II CONGRESSO NACIONAL DE ENGENHARIA MECÂNICA
II NATIONAL CONGRESS OF MECHANICAL ENGINEERING
12 a 16 de Agosto de 2002 - João Pessoa – PB

COMPARISON BETWEEN METHODS TO EVALUATION J-INTEGRAL ON 2-D FRACTURE PROBLEMS

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Abstract. *Methods of analysis for a two-dimensional cracked solid to evaluate the stress intensity factor (SIF) along the crack tip have been studied using the finite element method in conjunction with the J-integral method. The SIF was obtained by the J-integral method, which has been described in two approaches. The first approach considers a line integral path and the second considers the equivalent domain integral (EDI). The displacement field near the crack tip was modeled using isoparametric singular elements (quarter point elements). Two values of crack depth ratio and different lengths of the quarter point elements were tested to verify the accuracy of the developed routines. These routines were implemented in a commercial finite element code (FEM code). The results were analyzed and compared by analytical expression developed to geometry in study.*

Keywords: *Stress Intensity Factor, J Integral, Line Integral Path, Equivalent Domain Integral-EDI, Finite Element Method.*

1. INTRODUCTION

Computers have had an enormous influence in virtually all branches of engineering, and fracture mechanics is no exception. The use of fracture mechanics during the design process has been increasingly used during last years, because numerical modeling has become an indispensable tool in fracture analysis, since relatively few practical problems have closed-form analytical solutions (Anderson, 1995 and Erdogan, 2000). The evaluation of stress intensity factors by means of numerical methods (finite element method and boundary element method) are widely used. In the finite element method a number of techniques have been proposed for evaluating stress intensity factors. A very important phase in these techniques is the representation of the crack tip singularity. To represent adequately the singular stress-strain field near the crack tip, a modified isoparametric element was introduced by Henshell and Shaw (1975) and Barsoum (1976). They noticed that by displacing the midside-node of an eight-noded quadrilateral element to the quarter point position the element stress-strain field naturally exhibits a square root singularity.

Basically, there are two groups of estimation methods: those based on displacement or stress matching methods (displacement or stress correlation methods) and those based on energy methods (total energy method, stiffness derivative formulation, mapping technique, J-integral method, energy domain integral or crack closure integral method), (Antunes et al 1999 and Guinea et al 2000). Several algorithms for the determination of stress intensity factors and verification of crack propagation were applied and tested by various convergence criteria by Owen and Fawkes (1983); Raju and Shivakumar (1990); Woo et al (1998); Araújo et al (2000); Santos et al (2000); Santos and Carvalho (2001) and Kim and Paulino (2002). In this work was studied the J-integral method to determine the stress intensity factors.

Two different approaches were implemented in the post-processor of a commercial FEM code (ANSYS®). The first approach is based in the methodology developed by Owen and Fawkes (1983) that used a contour path around the crack tip and the contribution to the J-integral from a individual element was evaluated in the gaussian points for $\xi = \xi_p$ or $\eta = \eta_p$. The second approach is based in the methodology proposed by Raju and Shivakumar (1990) where line J-integrals were converted to equivalent area or domain integrals by the divergence theorem that is called the *Equivalent Domain Integral – EDI*. The results are presented for a closed-form solution problem under mode I of fracture conditions. The precision of the methodologies is discussed and analyzed. Different geometry configurations and mesh refinement were tested to verify the applicability of the developed routines.

2. J-INTEGRAL

As proposed by Rice (1968), the J-integral is defined by familiar expression:

$$J = \int_{\Gamma} [W n_1 - T_i \frac{du_i}{dx_1}] ds \quad \textbf{(1)}$$

where $W = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij}$ is the total strain energy density, $T_i = \sigma_{ij} n_j$ are the components of the traction vector, n_j are the components of the unit vector normal to the contour Γ, u_i are the displacement vector components and ds is a length increment along the contour Γ, Fig. (1).

