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A SIMPLE MULTIAXIAL FATIGUE CRITERION FOR HARD METALS

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Abstract. *The goal of the present paper is to propose a simple criterion to predict the fatigue strength under conditions of multiaxial, nonproportional loadings. The study reported here is focused on situations where the convex hull containing the stress path can be approximately represented by an ellipsoid, which is the case in a wide range of situations. Proportional and nonproportional, in-phase and out-of-phase cycling loads applied on a number of distinct hard metals are considered for the assessment of the model.*

Keywords: *Fatigue strength, equivalent shear stress, multiaxial out-of-phase loads.*

1.INTRODUCTION

Fatigue of engineering materials has been a major subject of study in mechanical sciences since the railway accident in 1842, near Versailles in France, caused by failure of the locomotive front axle. Several multiaxial fatigue criteria — including those proposed by Crossland (1956) and Bin Li et al. (2000), amongst others — can be written as:

$$\tau_{eq} + \kappa\, p_{max} \leq \lambda, \tag{1}$$

where τ_{eq} is the *equivalent shear stress amplitude*, p_{max} is the *maximum value of the hydrostatic stress* observed along the stress history, while κ and λ are material parameters. The basic difference among such criteria is concerned with the definition of the equivalent shear stress amplitude. In Crossland criterion, the equivalent shear stress is essentially the radius of the minimum hypersphere circumscrybing the stress path in the deviatoric stress space. It gives very good predictions of fatigue strength under in-phase loading conditions, when assessed in confrontation with experimental results reported in the literature The criterion proposed by Bin Li et al. (2000) , on the other hand, performs very well under both in-phase and out-of-phase conditions and computes τ_{eq} as:

$$\tau_{eq} := \sqrt{\sum_i R_i^2}. \tag{2}$$

where R_i, are the principal axes of the minimum ellipsoid circunscribing the stress path in the deviatoric stress space. Both criteria present as a drawback the need for quite elaborated optimization algorithms in order to determine the required variables.

In this paper, we propose a new and very simple expression for the equivalent shear stress amplitude τ_{eq} which, when incorporated into inequality (1), defines a multiaxial fatigue criterion for hard metals, very simple to compute, but still providing very good results for a wide range of in-phase and out-of-phase loading cases.

2. THE EQUIVALENT SHEAR STRESS AMPLITUDE

High cycle fatigue degradation takes place if, at mesoscopic level, the material point reaches a state of plastic shakedown, leading to the formation of persistent slip bands, even if the material shows an elastic behaviour at macroscopic level. If, on the other hand, the material point attains a state of elastic shakedown at mesoscopic level, then fatigue failure does not occur. This interpretation of fatigue failure is in agreement with experimental observations reported by Sines and Ohgi (1981), for instance, in which it was shown that superimposed static shear stresses do not influence the fatigue limit of metallic materials. In this setting, we assume that one of the variables governing the fatigue phenomenon is the *microscopic deviatoric stress tensor* $\mathbf{X}$, defined as:

$$\mathbf{X} := \mathbf{S} - dev(\boldsymbol{\rho}), \tag{3}$$

where $\mathbf{S}$ is the *deviatoric stress tensor*, $\boldsymbol{\rho}$ is the *stabilized residual stress tensor* after the state of shakedown is achieved and $dev(\boldsymbol{\rho}) := \boldsymbol{\rho} - \frac{1}{3}(tr\,\boldsymbol{\rho})\,\mathbf{I}$ accounts for the deviatoric part of $\boldsymbol{\rho}$.

If Dev^3 denotes the space of symmetric deviatoric tensors from $\mathbb{R}^3$ to $\mathbb{R}^3$ and $\{\mathbf{N}_i,\, i = 1, \ldots, 5\}$ is an arbitrarily chosen orthonormal basis for such space, then each microscopic deviatoric stress state $\mathbf{X}(t)$ can be written as:

$$\mathbf{X}(t) = \sum_{i-1}^{5} x_i(t)\,\mathbf{N}_i. \tag{4}$$

The components $x_i(t)$ of $\mathbf{X}(t)$ in the basis of Dev^3 given, for instance, by:

$$\begin{gathered}
\mathbf{N}_1 = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{-1}{\sqrt{6}} & 0 \\ 0 & 0 & \frac{-1}{\sqrt{6}} \end{pmatrix}, \quad \mathbf{N}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix}, \\
\mathbf{N}_3 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{N}_4 = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}, \quad \mathbf{N}_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix},
\end{gathered} \tag{5}$$

are:

$$\begin{gathered}
x_1(t) = \sqrt{\frac{3}{2}}\,X_{xx}(t), \quad x_2(t) = \frac{1}{\sqrt{2}}\,(X_{yy}(t) - X_{yy}(t))\,, \\
x_3(t) = \sqrt{2}\,X_{xy}(t), \quad x_4(t) = \sqrt{2}\,X_{xz}(t), \quad x_5(t) = \sqrt{2}\,X_{yz}(t);
\end{gathered} \tag{6}$$

