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THERMALLY DEVELOPING LAMINAR FLOW OF POWER-LAW NON-NEWTONIAN FLUIDS INSIDE RIGHT TRIANGULAR DUCTS

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Abstract. *Laminar forced convection of power-law non-Newtonian fluids inside ducts with arbitrarily shaped cross-sections is analytically studied by extending the Generalized Integral Transform Technique (GITT). The analysis is illustrated through consideration of right-angularly triangular duct subjected to constant wall temperature boundary condition. Reference results are established for quantities of practical interest (as dimensionless average temperature and Nusselt numbers) within the thermal entry region for different values of power-law behavior index and apex angles of the duct. Critical comparisons are also made with results available in the literature obtained through numerical and analytical-numerical approaches.*

Keywords: *Laminar triangular duct flow, Power-law non-Newtonian fluids, Laminar forced convection, Integral transform solution.*

1. INTRODUCTION

Analytical and numerical heat transfer solutions for thermally fully developed and thermally developing laminar flow inside non-circular ducts are of considerable interest, mainly, to the design of compact heat exchangers and several other low Reynolds number flow heat exchange devices, as pointed out in different articles and textbooks (Shah and London, 1978; Sundén and Faghri, 1998). Industrial applications in which processing of materials behaving as non-Newtonian fluids are those commonly encountered in the chemical, food processing and pharmaceutical industries (Shah and Focke, 1988) which undergo thermal processing in heat exchange equipment, and in these applications the power-law model can describe adequately the rheology of such fluids. In this context, the establishment of benchmark results through analytical-numerical solutions for power-law fluids inside non-circular ducts is desirable for reference purposes and validation of direct numerical schemes, and in addition, a survey of the literature reveals a limited amount of works about heat and fluid flow of non-Newtonian fluids in right-angularly triangular ducts is available, and most contributions deal with purely numerical or approximate approaches (Shah and London, 1978; Haji-Sheikh et al., 1983).

Particularly, the case of a non-circular duct constitutes an example of the difficulties associated with solving multidimensional convection problems, requiring costly numerical solutions limited to regions away from the inlet (longer ducts). The exact solution of such a problem, through classical analytical methods (Mikhailov and Ösizik, 1984) is inhibited due to the non-separable nature of the related eigenvalue problem. The present work aims at applying the so-called Generalized Integral Transform Technique (GITT) (Cotta, 1993) in order to avoid the difficulties associated with the non-separable eigenvalue problem, consequently to give an accurate and reliable analysis to allow for the solution of this formally transformable but non-separable problem, providing an efficient algorithm for numerical computations.

The problem considered is that of a right-angularly triangular duct subjected to a constant wall temperature to illustrate the powerfulness of this hybrid approach. An analysis of convergence is made and a set of benchmark results established for quantities of practical interest, such as dimensionless average temperature, local Nusselt numbers, within a wide range of the dimensionless axial coordinate, different power-law indices and apex angles. Comparisons are then critically performed with previously reported results (Shah and London, 1978) from direct numerical approaches, from Galerkin-type functions (Haji-Sheikh et al., 1983) and from hybrid analytical-numerical approach (Aparecido and Cotta, 1992) for both, fully developed and thermally developing regions.

2. ANALYSIS

Laminar flow of a non-Newtonian power-law fluid inside a right-angularly triangular duct of sides a and b, according to Fig. 1, is considered. The velocity profile is taken as fully developed and the duct walls are subjected to a constant temperature, so that the dimensionless energy equation for constant property flow, neglecting axial conduction and viscous dissipation, in thermally developing flow is written as:

$$U(X,Y)\frac{\partial\theta(X,Y,Z)}{\partial Z}=\frac{\partial^2\theta(X,Y,Z)}{\partial X^2}+\frac{\partial^2\theta(X,Y,Z)}{\partial Y^2},\ \text{in}\quad Z>0,\ 0<X<X_1(Y), 0<Y<\beta \qquad (1.a)$$

with inlet and boundary conditions given, respectively, as follows:

$$\theta(X,Y,0)=1,\qquad 0\le X\le X_1(Y),\qquad 0\le Y\le\beta \qquad (1.b)$$

$$\theta(0,Y,Z)=0;\quad \theta(X_1(Y),Y,Z)=0,\quad Z>0 \qquad (1.c,d)$$

$$\theta(X,0,Z)=0;\quad \theta(X,\beta,Z)=0,\quad Z>0 \qquad (1.e,f)$$

where in Eqs. (1) above the following dimensionless groups were employed:

