A FINITE ELEMENT FORMULATION FOR THE MOMENTUM AND ENERGY TRANSFER IN POROUS MEDIA USING A STABILIZED METHODOLOGY

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Abstract

A mixture theory approach has been employed to study the heat convection in a flow through a saturated rigid porous medium, generating a system of coupled partial equations. The employed method is built in order to remain stable and accurate even for very high advective regimen. Taking advantage of appropriated upwind strategies, the numerical method employed has generated stable and accurate approximations for fluid constituent velocity and pressure fields as well as for both solid and fluid constituents' temperatures even for very high Péclet flows. Some two-dimensional simulations of a nonisothermal flow of a Newtonian incompressible fluid through a porous channel bounded by two impermeable flat plates have been performed.

Keywords: Porous media, mixture theory, forced convection, finite element approximations, mixed formulations, GLS method.

1. INTRODUCTION

Transport phenomena in porous media present countless relevant applications among which: geomechanics, petroleum and mining industries, sintering technologies, biomechanics, not to mention problems that impact the energy self-sufficiency and the environmental state. In this article, the momentum and energy transport in a saturated flow through a rigid porous medium has been studied by employing a local model based on a *Continuum Theory of Mixtures* (Atkin and Craine, 1976; Bedford and Drumheller,1983). The mixture is a superposition of continuous constituents (each of them occupying its whole volume) standing for the fluid and the porous medium; the fluid constituent is assumed Newtonian and incompressible, while the solid constituent, representing the porous medium, is supposed rigid, homogeneous, isotropic and at rest

(Martins-Costa et al., 1992; Costa-Mattos etal., 1995). The mixture theory leads to an apparent thermomechanical independence, requiring momentum and energy generation terms to account for the thermomechanical coupling among the constituents in the balance equations. Since the solid constituent is rigid and at rest it suffices to solve mass and momentum conservation equations for the fluid constituent of the mixture, while the energy equation must be solved for both constituents. These equations, combined with constitutive assumptions satisfying the material objectivity and the Second Law of Thermodynamics, describe the heat convection in a porous medium.

Numerical simulations of incompressible flows suffer from two major difficulties: finite elements need to compatibilize velocity and pressure subspaces satisfying the Babuška-Brezzi mathematical condition and the instability inherent to central discretization schemes - either by Galerkin formulation or by central difference stencil - to approximate high advective dominated flows. Most of Galerkin method limitations may be overcome by the so called stabilized methods (Hughes and Franca, 1987 and references therein), which consists of adding mesh-dependent terms to the usual Galerkin formulation, which are functions of the residuals of the Euler-Lagrange equations evaluated elementwise. Since these residuals are satisfied by the exact solutions, consistency is preserved in these methods. The perturbation terms are designed to enhance stability of the original Galerkin formulation without upsetting consistency.

In this paper a stabilized finite element method - built in to inherit the good stability features shown by the stabilized methods already proposed for the Stokes problem (see, for instance, Franca et al., 1992; Sampaio, 1991) - has been introduced to simulate forced convection in porous media. Taking advantage of an improved design of the stability parameters (Franca et al., 1992), the method remains stable even for very high advective flows (Franca and Frey, 1992); thereby there is no need to satisfy the above mentioned Babuška-Brezzi condition.

2. MIXTURE THEORY MODELING

Since the solid constituent is rigid and at rest it suffices to solve mass and momentum conservation equations for the fluid constituent of the mixture. Therefore, the local form of the mass and momentum conservation may be stated as (Atkin and Craine, 1976)

$$\frac{\partial \rho_f}{\partial t} + (\nabla \rho_f)\mathbf{u}_f + \rho_f \nabla \cdot \mathbf{u}_f = 0 \tag{1}$$

$$\rho_f \left[\frac{\partial \mathbf{u}_f}{\partial t} + (\nabla \mathbf{u}_f)\mathbf{u}_f \right] = \nabla \cdot \boldsymbol{\sigma}_f + \mathbf{m}_f + \rho_f \mathbf{f}_f \tag{2}$$

in which $\mathbf{u}_f$ represents the fluid constituent velocity and ρ_f its mass density, so that $\rho_f = \rho\varphi$; where φ is the fluid fraction, coincident with the porosity, for saturated flows and ρ is the actual fluid mass density, measured in a continuum mechanics viewpoint. The momentum source, which accounts for the mechanical coupling between both constituents is represented by the interaction force $\mathbf{m}_f$ applied on the fluid constituent by the solid one, $\boldsymbol{\sigma}_f$ is the partial stress tensor acting on the fluid constituent and $\mathbf{f}_f$ the body force acting on it.