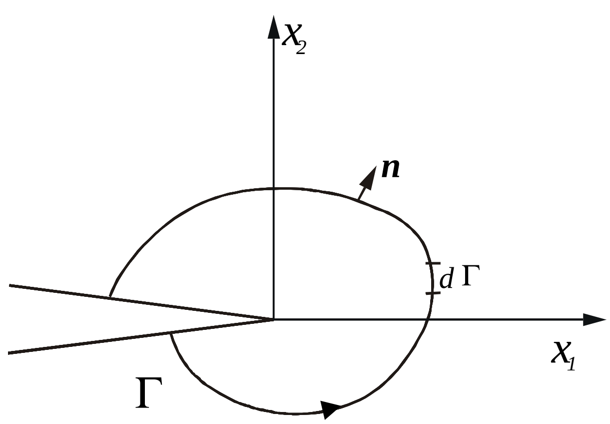

Figure 1. Contour around the crack tip

In the case of a homogeneous, isotropic, linear elastic material surrounding the crack tip, the relation among the component of the J-integral and mode I stress intensity factors is established as

$$J = \frac{k+1}{8\mu} K_I^2 \quad \textbf{(2)}$$

where $\mu = E/2(1+\nu)$ is the shear modulus; $\kappa = 3-4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for plane stress; ν is the Poisson's ratio and E is the modulus of elasticity.

Line Integral

How described by Owen and Fawkes (1983), since the integral is path-independent, the path can be conveniently chosen to coincide with the line $\xi = \xi_p =$ constant, shown in Fig. (2).

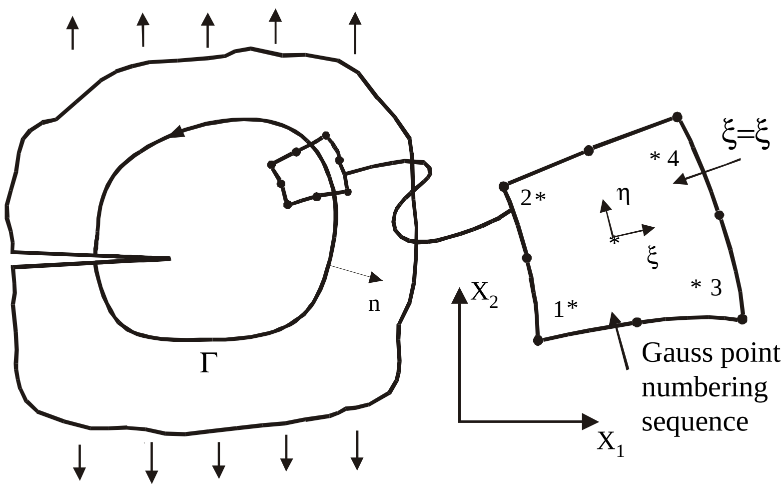

Figure 2. Contour path for J-integral evaluation, (Owen and Fawkes, 1983)

The contribution to the J-integral from an individual element is defined by:

$$J_{line} = \int_{-1}^{+1} \left\{ \frac{1}{2}(\sigma_{11}\frac{\partial u_1}{\partial x_1} + \sigma_{12}(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}) + \sigma_{22}\frac{\partial u_2}{\partial x_2})\frac{\partial u_2}{\partial \eta} + \right.$$
$$\left. - \left[(\sigma_{11}n_1 + \sigma_{12}n_2)\frac{\partial u_1}{\partial x_1} + (\sigma_{12}n_1 + \sigma_{22}n_2)\frac{\partial u_2}{\partial x_1} \right] \sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2} \right\} d\eta \tag{3}$$

The integration in Eq. (3) must be undertaken numerically. In particular

$$J_{line} = \sum_{q=1}^{NGAUS} I(\xi_p, \eta_q) W_q \tag{4}$$

in which the integrand *I* is evaluated at the Gaussian sampling points ξ_p, η_q and W_q is the weighting factor corresponding to η_q. The Cartesian derivatives of the displacement components required in Eq. (4) are given by

$$\frac{\partial u_i}{\partial x_j} = \sum_{q=1}^{n} \frac{\partial N_q}{\partial x_1}(u_i)_q \tag{5}$$

in which u_i are the displacements of the nodes of the elements and the cartesian derivatives of the elements shape functions are given by

