A MECHANICAL MODEL WITH INTERNAL LOOPS FOR PSEUDOELASTIC MATERIALS

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Abstract
The goal of the present paper is to propose a new macroscopic mechanical model for pseudoelastic materials. In particular, we are interested in the description of path dependent internal hysteresis loops detected in experimental observations. The model is written in the framework of one-dimensional media and considers, as state variables, the transformation strain and two hardening parameters, one associated with the austenite-to-martensite transformations and other associated with reverse transformations. Comparisons with experimental data are performed so as to assess the model.

Keywords: Pseudoelasticity, hysteresis, phase transformations.

1. INTRODUCTION

Pseudoelasticity or superelasticity accounts for the ability of certain metallic alloys to recover extremely large strains (of the order of 10%), as a result of stress induced reversible phase transformations from austenite to martensite. Due to their remarkable mechanical properties, superelastic materials have been considered for many applications such as, for instance, eyeglass frames, medical guidewiress, hingeless laparoscopic surgical instruments, and damping devices.

Mechanical models describing the phenomena have been proposed by many authors since the seventies. Ericksen (1975) considered nonmonotonous stress-strain relations for the description of discontinuities in the strain field, which were associated with phase interfaces. Abeyaratne & Knowles (1988) considered kinetic laws describing the motion of phase interfaces subjected to thermodynamic admissibility rules. Alternative models have been proposed by Frémond (1987), Graesser & Cozzarelli (1994), Leclercq & Lexcellent (1996), Auricchio et al. (1997), Souza et al. (1998), Govindjee & Hall (1999), amongst many others.

We propose, within the setting of one-dimensional media subjected to small strains, a new phenomenological model for the description of the mechanical behavior of pseudoelastic materials. The model is strongly influenced by classical descriptions of the elastoplastic behavior, but on the other hand it includes new features as, for instance, its ability to describe, with good accuracy, internal hysteresis loops observed experimentally when the material is subjected to complex loading conditions.

2. THE MECHANICAL MODEL

From the macroscopic point of view, martensitic phase transformations associated with pseudoelastic behavior can be characterized by the *transformation strain* ε_T, which is defined as $\varepsilon_T := \xi_m \varepsilon_M$, where $\xi_m \in [0,1]$ is the volume fraction of the martensite in the medium, while ε_M is the strain observed when the material undergoes a complete transformation from austenite to martensite. The stress σ is supposed to be a linear function of the elastic strain $\varepsilon - \varepsilon_T$:

$$\sigma = E\,(\varepsilon - \varepsilon_T), \tag{1}$$

where E is the Young modulus of the material and ε is the linear measure of total strain.

Transformation from austenite to martensite is associated with the following constraint:

$$f_{AM}(\sigma, \varepsilon_T, \alpha_{AM}) := |\sigma| - [\sigma_T(\varepsilon_T) + \kappa_{AM}(\alpha_{AM})] \le 0, \tag{2}$$

where $\sigma_T(\varepsilon_T) := k\,|\varepsilon_T| + \sigma_{T0}$ is the so called *transformation stress*, k and σ_{T0} are material parameters, while $\kappa_{AM}(\alpha_{AM})$ is a nonlinear function of the *austenite-to-martensite hardening variable* α_{AM}: stress induced phase transformation from austenite to martensite can occur only when the stress σ is such that equality is attained in (2). Otherwise, either elastic behavior or reverse transformation takes place.

On the other hand, transformation from martensite back to austenite is related to the inequality constraint:

$$f_{MA}(\sigma, \varepsilon_T, \alpha_{MA}) := [\sigma_T(\varepsilon_T) - \kappa_{MA}(\alpha_{MA})] - |\sigma| \le 0, \tag{3}$$

where $\kappa_{MA}(\alpha_{MA})$ is a nonlinear function of the *martensite-to-austenite hardening variable* α_{MA}: here again, phase transformation can happen only if equality is verified in (3) or otherwise either elastic behavior or direct transformation occurs.

The constraint (2) makes sense only if there is austenite to be transformed into martensite. Analogously, the constraint (3) is defined only if there is martensite present in the material.

