AN EXPERIMENTAL ANALYSIS OF THE CHAOTIC MOTION OF A NONLINEAR PENDULUM

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Abstract

The experimental analysis of nonlinear dynamical systems furnishes a scalar sequence of measurements and it is interesting to analyze it using state space reconstruction and other techniques related to nonlinear analysis. The noise contamination is unavoidable in cases of data acquisition and, therefore, it is important to recognize the techniques that can be employed for a correct identification of chaos. The present contribution discusses the analysis of chaotic motion of an experimental nonlinear pendulum, considering state space reconstruction, frequency domain analysis and the determination of Lyapunov exponents. A procedure to construct Poincaré map of the signal is presented. Results show that it is possible to identify chaos employing proper procedures without contemplates any kind of filters.

Key-words: Chaos, Nonlinear Dynamics, Nonlinear Pendulum.

1. INTRODUCTION

An approach to deal with the response of dynamical system is based on the analysis of data derived from an experiment. The experimental analysis of nonlinear dynamical systems furnishes a scalar sequence of measurements. Therefore, a time series associated with system dynamics is available and it is interesting to analyze it using state space reconstruction and other techniques related to nonlinear analysis. The noise contamination is unavoidable in cases of data acquisition and noise suppression is, therefore, essential in signal processing, especially in chaos analysis. Many studies are devoted to evaluate noise suppression and its effects in the analysis of chaos however, there are a small number of reports devoted to the effects of the system noise on chaos (Ogata *et al.*, 1996).

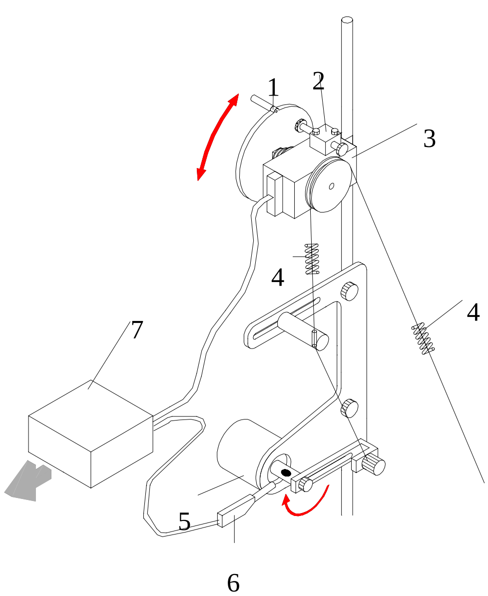
The analysis of nonlinear dynamical systems from time series involves state space reconstruction. The method of time delay method has proven to be a powerful tool to analyze chaotic behavior of dynamical system. Ruelle (1979), Packard *et al.* (1980) and Takens (1981) introduced the basic idea of this method and the main problem arising is the determination of the embedding parameters.

Nonlinear analysis also involves the determination of quantities, known as dynamical invariants, which are important to identify chaotic behavior. Lyapunov exponents and system dimension are some examples. Lyapunov exponents evaluate the sensitive dependence on initial conditions estimating the exponential divergence of nearby orbits. These exponents have been used as the most useful dynamical diagnostic tool for chaotic system analysis. Signs of the Lyapunov exponents provide a qualitative picture of the system's dynamics and any system containing at least one positive exponent presents chaotic behavior.

The present contribution discusses the analysis of chaotic motion of an experimental nonlinear pendulum, considering state space reconstruction, frequency domain analysis and determination of Lyapunov exponents. In these analyses, it is necessary to present a procedure to construct a Poincaré map of the signal, which is also presented. Since the greater exponent is the most important to diagnose chaotic motion, algorithms proposed by Rosenstein *et al.* (1993) and Kantz (1994) are conceived. Results show that it is possible to identify chaos employing proper procedures without contemplates any kind of filters.