$$\frac{\partial N_q^{(e)}}{\partial x_j} = \frac{\partial N_q^{(e)}}{\partial \xi}\frac{\partial \xi}{\partial x_j} + \frac{\partial N_q^{(e)}}{\partial \eta}\frac{\partial \eta}{\partial x_j} \quad (j = 1,2) \tag{6}$$

in which the terms $\partial\xi/\partial x_j$, $\partial\eta/\partial x_j$ may be obtained from the inverse of the jacobian matrix given by

$$[Jac]^{-1} = \begin{bmatrix} \frac{\partial \xi}{\partial x_1} & \frac{\partial \eta}{\partial x_1} \\ \frac{\partial \xi}{\partial x_2} & \frac{\partial \eta}{\partial x_2} \end{bmatrix} = \frac{1}{Det[Jac]}\begin{bmatrix} \frac{\partial x_2}{\partial \eta} & -\frac{\partial x_2}{\partial \xi} \\ -\frac{\partial x_1}{\partial \eta} & \frac{\partial x_1}{\partial \xi} \end{bmatrix} \tag{7}$$

The total value of J-integral is given by summing the contribution of all elements forming the integral path and the stress intensity factors are finally given by Eq. (3). Since the contour integral is accumulated from the paths $\xi = \xi_p$ = constant through neighbouring elements as shown in Fig. (2), this places restrictions on both the finite element mesh and the order of nodal numbering of elements through which the integral path is to pass.

Equivalent Domain Integral

As described by Raju and Shivakumar (1990), using the divergence theorem which can convert the line integrals into an area or a domain integral. Consider two contours Γ_0(OABCO) and Γ_1(ODEFO) around the crack tip as shown in Fig. (3). The two contours will enclose an area DEFCBAD.

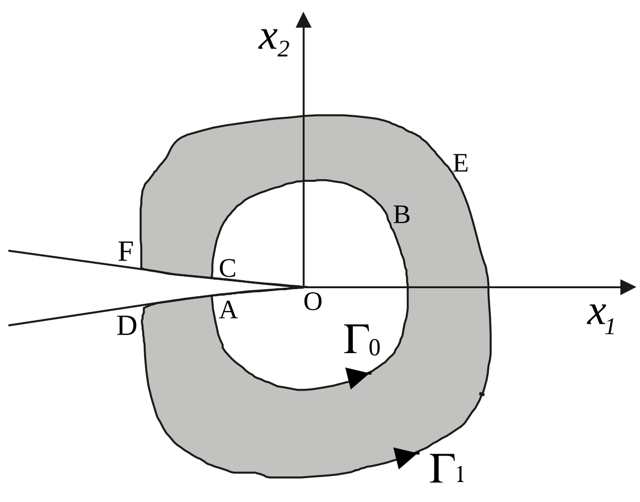

Figure 3. Contours around the crack tip, (Raju and Shivakumar, 1990)

By multiplying the integral over Γ_0 by unity an the integral over Γ_1 by zero, J can be expressed as

$$J = 1\int_{\Gamma_0}[Wn_1 - T_i \frac{du_i}{dx_1}]d\Gamma - 0\int_{\Gamma_1}[Wn_1 - T_i \frac{du_i}{dx_1}]d\Gamma \tag{8}$$

This manipulation is performed to convert the line integrals into area or a domain integral. In Eq. (8), this can be expanded as, Raju and Shivakumar, (1990)

$$J = 1\int_{ABC} bd\Gamma - 0\int_{DEF} bd\Gamma + 1\int_{CO} bd\Gamma + 1\int_{OA} bd\Gamma \tag{9}$$

where $b = Wn_1 - T_i \frac{du_i}{dx_1}$

$$J = \left[-\int_{CBA} bqd\Gamma - \int_{DEF} bqd\Gamma\right] + \int_{CO} bd\Gamma + \int_{OA} bd\Gamma \tag{10}$$

To interpolate two contours between Γ_0 and Γ_1 shown in Fig. (4), the continuous q-function, denoted by $q(x_1, x_2)$, is introduced with the property of