From (4), it is possible to describe the stress path in terms of a curve in $\mathbb{R}^5$, where each point $\mathbf{x}(t) \in \mathbb{R}^5$ can be expressed as:

$$\mathbf{x}(t) := (x_1(t)\ x_2(t)\ \ldots x_5(t))^T\,. \tag{7}$$

The set of all points $\mathbf{x}(t)$ describing the path of microscopic deviatoric stresses in $\mathbb{R}^5$ is represented by the symbol $\boldsymbol{\Delta}$.

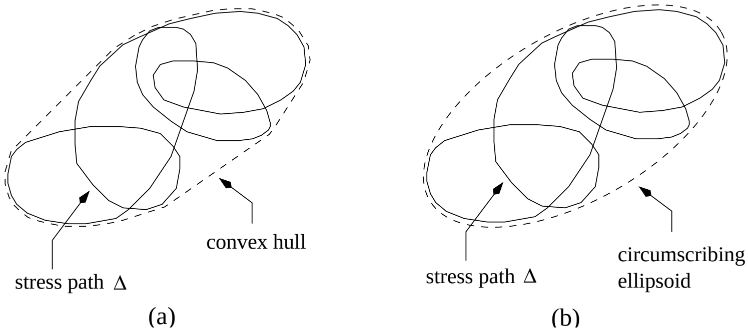

Figure 1. Characterization of the mechanical solicitation to fatigue: (a) convex hull, (b) circumscribing ellipsoid.

We assume here that only states $\mathbf{x}(t)$ defined upon the convex hull of $\mathbf{\Delta}$ must be considered for the characterization of the solicitation to fatigue (Fig. 1.a). In general, however, such convex hulls are difficult to determine. In other to overcome this difficulty, several approximations have been proposed in the literature. Crossland, for instance, considered the minimum circumscribing hypersphere as an approximation of the convex hull. Bin Li et al. (2000), on the other hand, proposed the minimum circumscribed ellipsoid as a measure of the fatigue damage. The use of ellipsoids as a measure of fatigue failure can provide satisfactory results in cases where the ellipsoid is a good approximation for the convex hull of $\mathbf{\Delta}$ (Fig. 1.b). In this case, τ_{eq} can be written as:

$$\tau_{eq} := \sqrt{\sum_{i=1}^{5} \lambda_i^2}, \tag{8}$$

where $\lambda_i, i = 1, \ldots, 5$ are the magnitudes of the principal semi-axes of the circumscribing ellipsoid, which has to be chosen so as to provide the minimum value of τ_{eq}. The drawback of the definition (8) comes from the fact that the ellipsoid itself and hence its semi-axes are difficult to determine. The next proposition enables an almost trivial calculation of τ_{eq}.

Proposition. *Given an ellipsoid $\mathcal{E}$ in $\mathbb{R}^m$ with centre located at the origin and an arbitrary orthonormal basis $\{\mathbf{n}_i,\ i = 1, \ldots, m\}$ of $\mathbb{R}^m$, let $\mathcal{P}$ be a rectangular prism circumscribing $\mathcal{E}$ such that its faces are orthogonal to each one of the basis elements. If λ_i, $i = 1, \ldots, m$ are the magnitudes of the principal semi-axes of $\mathcal{E}$ and a_i, $i = 1, \ldots, m$ denote the distances of the centre of the ellipsoid to the faces of the rectangular prism, then:*

$$\sum_{i=1}^{5} \lambda_i^2 = \sum_{i=1}^{5} a_i^2. \tag{9}$$

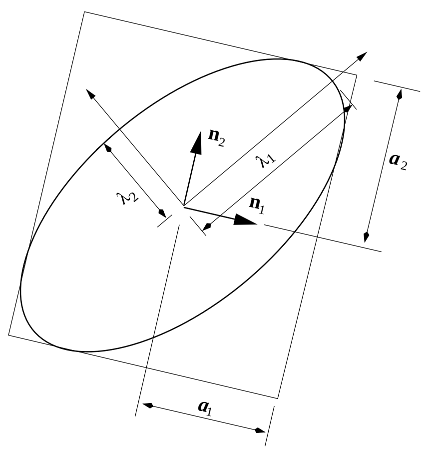

Figure 2. Ellipsoid in $\mathbb{R}^5$ and its arbitrarily oriented circumscribing rectangular prism.