$$\theta(X,Y,Z)=\frac{T(x,y,z)-T_w}{T_i-T_w};\quad X=\frac{x}{D_h};\quad Y=\frac{y}{D_h};\quad Z=\frac{z}{D_h Pe}; \qquad (2.a\text{-}d)$$

$$Pe=Re\,Pr=\frac{\rho cp}{k}u_m D_h;\quad U(X,Y)=\frac{u(x,y)}{u_m}; \qquad (2.e,f)$$

$$\alpha=\frac{a}{D_h};\quad \beta=\frac{b}{D_h};\ \gamma=\frac{2\beta}{2\alpha};\ X_1(Y)=\frac{x_1(y)}{D_h}=\alpha\left(1-\frac{Y}{\beta}\right) \qquad (2.g\text{-}j)$$

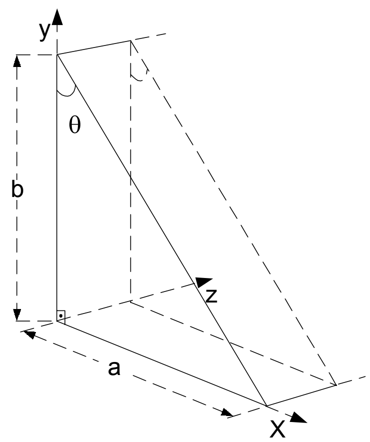

Figure 1 - Geometry and coordinates system for thermally developing right triangular duct flow.

The main dimensionless groups in Eqs. (2) above are: Pe (Péclet number), γ (aspect ratio), Re (Reynolds number) and Pr (Prandtl number). D_h is the hydraulic diameter defined as $D_h = 2\gamma a \Big/ \left(1+\gamma+\sqrt{1+\gamma^2}\right)$. The dimensionless velocity profile is given from the solution of momentum equations, for a non-Newtonian power-law fluid flowing within right triangular ducts, as an infinite series in the form (Chaves, 2001; Chaves et al., 2001):

$$U(X,Y) = \alpha\beta \frac{\sum_{k=1}^{\infty} K_k(X,Y)\tilde{u}_k(Y)}{\sum_{k=1}^{\infty} \tilde{h}_k \tilde{I}_k}; \quad \tilde{h}_k = \frac{4\sqrt{2}}{k\pi}; \quad \tilde{I}_k = \int_0^{\beta} \sqrt{X_1(Y)}\, \tilde{u}_k(Y) dY \tag{3.a-c}$$

$$K_k(X,Y) = \frac{\sin(\mu_k(Y)X)}{\sqrt{X_1(Y)/2}}; \quad \mu_k(Y) = \frac{k\pi}{X_1(Y)} \tag{3.d,e}$$

In Eqs. (3), the quantities $\tilde{u}_k(Y)$ represent the transformed potentials for the velocity field, which were numerically obtained by the application of the GITT approach (Chaves, 2001; Chaves et al., 2001), so that the integral in Eq. (3.c) must also be numerically obtained through appropriate subroutines to evaluate integrals of a cubic spline such as the CSITG from the IMSL Library (1989).

Due to the non-separable nature of the velocity profile given in Eq. (3.a) and consequently, of the related eigenvalue problem needed to solve the energy equation through well-known analytical methods such as the classical integral transform technique (Mikhailov and Ösizik, 1984), an exact solution of problem (1) is not possible. On the other hand, with the advances on the so-called GITT approach for the hybrid analytical-numerical solution of this class of non-separable eigenvalue problem, it is possible to avoid these difficulties as now demonstrated (Aparecido and Cotta, 1992; Cotta, 1993). For this purpose, in order to alleviate the difficulties related to the eigenvalue problem and to permit the employment of the generalized integral transform technique, the following auxiliary eigenvalue problems are chosen:

$$\frac{d^2\psi_i(X,Y)}{dX^2} + \mu_i^2(Y)\psi_i(X,Y) = 0; \; 0 < X < X_1(Y) \tag{4.a}$$