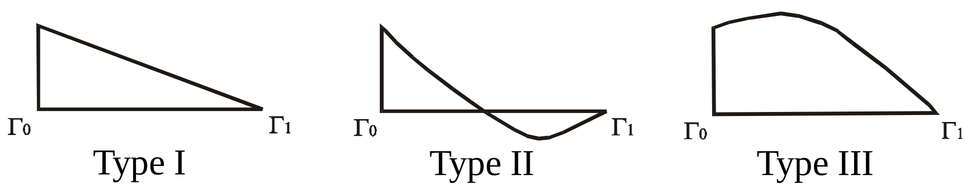

Figure 4. Example of q-functions

It can be that $q(x_1, x_2) = 1$ on the Γ_{ABC} and $q(x_1, x_2) = 0$ on the Γ_{DEF}. Invoking the divergence theorem, the closed line contour integral can be converted to a domain integral as:

$$-\int_{DEFCBAD} bqd\Gamma = -\int_{DEFCBAD} [Wn_1 - \sigma_{ij} \frac{\partial u_i}{\partial x_1} n_j]qd\Gamma = -\int_A [\frac{\partial (Wq}{\partial x_1} - \frac{\partial (\sigma_{ij}(\partial u_i / \partial x_1)q}{\partial x_j}]dA \tag{11}$$

$$J_{domain} = -\int_A [W\frac{\partial q}{\partial x_1} - \sigma_{ij}\frac{\partial u_i}{\partial x_1}\frac{\partial q}{\partial x_j}]dA - \int_A [\frac{\partial W}{\partial x_i} - \sigma_{ij}\frac{\partial \varepsilon_{ij}}{\partial x_1}]qdA \tag{12}$$

It can be notice that the second term of the above integral expression vanishes for elastic problems. The domain integral Eq. (12) can be rewritten in a form convenient to finite element computations as

$$J_{domain} = \int_{-1}^{+1}\int_{-1}^{+1} [W\frac{\partial q}{\partial x_1} - \sigma_{ij}\frac{\partial u_i}{\partial x_1}\frac{\partial q}{\partial x_2}]Det[Jac]d\xi d\eta \tag{13}$$

Once the q-function, denoted by $q(\xi, \eta)$, is defined, as shown in Fig. (4) the partial derivatives of q, can be easily computed by jacobian matrix as

$$\begin{Bmatrix} \dfrac{\partial q}{\partial x_1} \\ \dfrac{\partial q}{\partial x_2} \end{Bmatrix} = [Jac]^{-1} \begin{Bmatrix} \dfrac{\partial q}{\partial \xi} \\ \dfrac{\partial q}{\partial \eta} \end{Bmatrix} \tag{14}$$

Therefore the final form of the domain integral J can be shown using gaussian quadrature as

$$J_{domain} = -\sum_{p=1}^{NGAUS} \sum_{q=1}^{NGAUS} [[W \frac{\partial q}{\partial x_1} - \sigma_{ij} \frac{\partial u_i}{\partial x_1} \frac{\partial q}{\partial x_2}] Det[Jac] W_q W_p \tag{15}$$

where NGAUS is the number of Gauss point used in each direction points ξ_p, η_q and W_p and W_q are the weighting factors, and $Det[Jac]$ is the determinant of the jacobian matrix [*Jac*]. In this study, 2 x 2 gauss points were used for computation both line integral J_{line} and domain integral J_{domain} Equations 4 and 13, respectively.

3. NUMERICAL RESULTS AND DISCUSSIONS

The above procedures for evaluating the stress intensity factor K_I (equations 4 and 13), were applied to the well known geometry shown in Fig. (5), a center cracked plate tension specimen (CCT). The geometry was analyzed in generalized plane stress, width (2W=0.2 m) and crack depth ratios (a/W=0.5 and a/W=0.1). The material parameters were the Young's module (E=30 GPa) and Poisson's ratio (ν = 0.3). The stress applied was σ=100 MPa. The mesh was generated using eight-node isoparametric elements and the crack tip was modeled using the singular element (quarter point), (Ansys, 1995).

