ON POLAR DECOMPOSITION AND SUM DECOMPOSITION OF NONLINEAR GEOMETRIC FIELD THEORY OF CONTINUUM MECHANICS

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Abstract

This work addresses the fundamentals of nonlinear geometric field theory of continuum mechanics. Two decomposition theories: the classical *polar decomposition* (**F**=**RU**=**VR**) and the *sum decomposition* (**F**=**S**[strain]+**R**[rotation]) for the deformation gradient (**F**) are introduced. Those two theorems both give definitions for finite strain and local rotation. However the strain defined by the classical polar decomposition in the initial configuration of an embedding co-moving coordinate reference system is non-unique due to the non-commutative property of matrix products but the one defined by the sum decomposition in the deformed configuration is unique. The conception of rotation in polar decomposition is referred to the rigid body rotation of a mass particle while the one in sum decomposition is related to the curl of the particle. The non-unique strain and rigid body rotation conception is contributed to that the classical polar decomposition theorem is set in mathematics but not in physics of deformation. Some works of successfully applying the sum decomposition theorem have been mentioned at the end of this paper. This work suggests that the sum decomposition theorem will show its especial value in bio-mechanics field where large deformation and finite rotation are generally involved.

Keywords: Non-linear continuum mechanics, Polar decomposition, Sum decomposition, Large deformation and finite rotation, Co-moving coordinate system

1. INTRODUCTION

In non-linear continuum mechanics, to separate deformation from rotation out of a given displacement field is an important study subject, as rotation would result large and unreal strain. It is well known that when a body deforms, each one of the small segments passing through a point in the body will be stretched and rotate. For a large displacement field, which is generally involved in finite rotation, the formula for small strain is not applicable, therefore, definition for large strain must be given reasonably. Until now, there remain only three definitions of finite strain which are rigorous in mathematical character, they are the finite strains (a) defined by metric tensor, the so called Green's strain, (b) defined by polar decomposition theorem (Truesdell & Noll, 1965) and (c) defined by the sum decomposition theorem (Chen, 1979).

As for Green's strain, the finite strain and finite rotation are defined separately, it is not compatible in mathematical sense. Moreover, Green's strain is not suitable for engineering measure due to the length variation of a small segment emerges in quadratic form $(ds/ds_0)^2$.

Truesdell and Noll (1965) presented a method to decompose strain and rotation from a given deformation gradient field, the so called classical polar decomposition. The presentation of the decomposition is based only on mathematics consideration i.e. the deformed configuration of a mass particle can be achieve by a pure stretch transformation proceeding to a rigid rotation transformation or vice versa. This decomposition unavoidably leads two distinct strains respectively to strain pre-rotation or rotation pre-strain. Generally, people applies the polar decomposition without questioning its physical reality. Almost all of deformation analysis software has applied the polar decomposition to solve large deformation problems.

Chen (1979) proposed a new decomposition theorem, named sum decomposition theorem, in which strain and local rotation occur at same time without order and the strain is determined uniquely. Moreover, local rotation is simply and analytically given, which is relative to the curl of the particle. Since then, some examples successfully applying the sum decomposition theorem have been reported sequentially.

At the beginning of this paper, the polar decomposition and the sum decomposition are introduced respectively. To show the difference between those two decompositions, four simple examples of finite deformation in plane are given. By comparison, the merits of the sum decomposition is shown. In the later part of the paper, the performance comparison between the software based on the sum decomposition and the ADINA based on the polar decomposition is given (Li & Chen, 1994). The results show that the software based on the S-R decomposition is much more efficient than the ADINA, even though no significant difference in the displacement magnitudes respectively obtained by the software and by the ADINA was found.

2. POLAR DECOMPOSITION AND SUM DECOMPOSITION

To describe any degree of large deformation and rotation of a deformable body, double coordinate systems are usually required. One is a fixed system X^i (inertial system) and the other is a co-moving system x^i (or natural system) which is embedded in the body. Let $\overset{\circ}{\mathbf{g}}_i$ be the local basis vectors of co-moving system before deformation; it changes to $\mathbf{g}_i$ after deformation (Fig. 1). The transformation $\overset{\circ}{\mathbf{g}}_i \rightarrow \mathbf{g}_i$ is realized in the form:

$$\mathbf{g}_i = F_i^{\,j}\, \overset{\circ}{\mathbf{g}}_j = \frac{\partial X^j}{\partial x^i}\, \overset{\circ}{\mathbf{g}}_j, \qquad F_i^{\,j} = \delta_i^{\,j} + u^j\big|_i \qquad (i,\ j = 1,\ 2,\ 3) \tag{1}$$

where $F_i^{\,j}$ is a function of deformation gradient and $u^j\big|_i$ denotes the covariant derivation of displacement component u^j, which is defined in the initial co-moving system $\overset{\circ}{\mathbf{g}}_i$, with respect to co-moving coordinate x^i.