Proof. Let b_1 be the unit ball in $\mathbb{R}^m$:

$$b_1 := \{\mathbf{y} \in \mathbb{R}^m;\ ||\mathbf{y}|| = 1\}, \tag{10}$$

where $||\mathbf{y}|| := \left(y_1^2 + y_2^2 + \ldots + y_m^2\right)^{1/2}$ is the classical Euclidean norm in $\mathbb{R}^m$. The ellipsoid $\mathcal{E}$ can be characterized as the set of points:

$$\mathbf{x} \in \mathbb{R}^m;\quad \mathbf{x} = \mathbf{L}\,\mathbf{y},\ \mathbf{y} \in b_1, \tag{11}$$

where $\mathbf{L} : \mathbb{R}^m \to \mathbb{R}^m$ is a symmetric, positive semi-definite matrix with eigenvalues given by the magnitudes λ_i, $i = 1, \ldots, m$ of the semi-axes of $\mathcal{E}$. On the other hand, the faces of the rectangular prism, orthogonal to a basis element $\mathbf{n}_i$ and located at distances a_i from the centre of the ellipsoid, can be characterized as the set of points:

$$\mathbf{x} \in \mathbb{R}^m;\quad \mathbf{x} \cdot \mathbf{n}_i = a_i. \tag{12}$$

Substitution of $\mathbf{x}$ from (11) into (12) gives:

$$\mathbf{L}\,\mathbf{y} \cdot \mathbf{n}_i = a_i, \tag{13}$$

or, equivalently:

$$\mathbf{y} \cdot \mathbf{L}\,\mathbf{n}_i = a_i, \tag{14}$$

The set of points $\mathbf{x} = \mathbf{L}\mathbf{y}$ satisfying (13) or $\mathbf{y}$ satisfying (14) are illustrated in Figs. 3.a and 3.b, respectively.

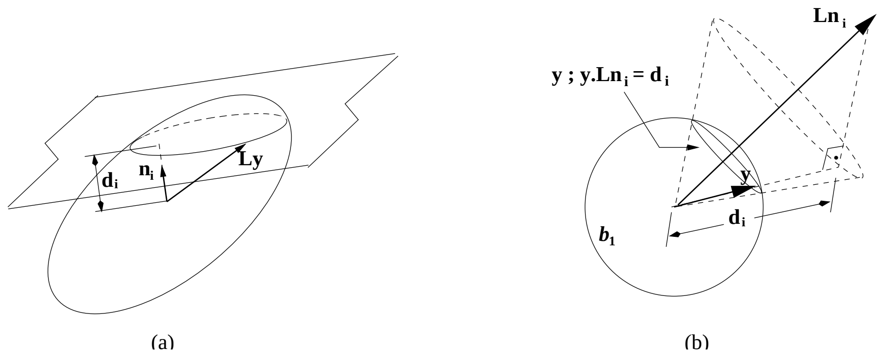

Figure 3. (a) Points $\mathbf{L}\,\mathbf{y}$ of the intersection between the ellipsoid and the hyperplane orthogonal to $\mathbf{n}_i$ satisfying relation (13); (b) Points $\mathbf{y} \in b_1$ satisfying relation (14).

The set of points $\mathbf{y} \in b_1$ satisfying (14) is unitary (meaning that the hyperplane defined in (12) is tangent to the ellipsoid $\mathcal{E}$) if $\mathbf{y}$ is parallel to $\mathbf{L}\,\mathbf{n}_i$, i.e.:

$$\mathbf{L}\,\mathbf{n}_i = ||\mathbf{L}\,\mathbf{n}_i||\,\mathbf{y}, \tag{15}$$

which, together with (14), implies that the distance of the hyperplane to the centre of the ellipsoid must be:

$$a_i = ||\mathbf{L}\,\mathbf{n}_i||. \tag{16}$$

From (16), we can write:

$$\sum_{i=1}^{m} ||\mathbf{L}\,\mathbf{n}_i||^2 = \sum_{i=1}^{m} a_i^2. \tag{17}$$