The mixture theory viewpoint requires constitutive assumptions for the partial stress tensor and the intraction force applied on the fluid constituent (Williams, 1978; Martins-Costa et al., 1992)

$$\boldsymbol{\sigma}_f = -\varphi p \mathbf{I} + 2\lambda\varphi^2 \mu \boldsymbol{\varepsilon}(\mathbf{u}_f) \tag{3}$$

$$\mathbf{m}_f = -\frac{\varphi^2 \mu}{K}\mathbf{u}_f \tag{4}$$

where μ is the actual fluid viscosity and K is actual the specific permeability of the porous medium, both regarded from a continuum mechanics viewpoint, λ is a scalar parameter depending on the porous matrix microstructure, p is the pressure acting on the mixture and $\boldsymbol{\varepsilon}(\mathbf{u}_f)$ is the symmetrical part of the fluid constituent velocity gradient.

The local form of the energy equation for each constituent is given by (Atkin and Craine, 1976)

$$\rho_\alpha\Big[\frac{\partial e_\alpha}{\partial t} + (\nabla e_\alpha)\mathbf{u}_\alpha\Big] = \rho_\alpha r_\alpha - \nabla\cdot\mathbf{q}_\alpha + \psi_\alpha + \boldsymbol{\sigma}_\alpha\cdot\boldsymbol{\varepsilon}(\mathbf{u}_\alpha) \qquad \alpha = s, f \tag{5}$$

in which e_α represent each constituent internal energy, r_α its energy generation per unit mass and $\mathbf{q}_\alpha$ the partial heat flux per unit time and unit area associated to the α-th constituent. The energy generation function, ψ_α, which is an internal contribution, represents the energy supply - per unit time and unit volume - to a given constituent, arising from its thermal interaction with the remaining constituents of the mixture (Martins-Costa et al., 1993).

Constitutive assumptions for the partial heat fluxes and energy generation function for the solid ($\mathbf{q}_s$ and ψ_s) and the fluid ($\mathbf{q}_f$ and ψ_f) constituents may be stated as (Martins-Costa et al., 1992)

$$\mathbf{q}_s = -\Lambda k_S(1-\varphi)\nabla\theta_s \tag{6}$$

$$\mathbf{q}_f = -\Lambda k_F\varphi\nabla\theta_f \tag{7}$$

$$-\psi_s = \psi_f = R(\theta_s - \theta_f) \tag{8}$$

where θ_s and θ_f represent the solid and the fluid constituents' temperatures, Λ an always positive parameter which may depend on both the internal structure and the kinematics of the mixture, k_S and k_F are the actual thermal conductivity of the solid and the fluid and R is a positive-valued parameter depending on both constituents' thermal properties and on the mixture internal structure.

3. FINITE ELEMENT MODELING

Substituting the constitutive eqs. (3), (4), (6)-(8) into the conservation eqs.(2) and (5), assuming low velocities and steady-state Stokes flow, we obtain the following boundary-value problem for incompressible flows through saturated porous media: *Given functions* $\mathbf{f}_f\colon\overline{\Omega}\to\mathbf{R}^3, r_f\colon\overline{\Omega}\to\mathbf{R}$ *and* $r_s\colon\overline{\Omega}\to\mathbf{R}$, *find the unknown fields*