The transformation strain ε_T and the hardening variables α_{AM} and α_{MA} define the set of internal variables of the thermodynamic system which evolves according to the following flow rules:

$$\dot{\varepsilon}_T = \dot{\gamma}_{AM}\frac{\partial f_{AM}}{\partial \sigma} + \dot{\gamma}_{MA}\frac{\partial f_{MA}}{\partial \sigma} = (\dot{\gamma}_{AM} - \dot{\gamma}_{MA})\,\frac{\sigma}{|\sigma|}, \tag{4}$$

$$\dot{\alpha}_{AM} = \dot{\gamma}_{AM} \quad \text{if} \quad \dot{\gamma}_{AM} > 0, \tag{5}$$

$$\dot{\alpha}_{MA} = \dot{\gamma}_{MA} \quad \text{if} \quad \dot{\gamma}_{MA} > 0. \tag{6}$$

where $\dot{\gamma}_{AM}$ and $\dot{\gamma}_{MA}$ are subjected to the following constraints:

$$\dot{\gamma}_{AM} \ge 0, \quad \dot{\gamma}_{AM}\, f_{AM} = 0, \qquad\qquad \dot{\gamma}_{MA} \ge 0, \quad \dot{\gamma}_{MA}\, f_{MA} = 0; \tag{7}$$

Further, state variables α_{AM} and α_{MA} are subjected to the rules:

$$\alpha_{AM} = 0, \quad \text{if} \quad \dot{\gamma}_{MA} > 0, \qquad \text{and} \qquad \alpha_{MA} = 0, \quad \text{if} \quad \dot{\gamma}_{AM} > 0, \tag{8}$$

Such resetting rules are the key point for the description of internal hysteresis loops.

Figure 1 illustrates a schematic stress-strain curve for a pseudoelastic material, as described by our model: let the specimen be loaded from a stress free configuration (point A), and let it be subjected to a strain driven tractive load. Since there is only austenite present in the material ($\varepsilon_T = 0$) and, below point B, strict inequality is verified in (2), the complementarity condition (7) imposes $\dot{\gamma}_{AM} = 0$, i.e., the material behaves elastically along line AB. Equality in (2) is verified along line BC and, from the same constraints (7), the consistency parameter $\dot{\gamma}_{AM}$ is allowed to attain values distinct from zero. Therefore, we conclude from (4) and (5) that the state variables ε_T and α_{AM} can evolve, i.e., phase transformation from austenite to martensite takes place. Between points C and D, as the specimen is unloaded, strict inequalities are observed in both expressions (2) and (3). Thus, from (4) and (5) we conclude that both $\dot{\gamma}_{AM}$ and$\dot{\gamma}_{MA}$ are equal, i.e., the material behaves elastically. Equality of (3) is observed along line DE, meaning that we can have $\dot{\gamma}_{MA} > 0$ and hence transformation of martensite back to austenite. If, at point E, the specimen is subjected to a new increase in the prescribed strain, then the material behaves elastically between points E and F. From F to G, tranformation from austenite to martensite is once more observed. It should be remarked that both direct and reverse transformations starts whenever the stress-strain curve crosses the dotted line corresponding to the transformation stress σ_T. This is justificated by the fact that, from conditions (8), the hardening variables α_{AM} and α_{MA} are reset to zero whenever ($\dot{\gamma}_{MA} > 0$) and ($\dot{\gamma}_{MA} > 0$), respectively.