$$\psi_i(0,Y) = 0; \qquad \psi_i(X_1(Y),Y) = 0 \tag{4.b,c}$$

and

$$\frac{d^2\phi_m(Y)}{dY^2} + \lambda_m^2\phi_m(Y) = 0; \; 0 < Y < \beta \tag{5.a}$$

$$\phi_m(0) = 0; \qquad \phi_m(\beta) = 0 \tag{5.b,c}$$

which are readily solved to yield eigenfunctions, eigenvalues, and normalization integrals as

$$\psi_i(X,Y) = \sin(\mu_i(Y)X); \qquad \phi_m(Y) = \sin(\lambda_m Y) \tag{6.a,b}$$

$$\mu_i = \frac{i\pi}{X_1(Y)}; \qquad \lambda_m = \frac{m\pi}{\beta} \tag{6.c,d}$$

$$N_i = \frac{X_1(Y)}{2}; \qquad M_m = \frac{\beta}{2}, \qquad i = 1,2,\dots,N; \; m = 1,2,\dots,N^*. \tag{6.e,f}$$

Eigenvalue problems (4) and (5) allow the development of the following integral transform pair:

$$\tilde{\tilde{\theta}}_{im}(Z) = \int_0^{\beta}\int_0^{X_1(Y)} K_i(X,Y)\, Z_m(Y)\theta(X,Y,Z)dXdY, \qquad \text{inversion} \tag{7.a}$$

$$\theta(X,Y,Z) = \sum_{i=1}^{\infty}\sum_{m=1}^{\infty} K_i(X,Y) Z_m(Y)\tilde{\tilde{\theta}}_{im}(Z), \qquad \text{transform} \tag{7.b}$$

where,

$$K_i(X,Y) = \frac{\sin(\mu_i(Y)X)}{\sqrt{N_i}} \text{ and } Z_m = \frac{\sin(\lambda_m Y)}{\sqrt{M_m}} \tag{7.c,d}$$

Equation (1.a) is now operated on with $\int_0^{\beta}\int_0^{X_1(Y)} K_i(X,Y)\, Z_m(Y)dXdY$ to yield, after employing the inversion formula (7.b), the following truncated system of coupled differential equations to compute the transformed potentials $\tilde{\tilde{\theta}}_{im}(Z)$:

$$\sum_{j=1}^{N}\sum_{n=1}^{N^*} D_{ijmn}\frac{d\tilde{\tilde{\theta}}_{jn}(Z)}{dZ} + H_{ijmn}\tilde{\tilde{\theta}}_{im}(Z) = 0, \quad Z > 0; \; j = 1, 2, ..., N; \; n = 1, 2, ... N^* \tag{8.a}$$

where

$$D_{ijmn} = \int_0^{\beta}\int_0^{X_1(Y)} K_i(X)K_j(X)\, Z_m(Y)Z_n(Y)U(X,Y)dXdY \tag{8.b}$$

$$H_{ijmn} = E_{ijmn}\delta_{ij} - F_{ijmn} - G_{ijmn} + \lambda_m^2\delta_{ij}\delta_{mn} \tag{8.c}$$

and

$$E_{ijmn} = \int_0^{\beta} \mu_i^2(Y) Z_m(Y) Z_n(Y) dY \qquad (8.d)$$

$$F_{ijmn} = 2\int_0^{\beta}\int_0^{X_1(Y)} K_i(X,Y)\frac{\partial K_j(X,Y)}{\partial Y} Z_m(Y)\frac{dZ_m(Y)}{dY} dXdY \qquad (8.e)$$

$$G_{ijmn} = \int_0^{\beta}\int_0^{X_1(Y)} K_i(X,Y)\frac{\partial^2 K_j(X,Y)}{\partial Y^2} Z_m(Y) Z_n(Y) dXdY \qquad (8.f)$$

while the transformed inlet condition becomes

$$\tilde{\tilde{\theta}}_{im}(0) = \tilde{\tilde{g}}_{im} = \frac{2\left[1+(-1)^{i+1}\right]}{(i\pi)\sqrt{\beta}} \int_0^{\beta} \sqrt{X_1(Y)}\, \mathrm{sen}(\lambda_m Y) dY \qquad (8.e)$$

In Eq. (7.b) each summation is associated with the eigenfunction expansion in a corresponding spatial coordinate, for computational purposes, the series solution given by Eq. (7.b) is, in general, truncated to a finite number of terms for it summation, in order to compute the potential $\theta(X,Y,Z)$. The solution convergence is verified by comparing the values for the potential obtained with the truncated series for different numbers of retained terms. Such number of terms is commonly user-supplied and even taken as the same for each summation.

