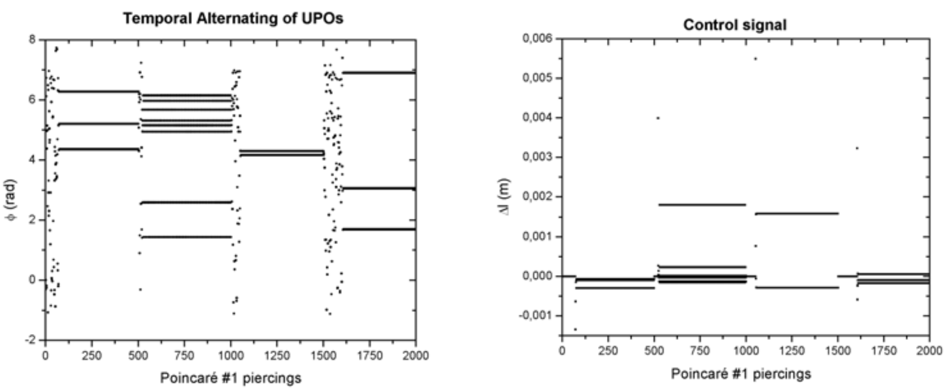
a (m)	b (m)	d (m)	D (m)	I (kg m^4)	k (N/m)	m (kg)
1.6 x 10^{-2}	6.0 x 10^{-2}	2.9 x 10^{-2}	9.2 x 10^{-2}	1.876 x 10^{-4}	4.736	1.6 x 10^{-2}

Values of the adjustable parameters ϖ and ζ are tuned to generate chaotic response in agreement to the experimental work done by Franca & Savi (2001). The Δl parameter has a null value for the system without control action. Therefore, using the parameters presented in Table 2, it is possible to use a fourth-order Runge-Kutta scheme in order to perform numerical simulations of the equations of motion.

Table 2 –Values of adjustable parameters.

ϖ (rad/s)	ζ (kg.m^2/s)	Δl (m)
5.15	5.575 x 10^{-5}	0

In order to explore the possibilities of alternating the stabilized orbits with small changes in the control parameter, one performs a simulation that aims the stabilization of the following UPOs related to a clean signal, with no noise: a period–3 UPO in the first 500 forcing periods, a period–8 UPO between 500 and 1000 forcing periods, a period–2 UPO between 1000 and 1500 forcing periods and a period–3 UPO, different from the first one, between 1500 and 2000 forcing periods. Figure 2 shows the system's dynamics in the Poincaré section #1 during the actuation. Notice that different times are needed for the system to achieve the desired stabilization on a particular UPO. This happens because one must wait until the trajectory comes close enough to a control point to perform the necessary perturbation, exploiting the ergodicity property of the chaos behavior. Moreover, it should be pointed out that, as expected, results show that unstable orbits are stabilized with small variations of control parameter after a transient, less than 2mm in this case.



(a) (b)

Figure 2 – Response under control.
(a) Temporal alternating of UPOs in Poincaré section #1. (b) Control signal.

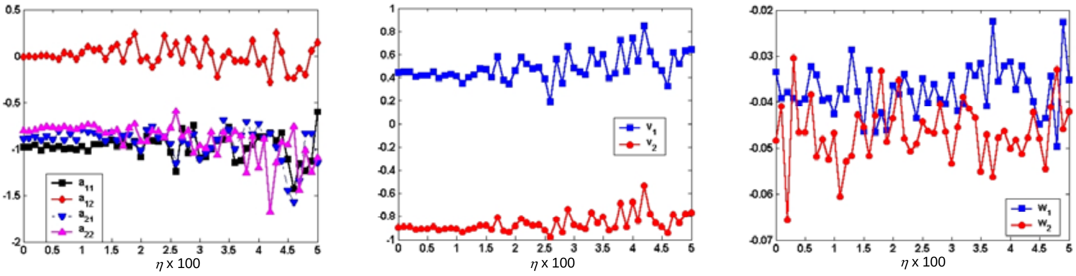
5. EFFECT OF NOISE ON THE CONTROL

In order to simulate experimental noisy data sets, a white gaussian noise is introduced in the signal, comparing results of control procedures with a clean time series, free of noise. In general, noise can be expressed as follows,

$$\begin{aligned} \dot{x} &= f(x,t) + \mu_d \\ z &= h(x,t) + \mu_o \end{aligned} \qquad (8)$$

where x represents state variables, while z represents the observed response. On the other hand, μ_d and μ_o are, respectively, dynamical and observed noises. Notice that μ_d has influence on system dynamics in contrast with μ_o. The noise level is parameterized by the root mean square value of the clean signal (RMS_{signal}). Therefore, the noise probability distribution variance is a fraction η of the RMS_{signal}, that is, $\eta = RMS_{signal}/RMS_{noise}$.