$\mathbf{u}_f: \overline{\Omega} \to \mathbf{R}^3, p: \overline{\Omega} \to \mathbf{R}, \theta_f: \overline{\Omega} \to \mathbf{R}$ *and* $\theta_s: \overline{\Omega} \to \mathbf{R}$, *such that*

$$\begin{aligned}
-2\lambda\varphi\mu\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u}_f) + \nabla p + \frac{\varphi\mu}{K}\mathbf{u}_f &= \mathbf{f}_f \qquad \text{in } \Omega \\
\nabla\cdot\mathbf{u}_f &= 0 \qquad \text{in } \Omega \\
\mathbf{u}_f\cdot\nabla\theta_f - \kappa_f\Delta\theta_f + \beta(\theta_f - \theta_s) &= r_f \qquad \text{in } \Omega \\
\kappa_s\Delta\theta_s + (\theta_s - \theta_f) &= r_s \qquad \text{in } \Omega \\
\mathbf{u}_f &= \mathbf{u}_g \qquad \text{on } \Gamma_g \\
\theta_f &= \theta_{f_g} \qquad \text{on } \Gamma_g \\
\theta_s &= \theta_{s_g} \qquad \text{on } \Gamma_g \\
\boldsymbol{\sigma}_f\mathbf{n} &= \boldsymbol{\sigma}_h \qquad \text{on } \Gamma_h \\
-\Lambda k_f\varphi\nabla\theta_f\cdot\mathbf{n} &= q_{f_h} \qquad \text{on } \Gamma_h \\
-\Lambda k_s(1-\varphi)\nabla\theta_s\mathbf{n} &= q_{s_h} \qquad \text{on } \Gamma_h
\end{aligned} \tag{9}$$

with the thermal porous diffusivities κ_f and κ_s and coefficient β being defined, respectively, by

$$\kappa_f = \frac{\Lambda k_f\varphi}{\rho_f c_f} \quad ; \quad \beta = \frac{R}{\rho_f c_f} \quad ; \quad \kappa_s = \frac{\Lambda k_s(1-\varphi)}{R} \tag{10}$$

and prescribed energy supplies r_f and r_s redefined as

$$r_f := \frac{r_f}{\rho_f c_f} \quad ; \quad r_s := \frac{r_s}{R} \tag{11}$$

3.1 Stabilized formulations

The finite element approximation of eqs.(9) is based on the following finite dimension subspaces,

$$W_h = \{\phi \in H_0^1(\Omega) \,|\, \phi_{|\mathbf{K}} \in P_m(K),\ K \in \mathcal{C}_h\} \tag{12}$$

$$\mathbf{V}_h = \{\mathbf{v} \in H_0^1(\Omega)^N \,|\, \mathbf{v}_{|K} \in P_k(K)^N,\ K \in \mathcal{C}_h\} \tag{13}$$

$$P_h = \{p \in \mathcal{C}^0(\Omega) \cap L_0^2(\Omega) \,|\, p_{|K} \in P_l(K),\ K \in \mathcal{C}_h\} \tag{14}$$

$$W_h^g = \{\phi \in H^1(\Omega) \,|\, \phi_{|K} \in P_m(K),\ K \in \mathcal{C}_h,\ \phi = \theta_{i_g} \text{ on } \Gamma_{\mathbf{g}}\} \qquad i = f, s \tag{15}$$

$$\mathbf{V}_h^g = \{\mathbf{v} \in H^1(\Omega)^N \,|\, \mathbf{v}_{|K} \in P_k(K)^N,\ K \in \mathcal{C}_h,\ \mathbf{v} = \mathbf{u}_g \text{ on } \Gamma_{\mathbf{g}}\} \tag{16}$$

where P_k, P_l and P_m denote, respectively, polynomial spaces of degrees k, l and m.

The following stabilized method may be introduced to represent the system (9), employing the definitions (10)-(11): *Find* $(\mathbf{u}_h, p_h, \theta_{fh}, \theta_{sh}) \in \mathbf{V}_h^g \times P_h \times W_h^g \times W_h^g$ *such that*

$$\begin{aligned}
B(\mathbf{u}_{fh}, p_h, \theta_{fh}, \theta_{sh}; \mathbf{v}, q, \phi_f, \phi_s) =& F(\mathbf{v}, q, \phi_f, \phi_s), \\
& (\mathbf{v}, q, \phi_f, \phi_s) \in \mathbf{V}_h \times P_h \times W_h \times W_h
\end{aligned} \tag{17}$$