2 - EXPERIMENTAL APPARATUS

The experimental data related to the response of the nonlinear pendulum is obtained from an apparatus discussed in this section. Consider a nonlinear pendulum depicted in Figure 1. This pendulum is constructed with a disc where there is a concentrated mass (1) that is connected to a rotary motion sensor (3). The dissipation is provided by a magnetic device (2), which may be adjusted to vary this effect. A motor-string-spring device (4-5) provides the excitation of the pendulum. The signal measurement is done with the aid of two transducers. The rotary motion sensor (3), *PASCO encoder CI-6538*, has 1440 orifices and a precision of 0.25^0. The magnetic transducer (6) is employed in order to generate a frequency signal associated with the forcing frequency of the motor, which is used to construct the Poincaré map of the signal. This apparatus is connected with an A/D interface, *Science Workshop Interface 500 (CI-6760)*, and then to a computer.



1. Disc with concentrated mass
2. Magnetic damping device
3. Motion sensor: PASCO CI-6538
4. Spring
5. DC Motor: ME – 8750
6. Magnetic transducer: TEKTRONIX
7. Science workshop interface: PASCO CI-6760

Figure 1 - Nonlinear pendulum apparatus.

In order to perform the analysis of the nonlinear pendulum, one conceives that the time series is a sequence of angular position measured from the experiment, $s = x = \theta$. The apparatus also permits to measure the angular velocity $y = \dot{\theta}$, which is used to construct the real phase space (x versus y), employed to perform a visual validation of the reconstructed phase space.

3 - CHAOTIC SIGNAL

The experimental pendulum dynamics is analyzed considering a chaotic signal with N = 30589 points, generated with a motor voltage V = 4.2V, a sample frequency Ω_s = 20Hz and a damping parameter ς = 0.0125. The time history evolution of part of the signal is shown in Figure 2.

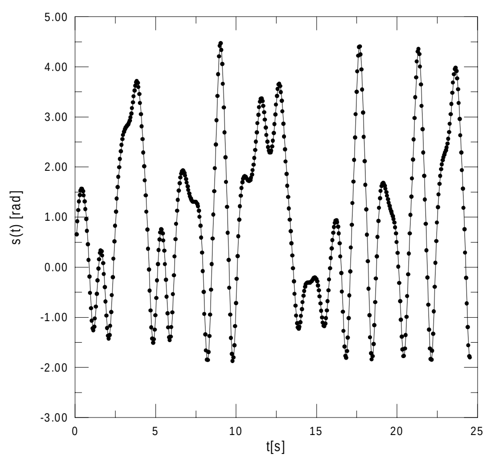


Figure 2 - Chaotic Signal.

Using the Fast Fourier Transform (FFT), it is possible to see that the fundamental frequency Ω_e = 0.8Hz is immersed in a continuous spectrum of frequencies. This behavior is typical in chaotic motion, nevertheless, it must be confirmed evaluating dynamical invariants. In this paper, Lyapunov exponents are considered. The forthcoming section contemplates the state space reconstruction.

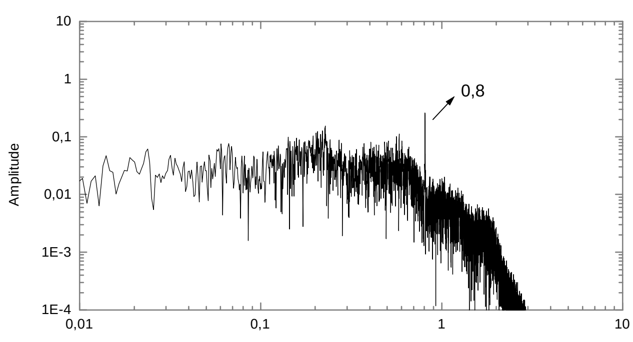


Figure 3 - *FFT* of the signal.