Basically, three different parameters are related to the SCC method: Jacobian matrix, sensitivity vector and maximum contraction direction. Figure 3 shows these parameters as a function of noise level. Notice that the increase of noise level causes an oscillatory behavior of these parameters. For noise levels with $\eta > 1.5\%$, there is a great level of uncertainties that can cause problems in SCC method efficiency for control purposes. Moreover, it is important to observe the high influence of noise associated with the estimation of the sensitivity vector.



(a) (b) (c)

Figure 3 – Influence of noise on control parameters.
(a) Jacobian matrix. (b) Maximum contraction vector. (c) Sensitivity vector.

The forthcoming analysis concerns to the effect of noise on the stabilization of a UPO. The increase of control stations is a useful procedure in order to avoid the effect of noise; however, it is limited by the time response of the system as discussed earlier. In order to study this behavior, an analysis based on a period-3 unstable orbit is carried out.

The increase of the number of control stations causes an increase in the value of control parameter. This behavior is related to the time response of the system. In order to compensate the decrease in the distance between two subsequent control stations, greater values of control parameters are needed. Figures 4 illustrate this behavior showing control parameters related to the stabilization of the period-3 orbit, for different number of control stations.

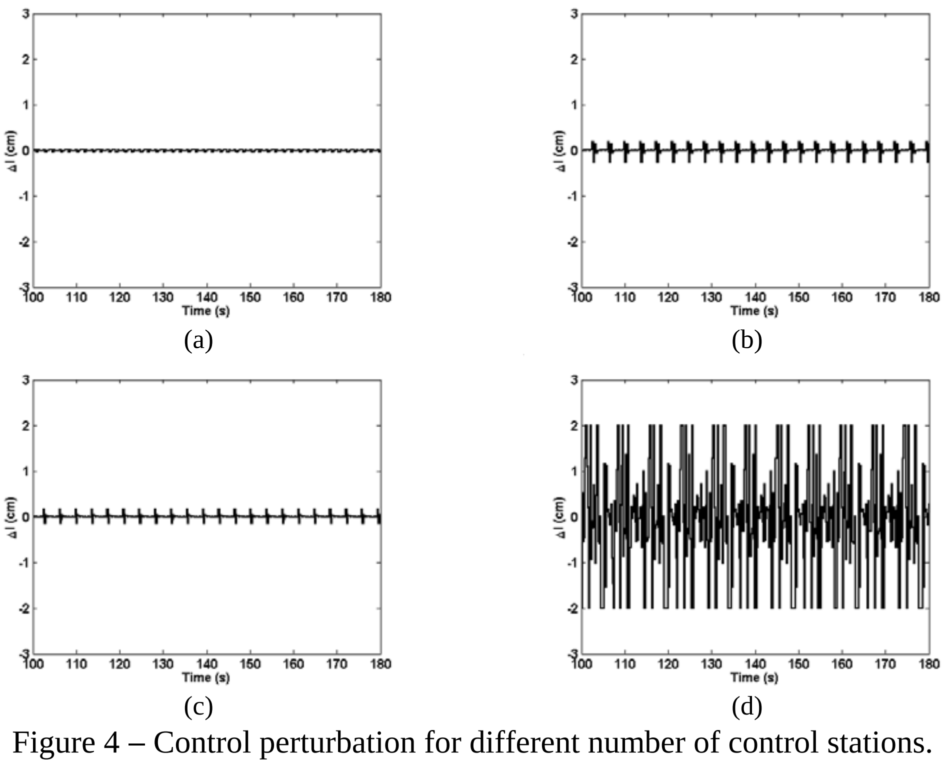


Figure 4 – Control perturbation for different number of control stations.
(a) $N = 3$. (b) $N = 4$. (c) $N = 5$. (d) $N = 6$.

At this point, the noise effect on the stabilization of the period-3 orbit is discussed. Different noise levels and number of control stations are considered. Firstly, a noise level $f = 0.6\%$ is treated. Figure 5 presents results related to the stabilization of the desirable orbit showing that, for this noise level, 3 control stations are sufficient to the stabilization. Moreover, it is should be pointed out that the same results can be obtained for $N = 4$, $N = 5$ and $N = 6$.