Then, the indices i and m related to the temperature field are reorganized into the single index p, while the indices j and n are collapsed into the new index q. The associated double sums are then rewritten as:

$$\sum_{i=1}^{N}\sum_{m=1}^{N^*} \rightarrow \sum_{p=1}^{NT} \quad ; \quad \sum_{j=1}^{N}\sum_{n=1}^{N^*} \rightarrow \sum_{q=1}^{NT} \qquad (9.a,b)$$

where

$i = \mathrm{int}[(p-1)/N] + 1$, $j = \mathrm{int}[(q-1)/N^*] + 1$, $m = p - (i-1)\,N$ and $n = q - (j-1)\,N^*$ (9.c-f)

The truncated version of system (8) is now rewritten in terms of these new indices as:

$$\sum_{q=1}^{NT} D_{pq}\frac{d\tilde{\tilde{\theta}}_q(Z)}{dZ} + H_{pq}\,\tilde{\tilde{\theta}}_p(Z) = 0, \quad Z > 0\,;\ p = 1, 2, ..., N \times N^*\,;\ q = 1, 2, ..., N \times N^* \qquad (10.a)$$

$$\tilde{\tilde{\theta}}_p(0) = \tilde{\tilde{g}}_p \qquad (10.b)$$

The coupled system of ordinary differential equations (10) is solved by efficient numerical algorithms for initial value problems, such as in subroutine IVPAG from the IMSL package (1989), with high accuracy. Then, after the transformed potentials are obtained, quantities of practical interest are determined from the analytic inversion formula (7.b), such as the dimensionless average temperature

$$\theta_{av}(Z) = \frac{1}{A_c}\int_{A_c} U(X,Y)\theta(X,Y,Z)dA \tag{11.a}$$

or

$$\theta_{av}(Z) = \frac{2\sqrt{2}}{\sqrt{\beta}\sum_{k=1}^{NV}\tilde{h}_k \tilde{I}_k}\sum_{p=1}^{NT}\sum_{k=1}^{NV} Q_{pk}\tilde{\tilde{\theta}}_p(Z) \tag{11.b}$$

where

$$Q_{pk} = \int_0^{\beta} \sin\left[\lambda_{m(p)}Y\right]\tilde{u}_k(Y)\delta_{i(p)k}dY; \quad \delta_{i(p)k} = \int_0^{X_1(Y)} K_i(X,Y)K_k(X,Y)dX \tag{12.a,b}$$

and the local Nusselt number can be calculated by making use of the temperature gradients at the wall integrated over the perimeter, or utilizing the axial gradient of the average temperature,

$$Nu_1(Z) = \frac{h(z)D_h}{k} = -\frac{1}{P_m^*\theta_{av}(Z)}\left[-\int_0^{\beta}\left.\frac{\partial\theta(X,Y,Z)}{\partial X}\right|_{X=0}dY + \int_0^{\beta}\left.\frac{\partial\theta(X,Y,Z)}{\partial X}\right|_{X=X_1(Y)}dY \right.$$
$$\left. -\int_0^{\alpha}\left.\frac{\partial\theta(X,Y,Z)}{\partial Y}\right|_{Y=0}dX + \int_0^{\beta}\left.\frac{\partial\theta(X,Y,Z)}{\partial Y}\right|_{X=X_1(Y)}dY\right] \tag{13.a}$$

or

$$Nu(Z) = -\frac{1}{4\theta_{av}(Z)}\frac{d\theta_{av}(Z)}{dZ} \tag{13.b}$$

3. RESULTS AND DISCUSSION

Now, results are presented in terms of dimensionless average temperature and Nusselt numbers along the axial coordinate, within the range of $Z = 10^{-4} - 1$, for apex different angles (θ = 10, 30 and 45°) and different power-law indices. System (10) was numerically solved with NT ≤ 400 and a relative user-prescribed tolerance of 10^{-8} in subroutine IVPAG from the IMSL package (1989).