with

$$
\begin{aligned}
B(\mathbf{u}_f,p,\theta_{fh},\theta_{sh};\mathbf{v},q,\phi_f,\phi_s) &= (2\lambda\varphi\mu\boldsymbol{\varepsilon}(\mathbf{u}_f),\boldsymbol{\varepsilon}(\mathbf{v})) - (\nabla\cdot\mathbf{v},p) - (\nabla\cdot\mathbf{u}_f,q) \\
&- (\frac{\varphi\mu}{\mathbf{K}}\mathbf{u}_f,\mathbf{v}) + (\nabla\cdot\mathbf{u}_f,\delta\nabla\cdot\mathbf{v}) \\
&+ (\mathbf{u}_f\cdot\nabla\theta_f,\phi_f) + (\kappa_f\nabla\theta_f,\nabla\phi_f) + (\beta(\theta_f-\theta_s),\phi_f) \\
&+ (\kappa_s\nabla\theta_s,\nabla\phi_s) + ((\theta_s-\theta_f),\phi_s) \\
&+ \sum_{K\in\mathcal{C}_h}\left(\nabla p + \frac{\varphi\mu}{\mathbf{K}}\mathbf{u}_f - 2\lambda\varphi\mu\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u}_f),\tau_{\mathbf{v}}(\mathrm{Re}_K)(-\nabla q - 2\lambda\varphi\mu\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{v}))\right)_K \\
&+ \sum_{K\in\mathcal{C}_h}\left(\mathbf{u}_f\cdot\nabla\theta_f - \kappa_f\Delta\theta_f + \beta(\theta_f-\theta_s),\tau_\phi(\mathrm{Pe}_K)(\mathbf{u}_f\cdot\nabla\phi_f - \kappa_f\Delta\phi_f)\right)_K
\end{aligned}
\tag{18}
$$

and

$$
\begin{aligned}
F(\mathbf{v},q,\phi_f,\phi_s) &= (\mathbf{f}_f,\mathbf{v}) + (\boldsymbol{\sigma}_h,\mathbf{v})_{\Gamma_h} + \sum_{K\in\mathcal{C}_h}\left(\mathbf{f}_f,\tau_{\mathbf{v}}(\mathrm{Re}_K)(-\nabla q - 2\lambda\varphi\mu\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{v}))\right)_K \\
&+ (r_f,\phi_f) + (q_{fh},\phi_f)_{\Gamma_h} + \sum_{K\in\mathcal{C}_h}(r_f,\tau_\phi(\mathrm{Pe}_K)(\mathbf{u}_f\cdot\nabla\phi_f - \kappa_f\Delta\phi_f))_K + (r_s,\phi_s)
\end{aligned}
\tag{19}
$$

with the stability parameters $\tau_{\mathbf{v}}$, τ_ϕ and δ defined by (Franca et al., 1992; Franca and Frey, 1992)

$$
\tau_i(X_i) = \frac{h_K}{2|\mathbf{u}_f|_p}\xi(X_i) \quad , \quad \text{with} \;\; i = \mathbf{v},\phi \quad X_{\mathbf{v}} = \mathrm{Re}_K \;,\; X_\phi = \mathrm{Pe}_K \tag{20}
$$

$$
\xi(X_i) = \begin{cases} X_i & , 0 \le X_i < 1 \\ 1 & , X_i \ge 1 \end{cases} \tag{21}
$$

$$
\mathrm{Re}_K = \frac{m_k|\mathbf{u}_f|_p h_K}{4\lambda\varphi\mu} \quad , \quad \mathrm{Pe}_K = \frac{m_k|\mathbf{u}_f|_p h_K}{2\kappa_f} \tag{22}
$$

$$
m_k^i = \min\left\{\frac{1}{3},\; 2\mathbf{C}_k^i\right\} \quad , \quad \text{with} \;\; i = \mathbf{v},\phi \tag{23}
$$

with $|\mathbf{u}_f(\mathbf{x})|_p$ representing the p-norm on $\mathbf{R}^n$, the constants $\mathcal{C}_k^\phi$ and $\mathcal{C}_k^{\mathbf{v}}$ defined as in Franca et al. (1992) and the δ-parameter defined as in Franca and Frey (1992).