Now, it can be shown (see Mamiya and Araújo (2002) for details) that the Frobenius norm of a matrix $\mathbf{L}$:

$$||\mathbf{L}||_F := \left(\sum_{i,j=1}^{m} \mathbf{L}_{ij}^2 \right)^{1/2} = \left(\sum_{i=1}^{m} \lambda_i^2 \right)^{1/2} \tag{18}$$

can be expressed alternatively as:

$$||\mathbf{L}||_F = \left(\sum_{i=1}^{m} ||\mathbf{L}\,\mathbf{n}_i||^2 \right)^{1/2} , \tag{19}$$

which, together with (17), provides the result (9). □

The aforementioned statement is of fundamental importance for the computation of τ_{eq} since it precludes the need to determine the principal semi-axes of the ellipsoid. More specifically, whenever the ellipsoid is a good approximation for the convex hull of the stress path $\boldsymbol{\Delta}$, instead of considering (8), the equivalent shear stress amplitude τ_{eq} can be simply computed as:

$$\tau_{eq} := \left(\sum_{i=1}^{5} a_i^2 \right)^{1/2} , \tag{20}$$

where, in the context of the present study, a_i, $i = 1, \ldots, 5$ are the *amplitudes of the components* $x_i(t)$ *of the microscopic deviatoric stresses* defined as:

$$a_i := \max_t |x_i(t)|, \; i = 1, \ldots, 5. \tag{21}$$

The procedure for computation of τ_{eq} can be summarized as follows:

- For each time instant t, compute the Cauchy stress tensor $\boldsymbol{\sigma}(t)$, its corresponding deviatoric stress states:

$$\mathbf{S}(t) = \boldsymbol{\sigma}(t) - \frac{1}{3}(tr\,\boldsymbol{\sigma}(t))\,\mathbf{I}; \tag{22}$$

  and its components in terms of an arbitrarily chosen orthonormal basis $\mathbf{N}_i$, $i = 1, \ldots, 5$:

$$\mathbf{S}(t) = \sum_{i=1}^{5} s_i(t)\,\mathbf{N}_i; \tag{23}$$

- Compute the amplitudes of the microscopic deviatoric stresses a_i, $i = 1, \ldots, 5$ as:

$$a_i := \frac{1}{2}\left(\max_t s_i(t) - \min_t s_i(t) \right), \; i = 1, \ldots, 5. \tag{24}$$

- Compute the equivalent shear stress amplitude τ_{eq} as:

$$\tau_{eq} := \left(\sum_{i=1}^{5} a_i^2 \right)^{1/2} . \tag{25}$$

It should be remarked that, as one of the consequences of the proposition, we do not have to determine the microscopic residual stress $dev(\rho)$:

$$dev(\boldsymbol{\rho})_i := \frac{1}{2}\left(\max_t s_i(t) + \min_t s_i(t)\right), \ i = 1, \ldots, 5. \tag{26}$$

during the computation of the equivalent shear stress amplitude.

3. ASSESSMENT OF THE CRITERION

Assessment of the proposed criterion in predicting fatigue strength under a high number of cycles was carried out by considering proportional and out-of-phase multiaxial fatigue experiments for a number of different materials. The data collected are reported in Tables 1 to 3 and correspond to experiments on hard metals ($1,3 \leq f_{-1}/t_{-1} < \sqrt{3}$) involving biaxial and triaxial stress states, where f_{-1} and t_{-1} are the fatigue limits under fully reversed bending and torsion, respectively. Biaxial data came from publications by Nishihara and Kawamoto (1945) (Table 1) and Zenner et al. (1985) (Table 2), while the triaxial tests (Table 5) were produced by Mielke (1980). The following nomenclature was adopted in these Tables: the subscript a stands for the amplitude of stresses, while m represents the mean value. As usual, σ and τ are normal and shear stresses, while β and γ contain information concerning phase angles. The stress values reported in each table correspond to the maximum combination of stresses that the specimen can stand without failing, up to a limit of 10^6 cycles.

To assess the quality of the results provided by our model, an error index I is defined as:

$$I = \frac{\tau_{eq} + \kappa\, p_{max} - \lambda}{\lambda} \times 100 \quad (\%), \tag{27}$$

which gives a measure of how close the prediction of the criterion is with respect to the experimental data. A negative I yields a non-conservative fatigue strength prediction. On the other hand, a positive I provides a conservative estimate while I=0 means a perfect prediction for the observed fatigue strength. Application of our model to the experimental data provided an error index which varied from -9.36% (Table 3) to 6.27% (Table 1) for all materials and loading conditions analysed. Similar results were obtained when Papadopoulos or Bin Li models were invoked. Papadopoulos mesoscopic criterion is based on the average measure of the accumulated plastic strain within an elementary volume. Calculation of I using Crossland criterion provided significantly poorer predictions. In this case I varied from -25.52% up to 1.45%.