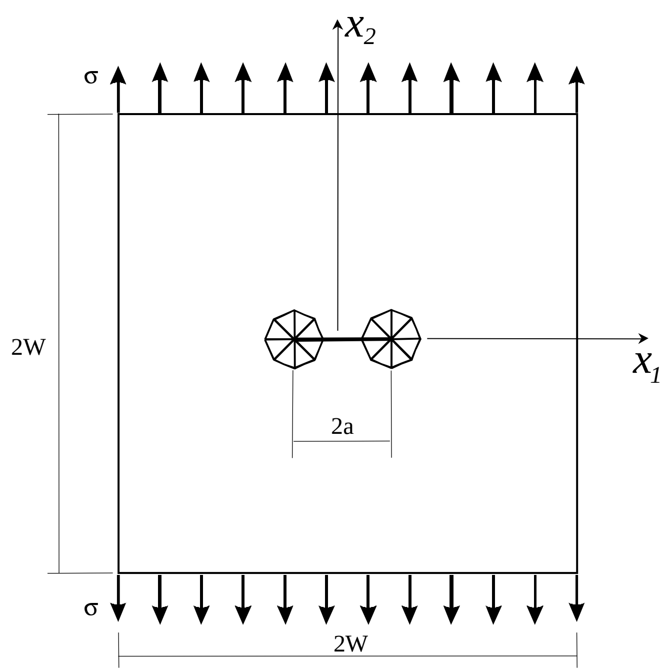

Figure 5. Center cracked plate tension specimen (CCT) geometry.

The values of K_I were confronted with the values calculated by a polynomial expression (Anderson, 1995)

$$K_I = \sigma\sqrt{\pi a}\left[\sec\left(\frac{\pi a}{2W}\right)^{1/2}\right] \quad \textbf{(16)}$$

Different levels of mesh refinement were used considering the number of element around the crack tip (N) and the length of the singular element (L). The length of element regular (b) was considered the same length of singular element (b = L). In the Fig. (7), shown the examples of mesh generated in the FEM code used.

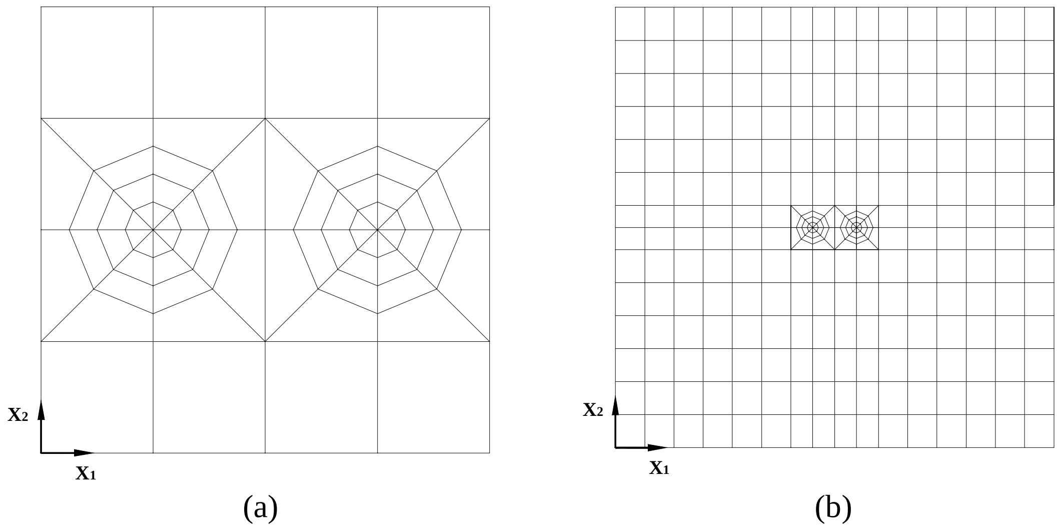

Figure 6. Mesh generated, with N=8 elements and a/L = 4. (a) a/W=0.5 (b) a/W=0.1.