2.1 Polar decomposition

In polar decomposition, $u^j\big|_i$ becomes conventional derivation of displacement component u^j i.e. u_i^j, and $x^i = X^i$ (the coordinates of the fixed system). The polar decomposition says that the deformation gradient can always be decomposed into a product of

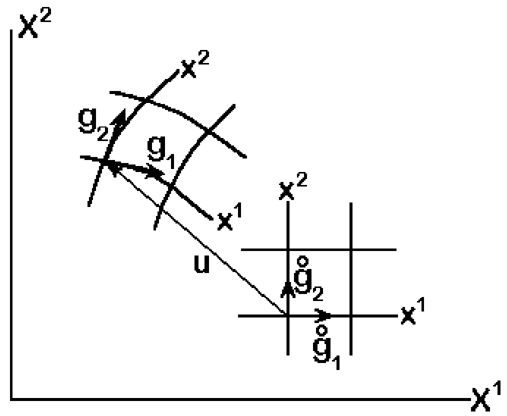

Figure 1 Transformation of the basis vectors of co-moving coordinates system before and after deformation

two matrices, a symmetric stretch matrix ***U*** or ***V*** and an orthogonal matrix ***R*** corresponding to a rigid principal axes rotation i.e.

$$\boldsymbol{F}=\boldsymbol{RU}=\boldsymbol{VR} \tag{2}$$

Equation (2) is interpreted in that the total deformation is obtained by first applying the stretch and then rotation (respect to the first equality) or vice versa (to the second equality). We see that this decomposition unavoidably leads two distinct stretch strains, in other words, the decomposition is non-unique. However, real deformation is that stretch and rotation occur at same time, no order is involved. Therefore Eq. (2) is only resulted from mathematical consideration instead from physics. Apart from the non-unique strain, the computation for the strain and rotation is quite complicate because no analytic formulae are found.

2.2 Sum decomposition (S-R)

The sum decomposition theorem (Chen, 1979) proved that: for a physically possible transformation induced by a deformable body point set, $F_i^{\,j}$ can be decomposed into a summed representation of a symmetrical transformation and an orthogonal transformation i.e.

$$F_i^{\,j} = S_i^j + R_i^j \tag{3}$$

where S_i^j and R_i^j are the components of strain tensor and rotation tensor respectively, which are determined as

$$S_j^i = \frac{1}{2}(u^i\big|_j + u^i\big|_j^T) - (1-\cos\vartheta)L_k^i L_j^k \tag{4}$$

$$R_j^i = \delta_j^i + L_j^i \sin\vartheta + (1-\cos\vartheta)L_k^i L_j^k \tag{5}$$

$$L_j^i = \omega_j^i / \sin\vartheta, \qquad \omega_j^i = \frac{1}{2}(u^i\big|_j - u^i\big|_j^T) \tag{6}$$

In the above $\delta_i^{\,j}$ is Kronecker identity tensor and L^i_j is a two order anti-symmetric tensor dual to the direction vector $\boldsymbol{l}$ of the rotation axis of local rotation, which is defined as

$$\boldsymbol{l} = \frac{1}{2\sin\vartheta}\,\mathrm{rot}\boldsymbol{u} \tag{7}$$

Equation (7) relates the local rotation angle ϑ to the curl of a mass particle, which is evaluated by

$$\begin{aligned}\vartheta &= \pm\arcsin(-\omega^i_j\omega^j_i)^{1/2} \\ &= \pm\arcsin\left\{\frac{1}{2}\sqrt{(u^1\big|_2 - u^1\big|_2^T)^2 + (u^2\big|_3 - u^2\big|_3^T)^2 + (u^3\big|_1 - u^3\big|_1^T)^2}\right\}\end{aligned} \tag{8}$$

The positive sign in Eq. (8) is used for counterclockwise rotation. For practical computation, the tensor components must be changed into physical components. Let $\bar{u}^j\big|_i$ denote the physical component of $u^j\big|_i$, then we have (Chen, 1988)

$$\bar{u}^j\big|_i = \sqrt{\frac{\mathring{g}_{(jj)}}{g_{(ii)}}}\,u^j\big|_i \tag{9}$$

(ii), (jj) indicate no sum over the double index, and the local rotation angle should be

$$\vartheta = \pm\arcsin\left\{\frac{1}{2}\sqrt{(\bar{u}^1\big|_2 - \bar{u}^1\big|_2^T)^2 + (\bar{u}^2\big|_3 - \bar{u}^2\big|_3^T)^2 + (\bar{u}^3\big|_1 - \bar{u}^3\big|_1^T)^2}\right\} \tag{10}$$