4. DISCUSSION AND CONCLUSION

A new multiaxial fatigue criterion which is very simple to implement has been proposed. Application of this criterion to a broad range of in-phase and out-of-phase loading conditions involving five different materials under biaxial and triaxial states of stress yielded an error index which never exceeded values lower than -9.36%. This essentially means that, in the worst scenario, our model predicts the specimen would stand a load 9.36% greater than the experimentally observed failure load combination. This can be considered a very good prediction within the high cycle fatigue regime. Further, these results are far better than the ones provided by the classical Crossland criterion and are as good as the results provided by Papadopoulos and Bin Li criteria. Moreover, the implementation of our criterion is of greater simplicity. On the other hand, application of our criterion is restricted to cases where the shape of the convex hull circunscribing the microscopic loading path in Dev^3 approximates well from an ellipsoid. Although this is a clear limitation, in practice there is a wide range of loading cases which fall within these conditions, such as components under dynamic loadings caused by a single source of excitation.

Table 1 — Experimental and predicted results for hard steel ($t_{-1} = 196.2$ MPa $f_{-1} = 313.9$ MPa).

σ_a (MPa)	σ_m (MPa)	τ_a (MPa)	τ_m (MPa)	β (°)	I Crossland	I Bin Li et al.	I Papadopoulos	I Our model
178.1	0	167.1	0	0	-2.28	-2.28	-2.3	-2.28
140.4	0	169.9	0	30	-2.63	-0.64	-0.6	-0.81
145.7	0	176.3	0	60	-4.52	3.1	3.1	2.93
150.2	0	181.7	0	90	-3.74	6.26	6.3	6.27
245.3	0	122.6	0	0	1.45	1.45	1.5	1.44
249.7	0	124.8	0	30	-1.26	3.28	3.3	3.17
252.4	0	126.2	0	60	-8.35	4.39	4.4	4.30
258.0	0	129.0	0	90	-17.81	6.69	6.5	6.70
299.1	0	62.8	0	0	0.92	0.92	0.9	0.92
304.5	0	63.9	0	90	-2.99	2.73	2.7	2.74

Table 2 — Experimental and predicted results for 34Cr4 ($t_{-1} = 256$ MPa $f_{-1} = 410$ MPa).

σ_a (MPa)	σ_m (MPa)	τ_a (MPa)	τ_m (MPa)	β (°)	I Crossland	I Bin Li et al.	I Papadopoulos	I Our model
314.0	0	157.0	0	0	-0.51	-0.55	-0.6	-0.55
315.0	0	158.0	0	60	-12.3	-0.11	-0.1	-0.19
316.0	0	158.0	0	90	-22.7	0.07	0.1	0.08
315.0	0	158.0	0	120	-5.1	-0.11	-0.1	-0.19
224.0	0	224.0	0	90	-8.38	5.15	5.2	5.15
380.0	0	95.0	0	90	-7.32	0.37	0.4	0.37
316.0	0	158.0	158.0	0	0.54	0.08	0.1	0.08
314.0	0	157.0	157.0	60	-12.3	-0.56	-0.6	-0.64
315.0	0	158.0	158.0	90	-21.8	-0.15	-0.1	-0.11
279.0	279.0	140.0	0	0	-6.38	-6.38	-6.4	-6.38
284.0	284.0	142.0	0	90	-25.52	-4.88	-4.8	-4.83
212.0	212.0	212.0	0	90	-9.4	3.38	3.4	3.41

Table 3 — Experimental and predicted results for 25CrMo4 ($t_{-1} = 260$ MPa $f_{-1} = 398$ MPa).

σ_{xa} (MPa)	σ_{xm} (MPa)	σ_{ya} (MPa)	σ_{ym} (MPa)	β (°)	τ_a (MPa)	γ (°)	I Crossland	I Bin Li et al.	I Papadopoulos	I Our model
261.0	340.0	261.0	170.0	0	0	0	-9.36	-9.36	-9.0	-9.36
275.0	340.0	275.0	170.0	60	0	0	-16.26	8.72	8.0	8.54
240.0	340.0	240.0	170.0	90	0	0	-5.367	6.12	6.0	6.01
196.0	340.0	196.0	170.0	180	0	0	-1.9	-1.9	-1.8	-1.91
220.0	340.0	0	170.0	0	110.0	60	-18.5	-8.96	-8.8	-9.01
233.0	340.0	0	170.0	0	117.0	90	-23.33	-4.18	-4.0	-4.13
155.0	340.0	0	170.0	0	155.0	60	-12.89	-5.71	-5.7	-5.81
159.0	340.0	0	170.0	0	159.0	90	-14.35	-3.6	-3.5	-3.56

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