The J-integral was evaluated along elements near the crack tip shown in Fig (7). The elements (domain) were used to evaluate the integrals (line integral and domain integral). The paths employed in solution to evaluate line integral passed through the two integration contours corresponding $\xi = \xi_1$ and $\xi = \xi_2$ of domain shown giving a total of two contours path. To compute the domain integral the *q*-function used was the type I being a simple linear function. This *q*-function was created by defining the values of *q* at nodes on the elements as discussed by Raju and Shivakumar (1990).

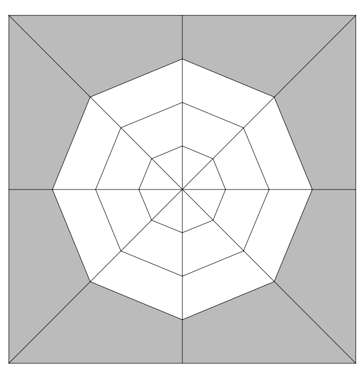

Figure 7. Elements selected to establish the path used in the J-integral calculation

The results, considering two crack depth ratio for the mode I of fracture, are shown in the Tabs. (1) and (2). In general can be observed that the results obtained through different methods are consistent and the accuracy on K_I was good for all the methods, well under 6% in module for almost all cases.

Table 1. Values of K_I ($MPa\sqrt{m}$) to a/W=0.5, ($K_{Itheoretical}$ = 47.132 $MPa\sqrt{m}$).

N	a/L	K_I (Line Integral ξ_1)	K_I (Line Integral ξ_2)	K_I (EDI)
8	4	49.489	49.561	49.484
	8	49.440	49.479	49.437
	16	49.503	49.543	49.499
	32	49.466	49.522	49.462
16	4	49.466	49.508	49.463
	8	49.478	49.511	49.475
	16	49.490	49.523	49.488
	32	49.584	49.636	49.580

Table 2. Values of K_I ($MPa\sqrt{m}$) to a/W=0.1, ($K_{Itheoretical}$ = 17.941 $MPa\sqrt{m}$).

N	a/L	K_I (Line Integral ξ_1)	K_I (Line Integral ξ_2)	K_I (EDI)
8	4	17.871	17.864	17.871
	8	17.817	17.849	17.814
	16	18.146	18.186	18.143
16	4	17.884	17.896	17.883
	8	18.195	18.221	18.193
	16	17.900	17.917	17.899

In the Fig. (9) and Fig. (10) are shown the percent difference from the results obtained to the stress intensity factor for the mode I of fracture. In these figures the effects of the size of singular element are illustrated. The figures show the results, with crack length element sizes ratio ranging from 4 to 32. A local refinement used in this work improve accurate estimation of K_I.

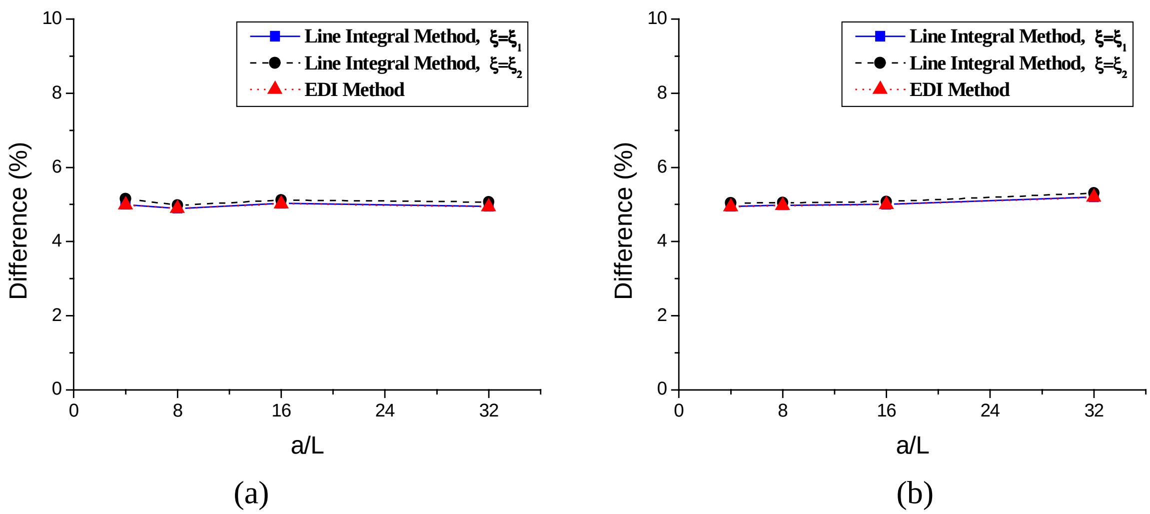

Figure 9. Difference in the values of stress intensity factor a/W=0.5. (a) N=8; (b) N=16