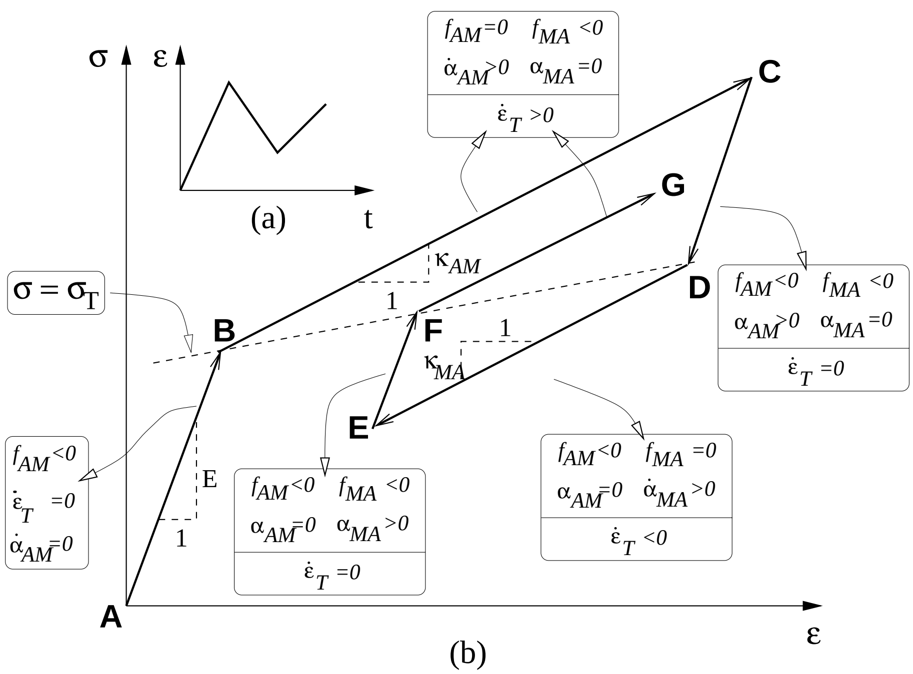

Figure 1: When subjected to the strain history in (a), the proposed constitutive model defines the stress-strain curve in (b).

3. NUMERICAL ASPECTS

The time discrete version of equations (1–8) is obtained via a *backward Euler* scheme and can be written as:

(i) *Stress-strain relation*:

$$\sigma_{n+1} := E\,(\varepsilon_{n+1} - \varepsilon_{T\,n+1}); \tag{9}$$

(ii) *Elastic domain*:

(ii.1) If $\varepsilon_{T\,n} = 0$ (austenite):

$$f_{AM\,n+1} := |\sigma_{n+1}| - [\sigma_T(\varepsilon_{T\,n+1}) + \kappa_{AM}(\alpha_{AM\,n+1})] \leq 0; \tag{10}$$

(ii.2) If $|\varepsilon_{T\,n}| = \varepsilon_M$ (martensite):

$$f_{MA\,n+1} := [\sigma_T(\varepsilon_{T\,n+1}) - \kappa_{MA}(\alpha_{MA\,n+1})] - |\sigma_{n+1}| \leq 0; \tag{11}$$

(ii.3) If $0 < |\varepsilon_{T\,n+1}| < \varepsilon_M$ (mixture of phases):

$$f_{AM\,n+1} \leq 0 \quad \text{and} \quad f_{MA\,n+1} \leq 0; \tag{12}$$

(iii) *Flow rules*:

$$\varepsilon_{T\,n+1} = \varepsilon_{T\,n} + (\Delta\gamma_{AM\,n+1} - \Delta\gamma_{MA\,n+1})\frac{\sigma_{n+1}}{|\sigma_{n+1}|}, \tag{13}$$

$$\alpha_{AM\,n+1} = \alpha_{AM\,n} + \Delta\gamma_{AM\,n+1} \quad \text{if} \quad \Delta\gamma_{AM\,n+1} \geq 0, \tag{14}$$

$$\alpha_{MA\,n+1} = \alpha_{MA\,n} + \Delta\gamma_{MA\,n+1} \quad \text{if} \quad \Delta\gamma_{MA\,n+1} \geq 0, \tag{15}$$

where $\Delta\gamma_{AM\,n+1} := \dot{\gamma}_{AM}\,(t_{n+1} - t_n)$ and $\Delta\gamma_{MA\,n+1} := \dot{\gamma}_{MA}\,(t_{n+1} - t_n)$ are subjected to contraints:

$$\Delta\gamma_{AM\,n+1} \geq 0, \quad \Delta\gamma_{AM\,n+1} f_{AM\,n+1} = 0, \tag{16}$$

$$\Delta\gamma_{MA\,n+1} \geq 0, \quad \Delta\gamma_{MA\,n+1} f_{MA\,n+1} = 0, \tag{17}$$

(iv) *Hardening resetting rules*:

$$\alpha_{AM\,n+1} = 0 \quad \text{if} \quad \Delta\gamma_{MA\,n+1} > 0, \tag{18}$$

$$\alpha_{MA\,n+1} = 0 \quad \text{if} \quad \Delta\gamma_{AM\,n+1} > 0. \tag{19}$$

The integration of the aforementioned set of equations is described by the algorithm below. Details on its derivation can be found in Ferreira et al. (2000).