Let the co-moving system before deformation be coincident with the fixed system, then

$$\mathring{g}_{ij} = \mathring{\mathbf{g}}_i\cdot\mathring{\mathbf{g}}_j = \delta_{ij} \tag{11}$$

and covariant derivation is identical to conventional derivation, i.e.

$$u^j\big|_i = \frac{\partial u^j}{\partial x^i} \tag{12}$$

Hence

$$\bar{u}^j\big|_i = \frac{1}{\sqrt{g_{(ii)}}}\frac{\partial u^j}{\partial x^i} \tag{13}$$

It is especially noted that ϑ is different from the rigid body rotation described by the polar decomposition theorem. When a body deforms, in general, the rotation of each line segment passing through a point differs from another line segment. So ϑ represents the mean rotation effect of all line segments passing through the point, a non-simple arithmetic mean effect. It might be said that ϑ scales the state of local rotation. As the advantages of mathematical uniqueness and physical reality, the S-R decomposition theorem has been used widely (Qin & Chen, 1988; Chen, 1989; Shang & Chen, 1989; Chen, 1989; Wang & Chen, 1991; Li & Chen 1994; Chen & Liu, 1995).

For a plane problem the physical components of strain tensor can be written as

$$\bar{S}_1^1 = \frac{\partial u}{\partial s_x} + (1 - \cos\vartheta) \tag{14}$$

$$\bar{S}_2^2 = \frac{\partial v}{\partial s_y} + (1 - \cos\vartheta) \tag{15}$$

$$\bar{S}_2^1 = \bar{S}_1^2 = \frac{1}{2}\left(\frac{\partial v}{\partial s_x} + \frac{\partial u}{\partial s_y}\right) \tag{16}$$

and local rotation angle is defined by

$$\vartheta = \pm \arcsin\left[\frac{1}{2}\left(\frac{\partial v}{\partial s_x} - \frac{\partial u}{\partial s_y}\right)\right] \tag{17}$$

where s_x and s_y are the arc lengths respectively along co-moving coordinate lines $x^1 = x$ and $x^2 = y$.

3. EXAMPLES OF PLANE FINITE DEFORMATION

Table 1 exhibits four examples of plane finite deformation to help ones to understand, by comparison, the finite strains and local rotations defined respectively by the polar decomposition theorem and the strain-rotation decomposition theorem. It should be noted that the finite strain defined by the strain-rotation decomposition is measured in the deformed configuration but the one by the polar decomposition is measured in the undeformed configuration, which has been seen clearly in the examples of uniaxial tensile and homogeneous dilation. In the later the transformation carries a square A: 3×3 to A': $4\sqrt{2} \times 4\sqrt{2}$ (Fig. 2), the dimension change: $\frac{l - l_0}{l} = \frac{4\sqrt{2} - 3}{4\sqrt{2}}$ shown as in the strain defined by the strain-rotation decomposition or $\frac{l - l_0}{l_0} = \frac{4\sqrt{2} - 3}{3}$ defined by the polar decomposition. For the fourth example, we obtain two distinct strain tensors from the polar decomposition. Which one is correct? If both of them is correct, then the strain energy per unit volume is unique?

Based on the sum decomposition, Li & Chen developed the so called UC software, then, used it to evaluate the displacement of a cantilever beam respectively subjected to a concentrated force at the free end or to a uniformly distributed load. They compared the efficiency of the UC with the ADINA. The displacements obtained from the UC and the ADINA are quite close, but the total number of time increment steps required by the ADINA is sixteen times the one by the UC. The detailed comparison is shown in Table 2. From the table, one can see that the UC software has less time expense in computation than the ADINA. It would be understood that the sum decomposition is more reasonable than the polar decomposition.

Table 1 Comparison of S-R and RU-VR (polar decomposition) measurements. $(\bar{x},\, \bar{y})$ is the transformed coordinates of a material point $(x,\, y)$ in the fixed system $\{x,\, y\}$; $\left[\varepsilon'_{ij}\right]_U$, the strain determined by the right stretch tensor $\boldsymbol{U}$ and $\left[\varepsilon'_{ij}\right]_V$, the one by the left stretch tensor $\boldsymbol{V}$, where $\boldsymbol{U}$ and $\boldsymbol{V}$ are defined by the polar decomposition theorem.