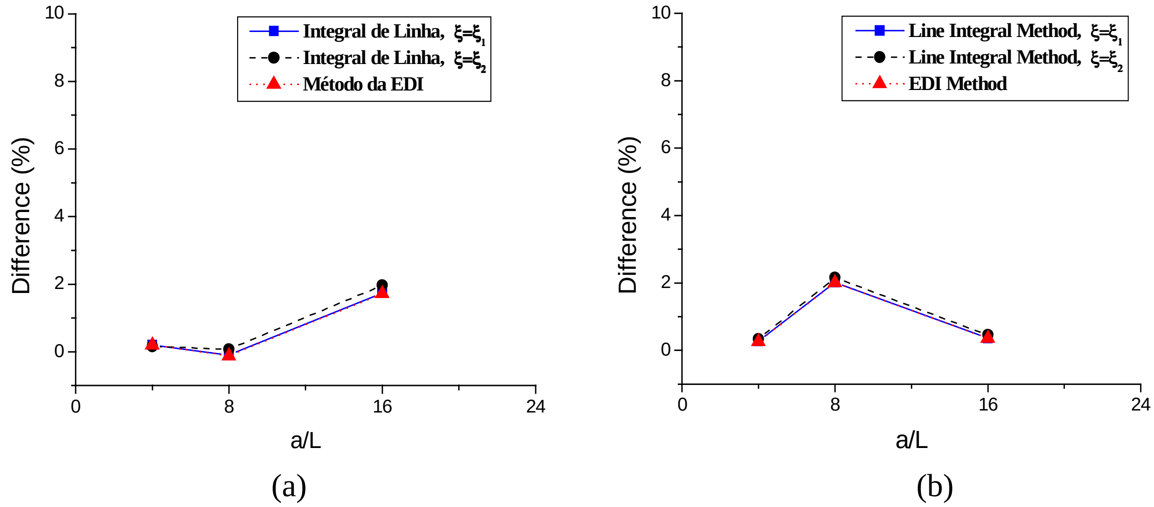

Figure 10. Difference in the values of stress intensity factor a/W=0.1. (a) N=8; (b) N=16

The observed accuracy in K_I values trough the methodologies studied are found to be consistent with the calculations by analytical expression and it's in agreement with the results obtained in the literature.

Using the J-integral methods by approach line integral and Equivalent Domain Integral – EDI the results were not affected by the mesh refinement in the cases using crack depth ratio a/W=0.5.

However it can be that results are consistent with the relation suggested by Raju (1987) and Araújo et al (2000) (a/L= 16 and a/W=0.1) when the maximum errors encountered was 1.8 % for K_I. It can be notice that results encountered in this wok was 2.2 % for the same geometry.

Can be observed that the number of elements around the crack tip did not influence results calculated. However the results are compatible with the obtained by Araújo et al (2000) when studied the influence of rosette type that involves the crack tip. They used a rosette with N=8 and N=12 elements to single edge crack tension specimen geometry and presented results converge, slowly, to the value of reference with the increase of number elements at the crack tip.

4. CONCLUSIONS

Methodologies based on the J-integral methods were used in this work very successfully. It was shown that the results obtained through different methods are consistent and the level of accuracy obtained were not dependent of mesh refinement used in the model. The observed inaccuracies stress intensity factors trough to crack depth ratio a/W=0.1 is found to be consistent with those from literature calculations considering the relation proposed by Raju (1987) a/L=16.

In general the values of the stress intensity factors did not influenced by the number of the singular elements around the crack tip.

Accurate results were obtained by the methodologies using the concept line integral and the concept of Equivalent Domain Integral. In these techniques very accurate results were obtained for the coarse meshes.