Remarks

1. When the stability parameters τ_v, τ_ϕ and δ are made equal to zero in eqs.(17)-(19), classical Galerkin formulation for the problem defined in eqs.(9) is obtained.
2. The usual Reynolds and Péclet grid numbers (Gresho and Chan, 1990) were modified by including the parameter m_k in eqs.(23), to account for the degree of interpolation employed. As a consequence advective-dominated flow regions are characterized by $\mathrm{Re}_K, \mathrm{Pe}_K > 1$ and diffusive-dominated ones by $\mathrm{Re}_K, \mathrm{Pe}_K < 1$, regardless the element considered.

4. NUMERICAL RESULTS

In this section numerical simulations of non-isothermal modified Stokes flow - defined by eqs. (9) and employing the stabilized formulation introduced in eqs. (17)-(19) are presented. Biquadratic *Serendipity* (Q2S) interpolations have been employed to approximate the fluid constituent velocity, the pressure and the fluid and the solid constituents' temperatures.

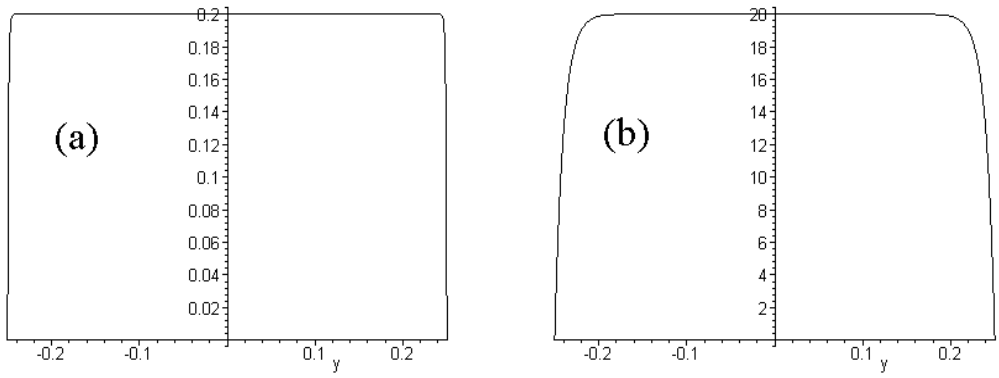

Figure 1. Fluid constituent velocity profiles:
(a) for $Da = 4 \times 10^{-6}$; (b) for $Da = 4 \times 10^{-4}$.

A very simple geometry is now being considered: the Newtonian fluid flows through a horizontal porous channel limited by impermeable and isothermal flat plates. A flat porous channel of aspect ratio $L/H = 2$ (with L denoting its length and H its width), thermal porous diffusivities $\kappa_f = 10^0$, 10^{-3} and 10^{-7} for the fluid constituent and $\kappa_s = 10^1$ for the solid one and the β-coefficient assuming the value $\beta = 10^{-2}$ has been simulated.

The following boundary condition have been employed in the simulation

$$\begin{cases} \mathbf{u}_f = 0 \text{ and } \theta_i = 1 & \text{for } 0 < x < L \ , \ y = \pm H/2 \\ \mathbf{u}_f = \mathbf{u}_f(y) \text{ and } \theta_i = 0 & \text{for } x = 0 \ , \ -H/2 < y < H/2 \\ \boldsymbol{\sigma}_f \cdot \mathbf{n} = 0 \text{ and } \theta_i = 1 \text{ or } \nabla\theta_i \cdot \mathbf{n} = 0 & \text{for } x = L \ , \ -H/2 < y < H/2 \end{cases} \tag{24}$$