3.1 - State Space Reconstruction

The basic idea of the state space reconstruction is that a signal contains information about unobserved state variables which can be used to predict the present state. Therefore, a scalar time series, $s(t)$, may be used to construct a vector time series that is equivalent to the original dynamics from a topological point of view. The state space reconstruction needs to form a coordinate system to capture the structure of orbits in state space which could be done using lagged variables, $s(t+\tau)$, where τ is the time delay. Then, considering an experimental signal, $s(n)$, n = 1, 2, 3,…, N, where $t = t_0 + (n\text{-}1)\Delta t$, it is possible to use a collection of time lags to create a vector in a D_e-dimensional space,

$$u(t) = \{s(t), s(t+\tau),..., s(t+(D_e-1)\tau)\}^T \tag{1}$$

The method of delays has become popular for dynamical reconstruction, however, the choice of the delay parameters, τ - time delay, and D_e - embedding dimension, has not been fully developed. Therefore, many researches have been developed to consider the better approaches to estimate delay parameters for different kinds of time series. In this paper, one considers the average mutual information method to determine time delay (Fraser, 1989) and the false nearest neighbors method to estimate embedding dimension (Kennel *et al.*, 1992). Results of this analysis are presented in Figure 4. Figure 4a shows the mutual information versus time delay, and the first minimum of the curve must be used as the time delay, furnishing τ = 6 x 0.05 = 0.30s. Figure 4b presents the curve of the percentage of false neighbor points versus embedding dimension, showing that the embedding dimension needs to be between 3 and 4.

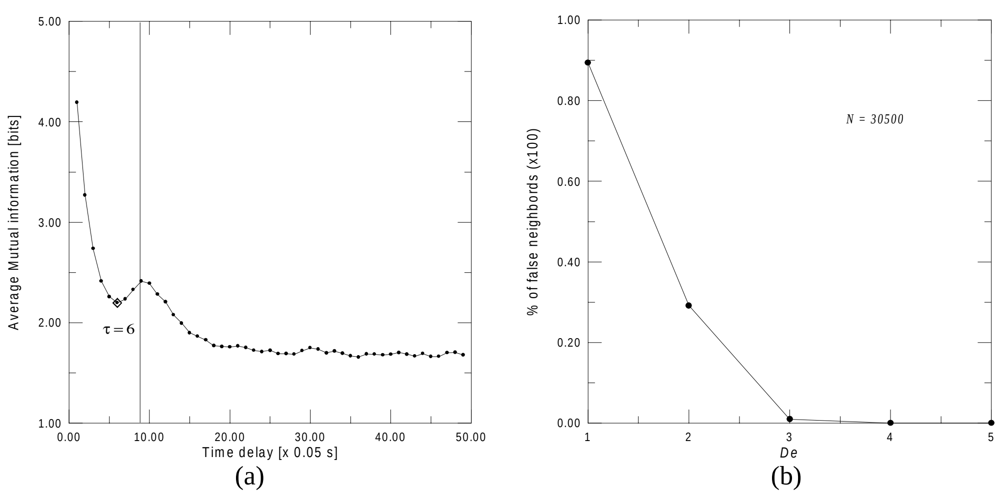


Figure 4 - Delay parameters: (a) Average mutual information versus τ; (b) % of false neighbors versus D_e.

After the determination of delay parameters it is possible to reconstruct the phase space. Figure 5a-b presents the reconstructed phase space projected in 2 and 3 dimension while Figure 5c presents the real phase space measured in the experiment. Both spaces are similar from a topological point of view (Takens, 1981), presenting just a small coordinate change from one to another.