ALGORITHM:

- Compute a *trial state* at time instant t_{n+1}:

$$\varepsilon^{trial}_{T\,n+1} := \varepsilon_{T\,n}, \quad \alpha^{trial}_{AM\,n+1} := \alpha_{AM\,n} \quad \text{and} \quad \alpha^{trial}_{MA\,n+1} := \alpha_{MA\,n}. \tag{20}$$

- Compute the corresponding *trial stress state* and the corresponding *trial yield functions*:

$$\sigma^{trial}_{n+1} := E\,(\varepsilon_{n+1} - \varepsilon^{trial}_{T\,n+1}), \tag{21}$$

$$f^{trial}_{AM\,n+1} := |\sigma^{trial}_{n+1}| - [\sigma_T(\varepsilon^{trial}_{T\,n+1}) + \kappa_{AM}\alpha^{trial}_{AM\,n+1}] \tag{22}$$

$$f^{trial}_{MA\,n+1} := [\sigma_T(\varepsilon^{trial}_{T\,n+1}) - \kappa_{MA}\alpha^{trial}_{MA\,n+1}] - |\sigma^{trial}_{n+1}| \tag{23}$$

- If $\big(|\varepsilon_{T\,n}| = 0$ and $f^{trial}_{AM\,n+1} \le 0\big)$, $\big(|\varepsilon_{T\,n}| = \varepsilon_M$ and $f^{trial}_{MA\,n+1} \le 0\big)$ or $\big(0 < |\varepsilon_{T\,n}| < \varepsilon_M,\ f^{trial}_{AM\,n+1} \le 0$ and $f^{trial}_{MA\,n+1} \le 0\big)$, then (*elastic step*):

$$\varepsilon_{T\,n+1} = \varepsilon_{T\,n}, \quad \alpha_{AM\,n+1} = \alpha_{AM\,n} \quad \text{and} \quad \alpha_{MA\,n+1} = \alpha_{MA\,n}. \tag{24}$$

else:

  - If $0 \le |\varepsilon_{T\,n}| < \varepsilon_M$, $f^{trial}_{AM\,n+1} > 0$ and $f^{trial}_{MA\,n+1} < 0$ (*transformation from austenite to martensite*):

    * Compute $\Delta\gamma_{AM\,n+1}$ from the nonlinear equation:

$$|\sigma^{trial}_{n+1}| - E\,\Delta\gamma_{AM\,n+1}\Bigg[\sigma_T\left(\varepsilon_{T\,n} + \Delta\gamma_{AM\,n+1}\frac{\sigma^{trial}_{n+1}}{|\sigma^{trial}_{n+1}|}\right) + \kappa_{AM}(\alpha_{AM\,n} + \Delta\gamma_{AM\,n+1})\Bigg] = 0, \tag{25}$$

    * Compute the new state variables:

$$\varepsilon_{T\,n+1} = \varepsilon_{T\,n} + \Delta\gamma_{AM\,n+1}\frac{\sigma^{trial}_{n+1}}{|\sigma^{trial}_{n+1}|}, \tag{26}$$

$$\alpha_{AM\,n+1} = \alpha_{AM\,n} + \Delta\gamma_{AM\,n+1}, \tag{27}$$

$$\alpha_{MA\,n+1} = 0. \tag{28}$$

  - else $\big(0 < |\varepsilon_{T\,n}| \le \varepsilon_M,\ f^{trial}_{AM\,n+1} < 0$ and $f^{trial}_{MA\,n+1} > 0\big)$ (*transformation from martensite to austenite*):

    * Compute $\Delta\gamma_{MA\,n+1}$ from the nonlinear equation:

$$\Bigg[\sigma_T\left(\varepsilon_{T\,n} - \Delta\gamma_{MA\,n+1}\frac{\sigma^{trial}_{n+1}}{|\sigma^{trial}_{n+1}|}\right) - \kappa_{AM}(\alpha_{AM\,n} + \Delta\gamma_{AM\,n+1})\Bigg] - |\sigma^{trial}_{n+1}| - E\,\Delta\gamma_{AM\,n+1} = 0, \tag{29}$$