Example	Transformation function	Strain and rotation of S-R decomposition	Strain and rotation of polar decomposition
Uniaxial tensile	$\bar{x} = (1+\lambda)x$ $\bar{y} = y$	$\left[S^i_j\right] = \begin{bmatrix} \dfrac{\lambda}{1+\lambda} & 0 \\ 0 & 0 \end{bmatrix}$ $\vartheta = 0$	$\left[\varepsilon'_{ij}\right] = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}$ $\theta = 0$
Pure shear	$\bar{x} = x + tg\gamma \cdot y$ $\bar{y} = tg\gamma \cdot x + y$	$\left[S^i_j\right] = \begin{bmatrix} 0 & sin\gamma \\ sin\gamma & 0 \end{bmatrix}$ $\vartheta = 0$	$\left[\varepsilon'_{ij}\right] = \begin{bmatrix} 0 & tg\gamma \\ tg\gamma & 0 \end{bmatrix}$ $\theta = 0$
Large rotation and dilation of a square	$\bar{x} = \frac{4}{3}x - \frac{4}{3}y$ $\bar{y} = \frac{4}{3}x + \frac{4}{3}y$	$\left[S^i_j\right] = \begin{bmatrix} \dfrac{4\sqrt{2}-3}{4\sqrt{2}} & 0 \\ 0 & \dfrac{4\sqrt{2}-3}{4\sqrt{2}} \end{bmatrix}$ $\vartheta = 45°$	$\left[\varepsilon'_{ij}\right] = \begin{bmatrix} \dfrac{4\sqrt{2}-3}{3} & 0 \\ 0 & \dfrac{4\sqrt{2}-3}{3} \end{bmatrix}$ $\theta = 45°$
Large rotation and distortion of a square	$\bar{x} = \frac{1}{2}x - \frac{3}{2}y$ $\bar{y} = \frac{5}{4}x + y$	$\left[S^i_j\right] = \begin{bmatrix} 0.1541 & 0.0482 \\ 0.0482 & 0.5255 \end{bmatrix}$ $\vartheta = 61.67°$	$\left[\varepsilon'_{ij}\right]_U = \begin{bmatrix} 0.3368 & 0.1596 \\ 0.1596 & 0.7965 \end{bmatrix}$ $\left[\varepsilon'_{ij}\right]_V = \begin{bmatrix} 0.5627 & -0.2793 \\ -0.2793 & 0.5627 \end{bmatrix}$ $\theta = 61.39°$

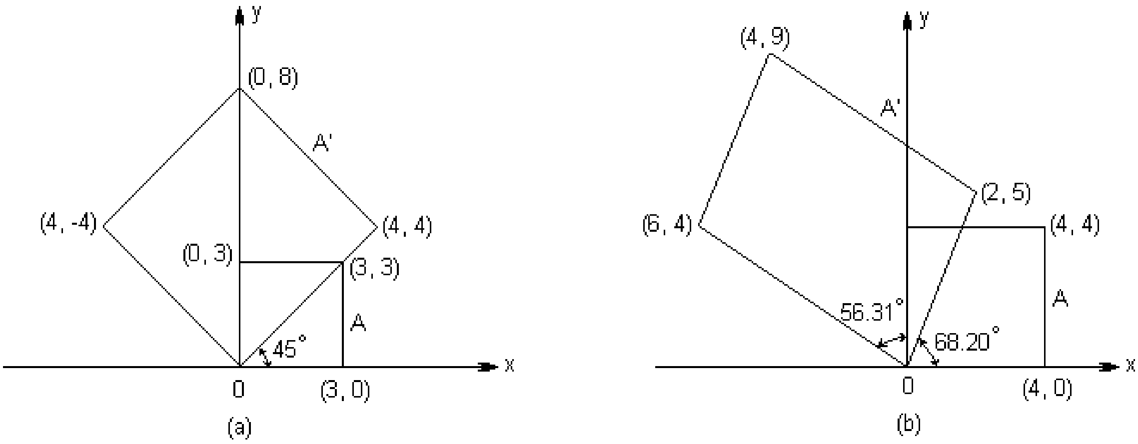

Fig. 2 (a) Large rotation and dilation of a square; (b) large rotation and distortion of a square.

Table 2 Comparison of efficiency of the UC software with the ADINA (Li & Chen, 1994); all the calculation were performed in double precision on a VAX 11/780 computer.

Cantilever beam subjected to a concentrated force at the free end				Cantilever beam under a uniformly distributed load			
Program	Load steps	No. Of iterations	CPU time (s)	Program	Load steps	No. Of iterations	CPU time (s)
UC	10	57	99	UC	10	61	99
ADINA	160	1172	900	ADINA	150	1040	673

4. DISCUSSION

As being set on mathematics consideration instead of physics, the polar decomposition only gives non-unique strains. The rigid rotation conceptually described by the polar decomposition does not exit in the process of deformation. The local rotation and stretch of a deformable particle occur always at same time without the order of stretch pre-rotation or rotation pre-stretch. Although the polar decomposition has used widely, its reasonableness in physics should be studied. Perhaps this question would be clarified when strain energy evaluation is required. Owing to its reasonableness in physics and simplicity in computation, the sum decomposition will obtain more application especially in bio-mechanics field where large deformation and rotation are generally involved.

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