To illustrate the convergence behavior of the present approach, Tab. (1) brings the convergence of the Nusselt number in thermal entry region (i.e., $Z = 10^{-1}$ and 1) for different power-law indices and θ = 45°. It is observed in this table an excellent convergence ratio, with practically three digits converged for both positions studied. The comparisons among the values of average temperatures and Nusselt numbers for n = 1.0 calculated in the present work and the values of Aparecido and

Cotta (1992) are graphically shown in Figs. 2 and 3, respectively, and it can be noticed that the values are in good agreement with each other, indicating that the numerical code developed here is well established.

Table 1. Convergence of the local Nusselt number for a right triangular duct (θ = 45°)

Nu(Z)			
Z = 0.01			
NT	**n = 0.50**	**n = 1.00**	**n = 1.50**
25	2.5529	2.4052	2.3578
100	2.5514	2.4030	2.3549
255	2.5510	2.4027	2.3546
400	2.5509	2.4026	2.3546
a	NA	2.4	NA
Z = 1			
25	2.5124	2.3599	2.3089
100	2.5096	2.3568	2.3055
255	2.5094	2.3567	2.3054
400	2.5093	2.3567	2.3053
a	NA	2.34	NA

NA - Not available, a - Shah and London (1978)

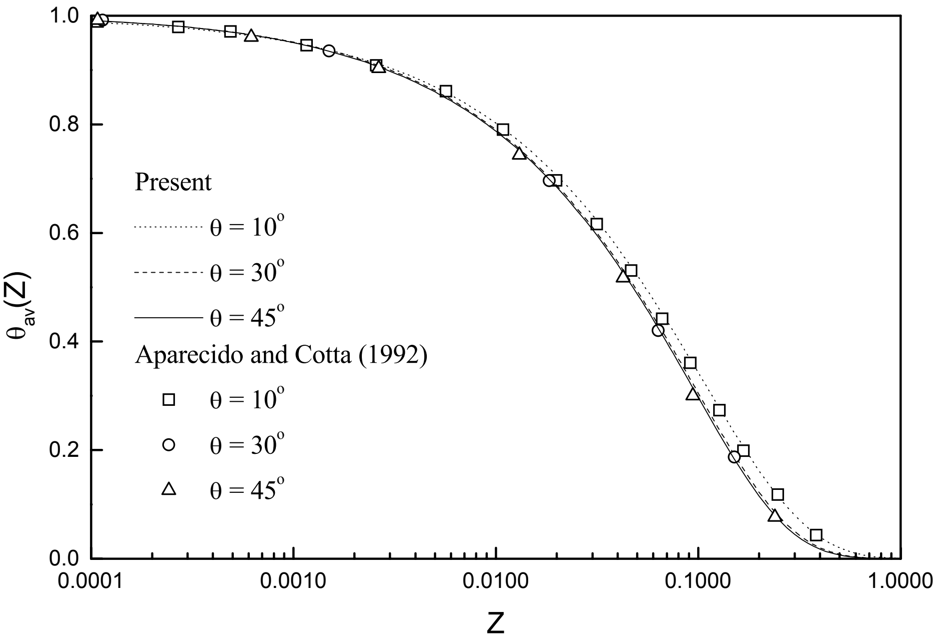

Figure 2. Comparison of dimensionless average temperature for different apex angles.