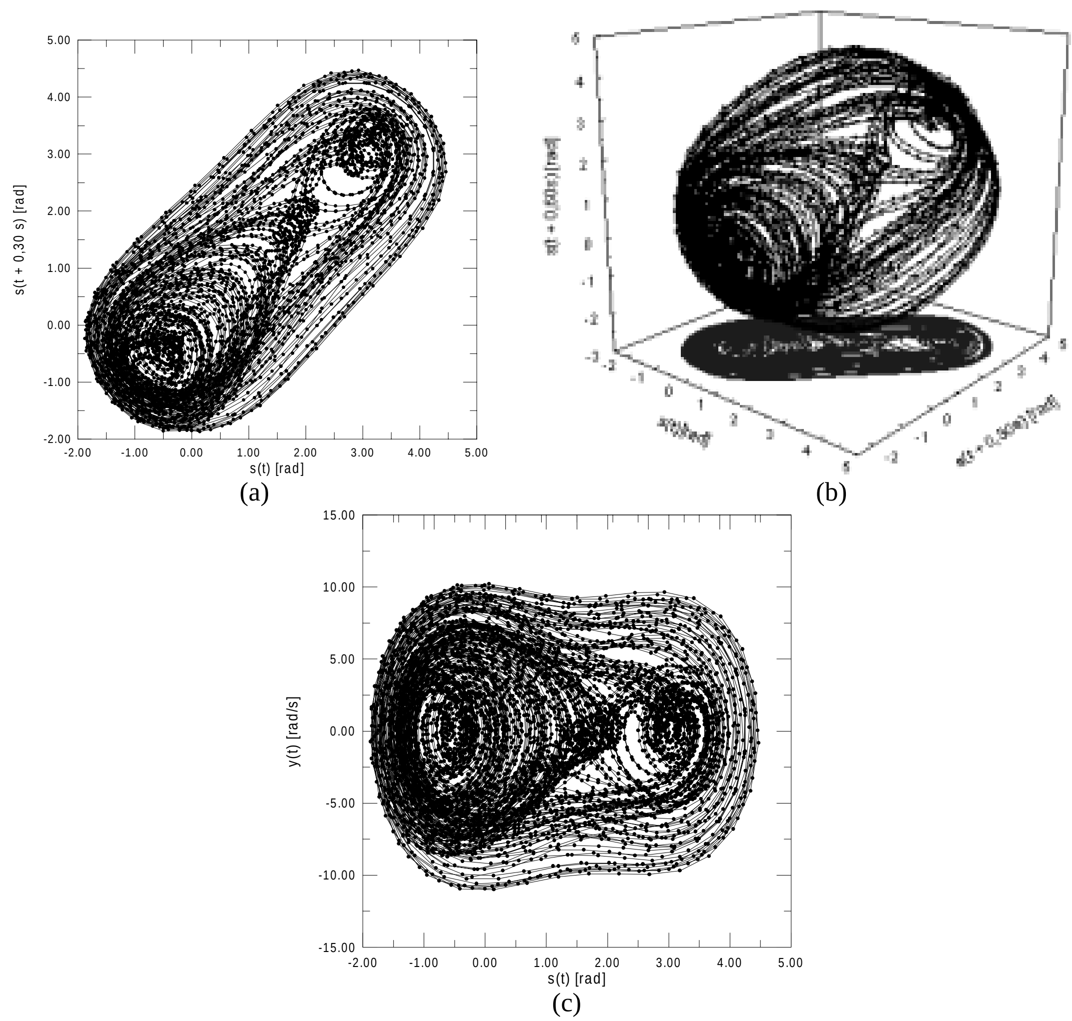


Figure 5 - Phase Space: (a) Reconstructed, 2-Dim; (b) Reconstructed, 3-Dim; (c) Real.

3.2 - Poincaré Maps

Poincaré map is an important tool to observe the response of a nonlinear system. Experimentally, this can be done in several ways. Moon (1992) presents a procedure where there is a signal converter, which store sampled data in a computer for display at a later time. Here, a similar procedure is conceived to generate two signals: one associated with the motion and the other associated with the forcing frequency. The forcing frequency signal is generated with the aid of a magnetic transducer, which generates electric pulses when a reference bolt, connected to the motor, passes near it. These pulses are compared with the motion signal and only the measures in these instants are contemplated. Then, a third signal is generated representing the Poincaré Map of the motion. Figure 6 presents the forcing frequency signal showing the points which is used to define the time instant where the motion signal must be considered. The Poincaré map defined by this procedure is presented in Figure 7. Figure 7a shows the reconstructed phase space while Figure 7b the real phase space. A strange attractor is clearly identified in this phase space showing a fractal-like structure. Nevertheless, it is useful to confirm this with the calculation of the dynamical invariants.