    * Compute the new state variables:

$$\varepsilon_{T\,n+1} = \varepsilon_{T\,n} + \Delta\gamma_{MA\,n+1}\frac{\sigma^{trial}_{n+1}}{|\sigma^{trial}_{n+1}|}, \tag{30}$$

$$\alpha_{MA\,n+1} = \alpha_{MA\,n} + \Delta\gamma_{MA\,n+1}, \tag{31}$$

$$\alpha_{AM\,n+1} = 0. \tag{32}$$

4. ASSESMENT OF THE MODEL

Next, we compare our model with experimental data reported by Sittner et al. (1995), who performed studies on the mechanical behavior of Cu 80% - Al 10% - Zn 5% - Mn 5% industrial polycrystalline shape memory alloy at temperature $A_f + 25°K (285°K)$. Figures 2 to 4 present stress-strain curves for tensile tests where the specimens were subjected to three distinct loading-unloading histories. We considered, for these numerical simulations, the following expressions for the nonlinear functions $\kappa_{AM}(\alpha_{AM})$ and $\kappa_{MA}(\alpha_{MA})$:

$$\kappa_{AM}(\alpha_{AM}) := a_{AM}\alpha_{AM} + b_{AM}\left[1 - exp(-c_{AM}\alpha_{AM})\right], \tag{33}$$

$$\kappa_{MA}(\alpha_{MA}) := a_{MA}\alpha_{MA} + b_{MA}\left[1 - exp(-c_{MA}\alpha_{MA})\right], \tag{34}$$

respectively, where a_{AM}, b_{AM}, c_{AM}, a_{MA}, b_{MA} and c_{MA} are material parameters. The following values of material properties were considered: $E = 30.7\ GPa$, $\sigma_{T0} = 120\ MPa$, $k = 18\ GPa$, $a_{AM} = 10\ GPa$, $b_{AM} = 45\ MPa$, $c_{AM} = 1600$, $a_{MA} = 10\ GPa$, $b_{MA} = 38\ MPa$, $c_{MA} = 1600$.

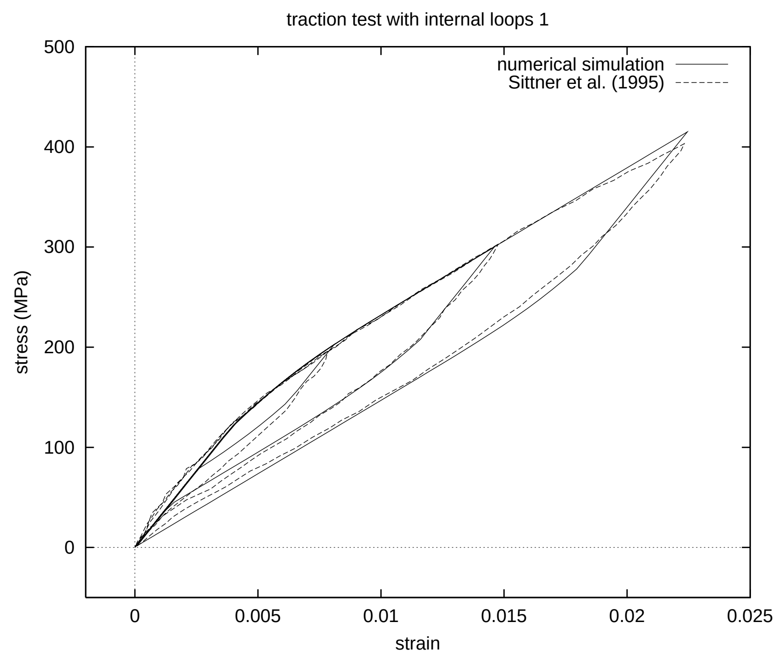

Figure 2: Traction test on Cu-Zn-Al-Mn polycrystalline specimen with distinct values of maximum strain on each cycle. Numerical simulation based on our model and experimental results from Sittner et al. (1995).

Numerical results are consistently in good agreement with experimental data. For the three cases of loading-unloading histories, the internal hysteresis loops could be replicated, both qualitatively and quantitatively.