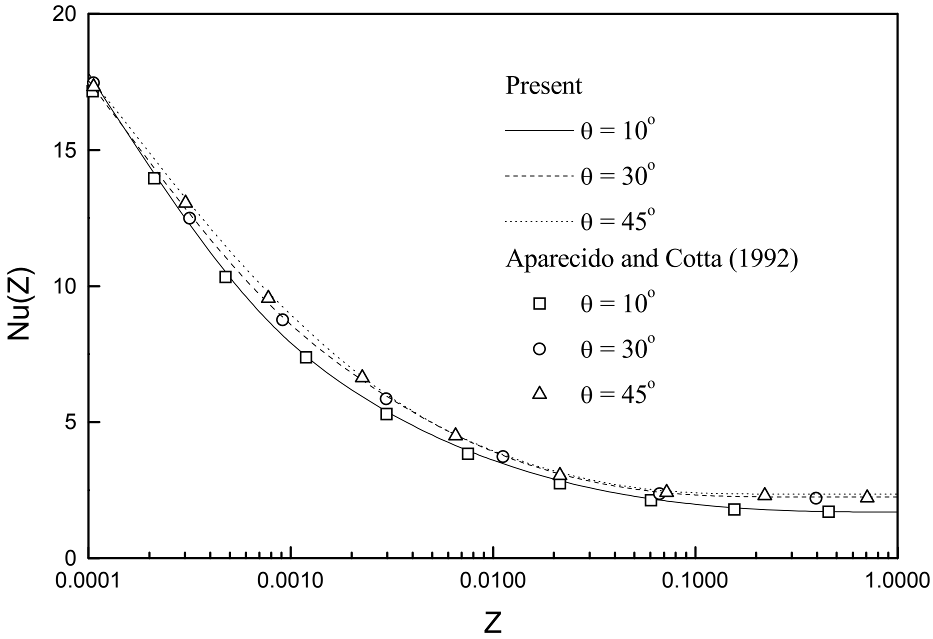

Figure 3. Comparison of local Nusselt numbers for different apex angles.

Figure 4 shows fully developed Nusselt numbers for n = 1.0 from this work and the results available in Haji-Sheikh et al. (1983) and Aparecido and Cotta (1992), where an excellent agreement was obtained for the full range of apex angles $0 \leq \theta \leq 45°$.

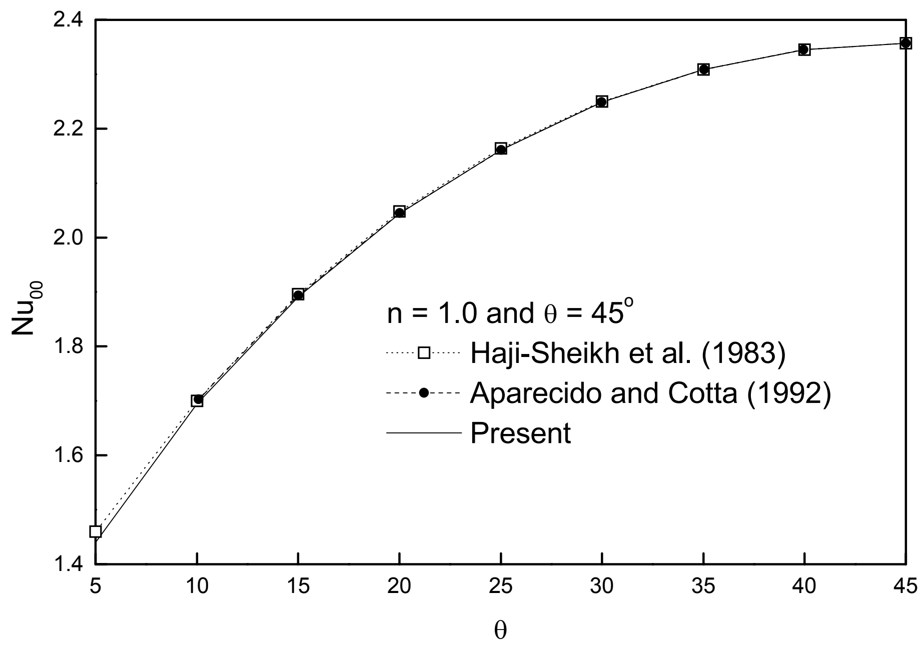

Figure 4. Comparison of fully developed Nusselt numbers with literature for n = 1.0 and different apex angles.

In Fig. 5 results of axial distribution of the dimensionless average temperature along the thermal entry region of a right triangular isosceles duct are presented for different power-law indices. It can be noticed a small influence of the power-law index in the dimensionless average temperature along the thermal entry region. In Fig. 6, it is also observed a little influence of the power-law index in the Nusselt numbers in the thermal entry region. The effect of power-law index in the average temperature is small, as can be verified in Fig. 6 by a slight increase in the average temperature when n > 1. However, in Fig. 6, it is observed an opposite behavior for the Nusselt numbers. These aspects can be explained by the fact when n > 1 the viscous effects near the wall diminish and, consequently, the thermal exchange is less intensified resulting in lower values for the Nusselt numbers when compared with those values for n < 1.