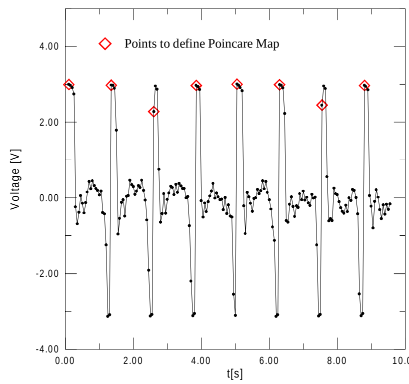
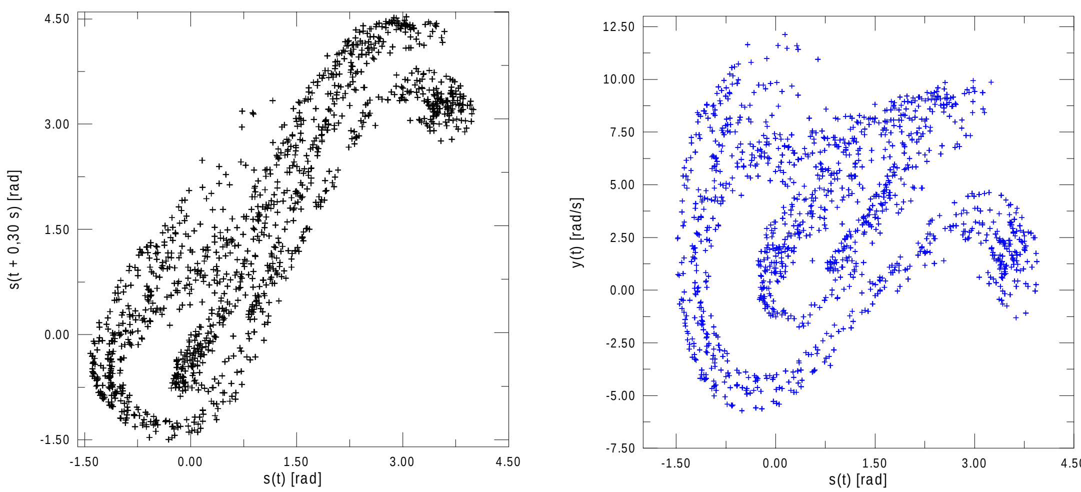


Figure 6 - Forcing frequency signal to define Poincaré Map.



(a) (b)

Figure 7 – Poincaré Map of a chaotic signal: (a) Reconstructed; (a) Real.

3.3 - Lyapunov Exponents

Lyapunov exponents evaluate the sensitive dependence on initial conditions by considering the exponential divergence of nearby orbits. Therefore, one needs to evaluate how trajectories with nearby initial conditions diverge. The dynamics of the system transform the D-sphere of states in a D-ellipsoid and, when there is a chaotic motion, a complex evolution exists. Mathematically, the Lyapunov exponents considers $d(t) = d_0\, b^{\lambda t}$, where b is a reference basis. The signs of the Lyapunov exponents provide a qualitative picture of the system's dynamics. The existence of positive Lyapunov exponents defines directions of local instabilities in the system dynamics.