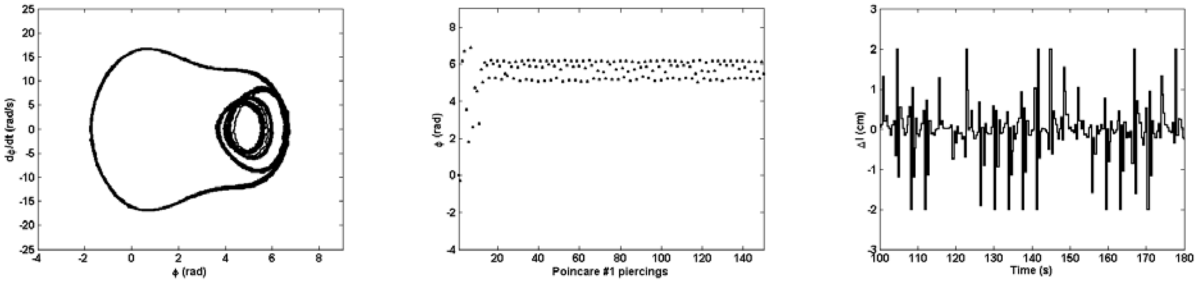


Figure 5 – Stabilization procedure for η = 0.6% with N = 3.
(a) State space. (b) Position on Poincaré section. (c) Control perturbation.

By increasing the noise level for η = 3.6%, the influence of the number of control stations is more pronounced. Figures 6-7 presents the process of stabilization for this noisy time series, showing the difficult related to this procedure. It should be pointed out that the stabilization is only effective for N = 6. For small number of control stations, the control procedure fails and the system diverges to other orbits.

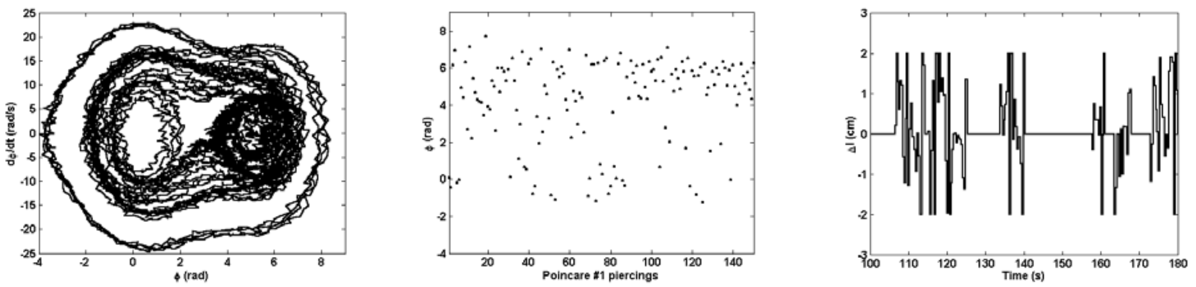


Figure 6 – Stabilization procedure for η = 3.6% with N = 3.
(a) State space. (b) Position on Poincaré section. (c) Control perturbation.

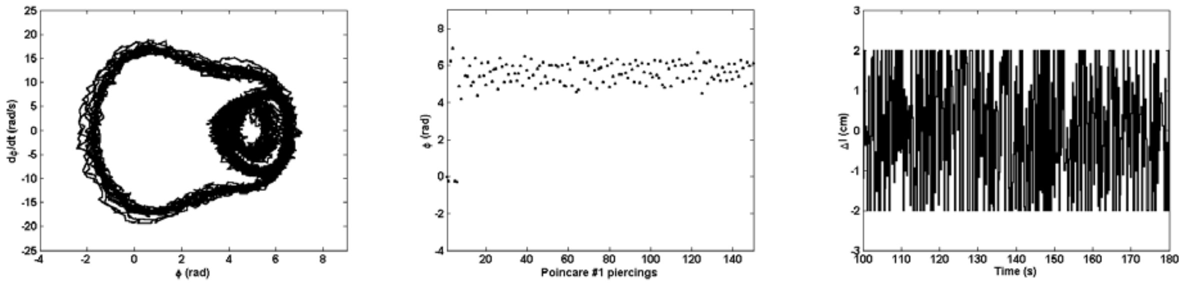


Figure 7 – Stabilization procedure for η = 3.6% with N = 6.
(a) State space. (b) Position on Poincaré section. (c) Control perturbation.

6. CONCLUSIONS

This contribution discusses the effect of noise on the control of chaos in a simulated nonlinear pendulum based on an experimental apparatus previously analyzed by Franca & Savi (2001) and Pinto & Savi (2003). In the first stage of the control process, the close-return method is employed to identify unstable periodic orbits (UPOs) embedded in the chaotic attractor. After that, the semi-continuous control (SCC) method is considered to stabilize desirable orbits. Least-square fit method is employed to estimate Jacobian matrixes and sensitivity vectors. Moreover, SVD decomposition is employed to estimate directions of unstable and stable manifolds in the vicinity of control points. Signals are generated by the numerical integration of the mathematical model. An analysis related to the effect of noise in controlling chaos is of concern. The stabilization of orbits related to noisy time series is more complex and the increase of control stations tends to increase the robustness of the control procedure. Nevertheless, it is important to notice that time response plays an important role in the control procedure, defining the maximum number of control stations.

7. ACKNOWLEDGEMENTS

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