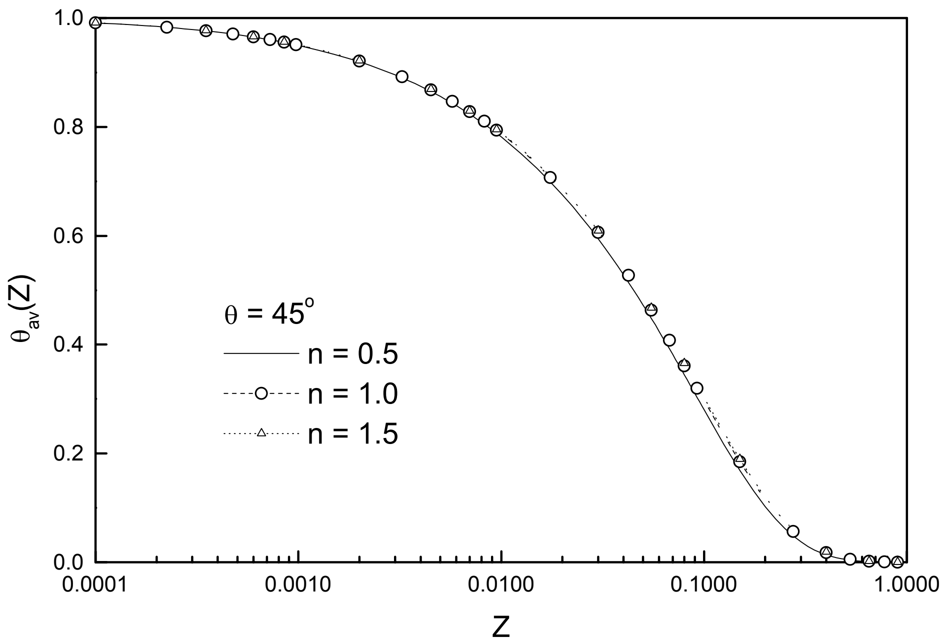

Figure 5. Dimensionless average temperature for different power-law indices.

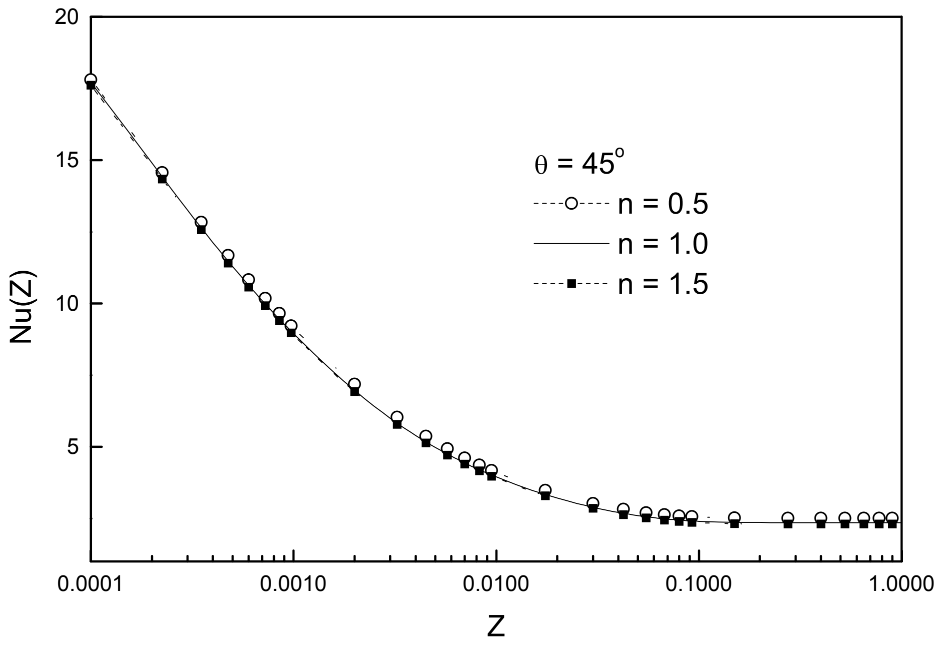

Figure 6. Nusselt numbers along the thermal entry region for different power-law indices.

4. CONCLUSIONS

The present approach demonstrated to be relatively cheap, within the range of NT considered. Numerical results were tabulated and graphically presented providing sets of benchmark for the local Nusselt numbers and dimensionless average temperature. The next step in the application of the present methodology involving the flow of non-Newtonian fluids will be concerned to the case of others irregularly shaped duct geometries as described by Cotta (1993) and Chaves (2001).

5. ACKNOWLEDGEMENTS

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