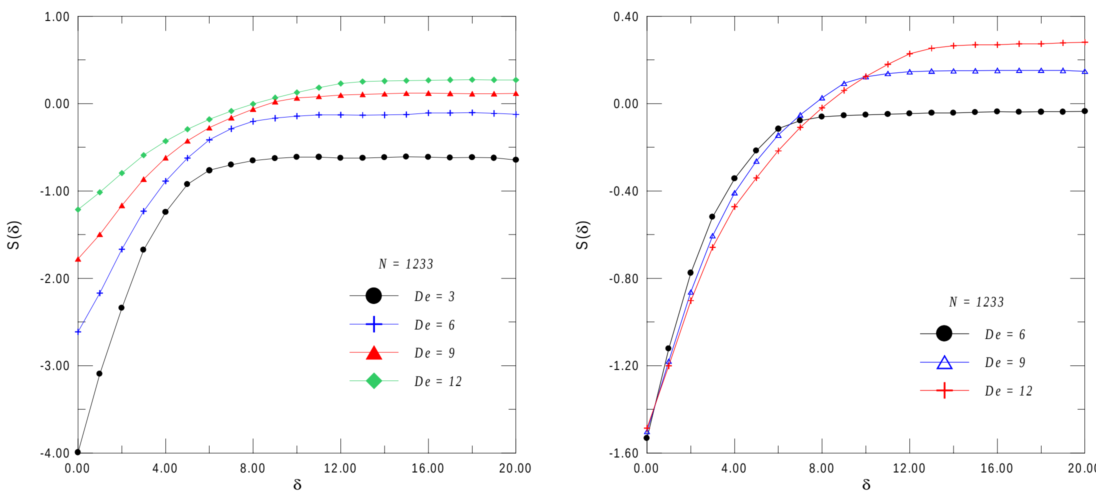
The determination of Lyapunov exponents of dynamical system with an explicitly mathematical model, which can be linearized, is well established from the algorithm proposed by Wolf *et al.* (1985). On the other hand, the determination of these exponents from time series is quite more complex. Basically, there are two different classes of algorithms: Trajectories, real space or direct method; and perturbation, tangent space or Jacobian matrix method.

Franca & Savi (2000) shows that algorithms due to Kantz (1994) and due to Rosenstein *et al.* (1993) permit to establish a difference between chaotic and periodic motion, and has no noise sensibility. Therefore, the present contribution considers these algorithms to estimate Lyapunov exponents. The algorithm proposed by Kantz (1994) uses the same idea of the one proposed by Wolf *et al.* (1985). Kantz (1994) considers that the divergence rate trajectories fluctuates along the trajectory, with the fluctuation given by the spectrum of effective Lyapunov exponents. The average of effective Lyapunov exponent along the trajectory is the true Lyapunov exponent and the maximum exponent is given by

$$\lambda(t) = \lim_{\varepsilon \to 0} \frac{1}{\delta} \ln\left(\frac{|u(t+\delta) - u_\varepsilon(t+\delta)|}{\varepsilon} \right) \tag{2}$$

where $|u(0) - u_\varepsilon(0)| = \varepsilon$ and $u(t) - u_\varepsilon(t) = \varepsilon v_u(t)$, with $v_u(t)$ representing the eigenvectors associated with the maximum Lyapunov exponent, λ_{max}; δ is a relative time referring to the time index of the point where the distance begin to be greater than ε, $\delta(0)$. Rosenstien *et al.* (1993) have proposed a similar algorithm where the distance between the trajectories is defined as the Euclidean norm in the reconstructed phase space and, also, they have used only one neighbor trajectory.

Employing these algorithms to the Poincaré map signal, it is possible to estimate the maximum Lyapunov exponent of the system. Figure 8 presents the curve $S(\delta)$ versus δ predicted by both algorithms using $\varepsilon = 1.6$ and $D_e = 3, 6, 9, 12$. After a linear regression, the algorithm due to Rosenstein *et al.* (1993) furnishes $\lambda = 0.468 \pm 0.059$ while the algorithm due to Kantz (1994) furnishes $\lambda = 0.177 \pm 0.024$. As expected, the system presents a positive exponent. Further studies show that the Rosenstein's algorithm may present problems evaluating periodic signals (Franca, 2000).



(a) (b)

Figure 8 - Chaotic signal: $S(\delta)$ curves. (b)Rosenstein *et al.*; (b)Kantz;

4. CONCLUSIONS

This contribution reports on the analysis of the chaotic motion of an experimental nonlinear pendulum. The phase space reconstruction is done employing the method of delay coordinates and delay parameters are estimated with the average mutual information method to determine time delay and the false nearest neighbors method to estimate embedding dimension.

A procedure to construct the Poincaré Map is developed and presents good results. The FFT analysis shows that the signal may be chaotic, however, it is necessary to evaluate dynamical invariants to assure this conclusion. Lyapunov exponents are used with this aim. The algorithm due to Kantz (1994) and due to Rosenstein *et al.* (1993) permit to identify chaotic motion, and has no noise sensibility. Nevertheless, further studies shows that Kantz's algorithm presents better results.

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