In addition, it was shown that the use of a commercial finite element package made possible to the stress intensity factor calculation directly from the post processor page. This was realized, because the software has a scripting language (Ansys Parametric Design Language - APDL) used to automate the analysis (Ansys, 1998).

5. ACKNOWLEDGEMENTS

The authors would like to thank the CNPq (Research Brazilian Agency) for the scholarship, and the authors would also like to thank the CAD/CAE Laboratory (EESC/USP) for the use of the license ANSYS® software.

6. REFERENCES

Anderson, T. L., 1995, "Fracture Mechanics – Fundamentals and Applications", CRC Press Inc., Boca Raton, EUA, 688 p.

ANSYS APDL Programmer´s Guide, 1998, SAS IP, Sep.

ANSYS User´s Manual, 1995, Procedures Volume. I, SAS IP, Aug.

Antunes, F. V., Ferreira, J. M. and Byrne, J., 1999, "Stress intensity factor calculation based on the work of external forces", International Journal of Fracture, Vol. 98, pp. 1-14.

Araújo, T. D., Bittencourt, T. N., Roehl, D. and Martha, L. F., 2000, "Numerical Estimation of Fracture Parameters Elastic and Elastic-Plastic Analysis", Proceedings of European Congress on Computational Methods in Applied Sciences and Engineering – ECCOMAS 2000, Barcelona, 11-14 Sep., Spain.

Barsoum, R., 1976, "On the Use of Isoparametric Elements in Linear Fracture Mechanics", International Journal for Numerical Methods in Engineering, Vol. 10, pp. 25-37.

Erdogan, F., 2000, "Fracture Mechanics", International Journal Solids and Structures", Vol. 37, pp. 171-183.

Guinea, G. V., Planas, J. and Elices M., 2000, "K_I Evaluation by the Displacement Extrapolation Technique", Engineering Fracture Mechanics, Vol. 66, pp. 243-255.

Henshel, R. D. and Shaw, K. G., 1975, " Crack Tip Finite Elements are Unnecessary", International Journal for Numerical Methods in Engineering", Vol. 9, pp. 495-507.

Kim, J. H. and Paulino, G. H., 2002, "Finite Element Evaluation of Mixed Mode Stress Intensity Factors in Functionally Graded Materials", International Journal for Numerical Methods in Engineering, Vol. 53, No. 8, pp. 1903-1935, Mar.

Owen, D. R. J. and Fawkes, A. J., 1983, " Engineering Fracture Mechanics: Numerical Methods and Applications", Pineridge Press Ltd., Swansea, U. K., 305 p.

Raju, I. S. and Shivakumar, K. N., 1990, "An Equivalent Domain Integral Method in the Two-Dimensional Analysis of Mixed Mode Crack Problems", Vol. 37, No. 4, pp. 707-725.

Raju, I. S., 1987, "Calculation of Strain-Energy Release Rates with Higher Order and Singular Finite Elements", Engineering Fracture Mechanics, Vol. 28, No. 3, pp. 251-274.

Rice, J. R., 1968, "A Path Independent Integral and the Approximate Analysis of Strain Concentration by Notches and Cracks", Journal of Applied Mechanics, Vol. 35, No. 2, pp. 379-386, Jun.

Santos, N. C. and de Carvalho, J., 2001, "Computational Methods for Stress Intensity Factors Determination", Proceedings of XVI Brazilian Congress of Mechanical Engineering - COBEM 2001, Mechanics and Structural Integrity, Vol. 12, pp. 61-70, 7-11 Nov., Uberlândia, Brazil.

Santos, N. C., de Carvalho, J. and Lirani, J., 2000, "Determination of the stress intensity factor by the finite element method", National Congress of Mechanical Engineering, 2000, 7-11 Ago., Natal, Brazil, in Portuguese.

Woo, K. S., Hong, C. H. and Shin, Y. S., 1998, "An Extended Equivalent Domain Integral Method for Mixed Mode Fracture Problems by the *p*-Version of FEM", International Journal for Numerical Methods in Engineering", Vol. 42, No. 5, pp. 857-884.

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