5. CONCLUDING REMARKS

A macroscopic model capable to describe the mechanical behavior of pseudoelastic materials is presented in this paper. Most of its ingredients are inherited, from the formal point of view, from classical plasticity. Due to the resetting rules considered in the model, complex hysteresis patterns can be replicated by the model with good accuracy.

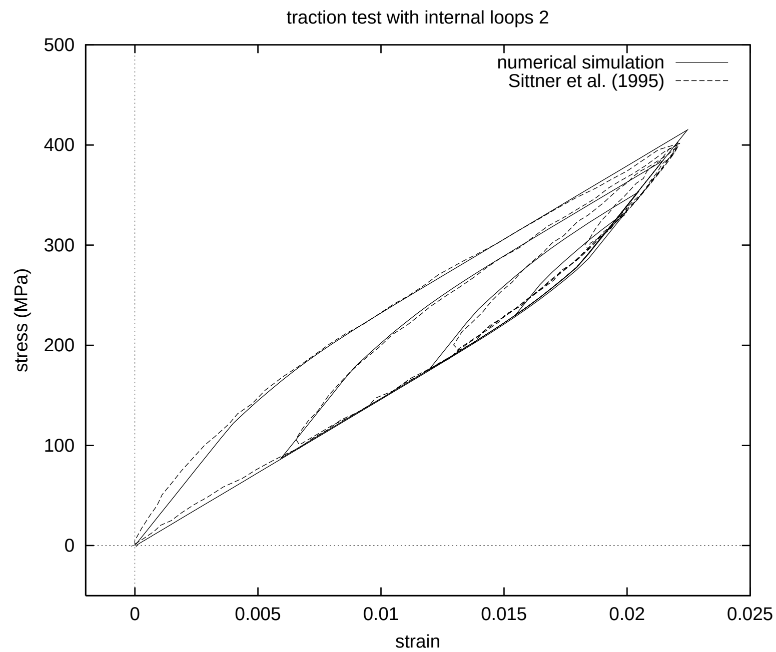

Figure 3: Traction test on Cu-Zn-Al-Mn polycrystalline specimen with distinct values of minimum strain on each cycle. Numerical simulation based on our model and experimental results from Sittner et al. (1995).

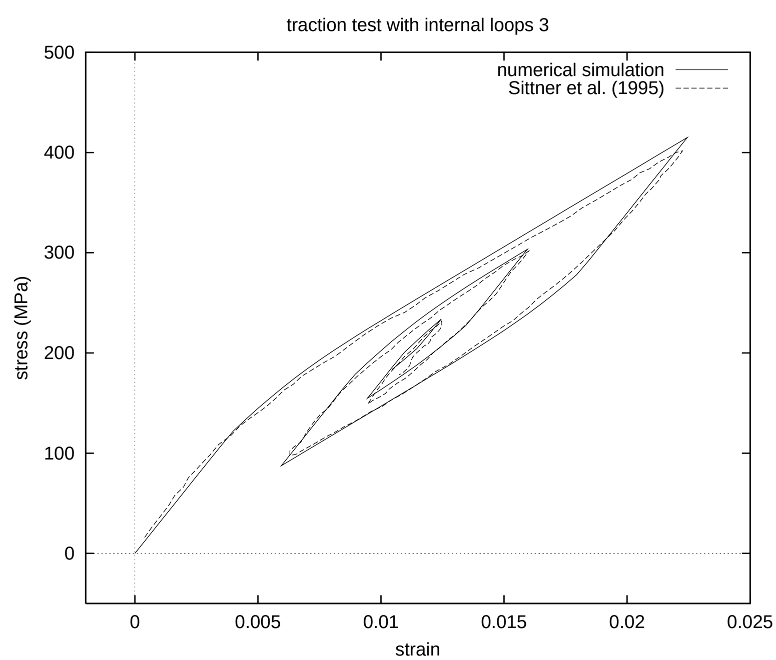

Figure 4: Traction test on Cu-Zn-Al-Mn polycrystalline specimen with decreasing values of maximum strain and increasing values of minimum strain prescribed along the loading-unloading cycles. Numerical simulation based on our model and experimental results from Sittner et al. (1995).

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