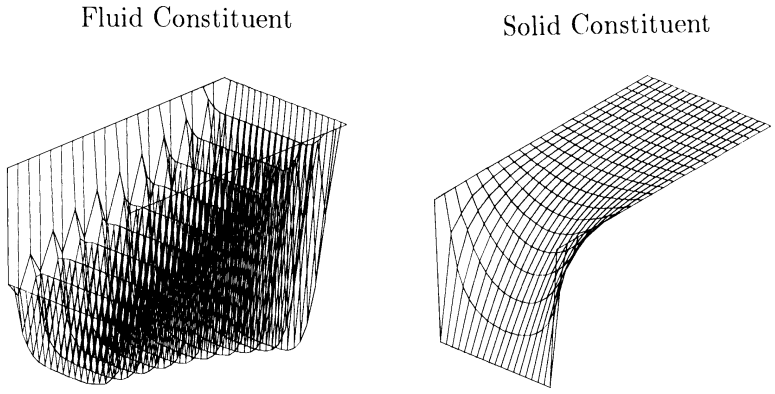

Figure 2. Galerkin method with Dirichlet conditions at the outlet

An almost flat velocity profile was obtained from the numerical approximation of the hydrodynamical problem, considering the classical no-slip condition on the impermeable

flat plates, the classical fully developed flow boundary condition at the inlet and free traction at the outlet, as shown in Figure 1. This profile presented a very good agreement with the analytical result presented in Martins-Costa et al. (1992) besides approaching the classical Darcy's law velocity expression (Bejan, 1987) as the channel width $H \to \infty$.

Taking the centerline velocity as the characteristic flow velocity and fixing the channel width as $H = 0.5$, we have the folowing porous Péclet numbers: *for* $\kappa_F = 1$, $\overline{\mathrm{Pe}}^I = u_F(0)H/\kappa_F = 5 \times 10^{-1}$; *for* $\kappa_F = 10^{-3}$, $\overline{\mathrm{Pe}}^{II} = 5 \times 10^2$; *for* $\kappa_F = 10^{-7}$, $\overline{\mathrm{Pe}}^{III} = 5 \times 10^6$.

Figures 2 and 3 illustrate the finite element approximation employing the stabilized method defined by eqs.(17)-(24) considering a fixed channel geometry, namely $L/H = 2$ and employing a uniform quadrilateral mesh. Elevation plots for $\kappa_F = 10^{-3}$ are shown employing the Galerkin and the stabilized method described in this paper with Dirichlet outflow boundary condition. Since the flow is parallel to the mesh one might conjecture that the Galerkin formulation might work, which was not confirmed here, while the stabilized one points out an excellent pattern with only small oscilations near the channel exit. The poor performance of Galerkin is due to the outflow condition employed that creates an outflow boundary layer which contaminates the Galerkin solution.

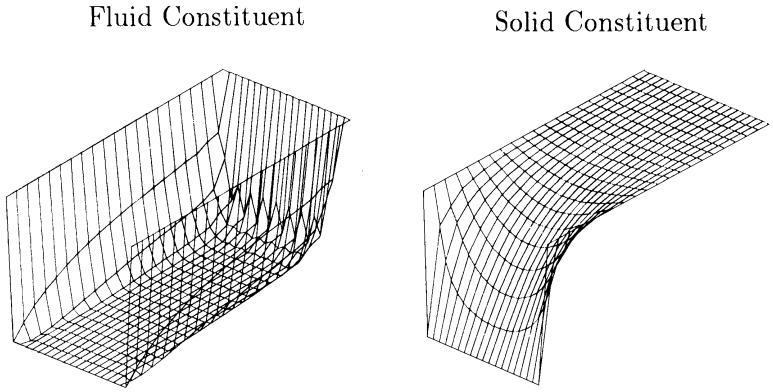

Figure 3. Stabilized method with Dirichlet conditions at the outlet

This methodology has also been tested for Neumann boundary conditions for a diffusive dominated flow (porous thermal diffusivity $\kappa_F = 1$), a high advective dominated flow ($\kappa_F = 10^{-3}$) and a very high advective dominated one, ($\kappa_F = 10^{-7}$). In all situations the finite element method has computed stable θ_f and θ_s-surfaces, indicating that the numerical method employed was capable to generate stable and accurate temperature approximations even for high Péclet flows, $10^3 \leq \overline{\mathrm{Pe}}_K \leq 10^7$.

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