FRACTURE AND FATIGUE ANALYSIS USING A GRADIENT-ENHANCED CONTINUUM DAMAGE THEORY

Heraldo S. Costa Mattos, Stella Maris Pires-Domingues
Laboratório de Mecânica Teórica e Aplicada, Departamento de Engenharia Mecânica, Universidade Federal Fluminense - UFF, Niterói - RJ - Brasil - 24210-240
Fulvio Giacomo Chimisso
Departamento de Materiais de Construção, Fundação Universidade do Rio Grande - FURG, Rio Grande - RS - Brasil - 96200-210
Fernando Alves Rochinha
Departamento de Engenharia Mecânica, Universidade Federal do Rio de Janeiro - EE-COPPE/UFRJ, CP 68503, Rio de Janeiro - RJ - Brasil - 21945-970

Abstract

This paper discuss the possibility of structural failure prediction using a theory of continuum media with microstructure. The proposed theory allows an adequate description of the strain-softening and localization behaviors due to the material degradation. This theory is similar to some other continuum damage theories for elastic materials that introduce higher order gradients of the damage variable in the constitutive model in order to avoid the ill-posedness in the post-localization range. The possibility and main features of such kind of approach are discussed through examples concerning elastic-plastic and brittle-elastic behaviors.

Keywords: Continuum Damage Mechanics; Fracture, Fatigue.

1- INTRODUCTION

Continuum Damage Mechanics uses a phenomenological approach to model the effect of microscopic geometric discontinuities induced by the deformation process (micro-cracks, micro-voids, so on) on the macroscopic behaviors of a structure. In continuum damage theories an internal variable related to the growth and coalescence of micro-defects before the macroscopic crack initiation (whose definition and physical interpretation may vary form one model to the other. Therefore, the problem becomes to establish the constitutive relations for the damage variable as a function of the other state variables.

Many different continuum damage theories have been proposed to describe the degradation process in elastic brittle materials. The local damage theories, Kachanov (1986) and Lemaitre *et al* (1990), often lead to a physically unrealistic description of strain localization phenomena. In general, due to the loss of ellipticity of the governing equations in the post-localization range, the resulting mathematical problems may present an infinite number of solutions with discontinuous fields of displacement gradients what leads to numerical difficulties of mesh-dependence, Knowles *et al*. (1978), Bazant *et al*. (1988) and Vree *et al* (1995). In order to avoid the loss of well-posedness in the post-localization range, some alternative non-local approaches were proposed, Costa-Mattos e*t al* (1992) and (1995), Frémond e*t al* (1996) and Domingues (1996), for example. Some of these alternative theories introduce higher order gradients of the damage variable in the constitutive model. Although such theory allow a mathematically correct modeling of the strain localization phenomena, they are usually considered very complex to handle from the numerical point of view.

The present paper deals with an alternative gradient enhanced theory in which the continuum is supposed to possess a microstructure. In the resulting nonlinear mathematical problems the coupling between damage and strain is circumvented by means of a splitting technique which allows to solve the nonlinear problem through a sequence of simpler linear problems, Domingues (1996). To demonstrate the capability of the proposed model some examples involving elastic and elastic-plastic problems are exploited.

2- MODELING - BASIC BALANCE EQUATIONS

A body is defined as a set of material points B which occupies a region Ω of the Euclidean space at the reference configuration. In this theory, besides the classical variables that characterize the kinematics of a continuum medium (displacements and velocities of material points), an additional scalar variable $\beta \in [0,1]$, is introduced. This variable is related with the links between material points and can be interpreted as a measure of the local cohesion state of the material. If $\beta = 1$, all the links are preserved and the initial material properties are preserved. If $\beta = 0$ a local rupture is considered since all the links between material points have been broken. The variable β is associated to the damage variable D by the following relation: $\beta = 1 - D$. Since the degradation is an irreversible phenomena, the rate $\dot{\beta}$ must be negative or equal to zero.

In this section are presented a summary of the basic balance equations of the modeling. Here, the appropriate conservation laws that governing the evolution of a continuous damageable body are postulated. To simplify the presentation, non mechanical external actions that may affect the cohesion of the material points are neglected. A more detailed discussion about the basic principles can be found in Costa-Mattos *et al* (1995).

The main difference from other damage theories are some additional terms in the balance of energy equations and a additional conservation law for the microscopic forces associated to β must be postulated. Thus, besides the classical balance relations for linear momentum and angular momentum, the evolution is supposed to be governed by the following local balance relations:

$$\operatorname{div} \mathbf{H} - F = 0 \tag{1}$$

$$\rho\dot{e} = \boldsymbol{\sigma}.\nabla\dot{\mathbf{u}} + \mathbf{H}.\nabla\dot{\beta} + F\dot{\beta} - \operatorname{div} \mathbf{z} + \rho r \tag{2}$$

where ρ is the density, $\boldsymbol{\sigma}$ is the stress tensor, e is the specific internal energy, $\mathbf{u}$ is the displacement, $\mathbf{z}$ is the heat flux vector, $\mathbf{H}$ is the microscopic internal force related to $\nabla\beta$, F is the microscopic internal force related to β and r is a heat source per mass and time. The equation (1) is the balance of microscopic forces and the equation (2) is the balance of energy.

3- CONSTITUTIVE EQUATIONS

The balance equation and the second law restriction are valid for any kind of process. A complete modeling requires additional information in order to characterize the behavior of each kind of material. In this section it is presented a general constitutive theory for damageable materials.

State variables: Under the hypothesis of small deformations and isothermal process, the local state of a damageable material is supposed to be a function of the total strain $\boldsymbol{\varepsilon} = \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right]$, of the cohesion variable β and its gradient $\nabla\beta$. To complete the presentation of the constitutive theory of a hypothetical material, initially only a scalar internal variable ζ related with other dissipative mechanisms is considered.

Free energy - state laws: Following the classical assumption of the Thermodynamic of Irreversible Processes, the free energy Ψ is supposed to be a function of the state variables with the following form:

$$\Psi(\boldsymbol{\varepsilon}, \beta, \nabla\beta, \theta, \zeta) = \hat{\Psi}(\boldsymbol{\varepsilon}, \beta, \nabla\beta, \zeta) + \frac{1}{2} k \nabla\beta . \nabla\beta \tag{3}$$

where $\hat{\Psi}$ is a differentiable function such that $\frac{\partial \hat{\Psi}}{\partial \beta} \geq 0$ and k is a positive function of the absolute temperature θ. The term $\left(k/2\right)\nabla\beta.\nabla\beta$ is so considered to give β a diffusive behavior, thus smoothing the field β on Ω.

The here called thermodynamic forces $(\boldsymbol{\sigma}, B^\beta, \mathbf{H}, s, B^\zeta)$ related to the state variables $(\boldsymbol{\varepsilon}, \beta, \nabla\beta, \zeta)$ are defined from the free energy by the state laws:

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} \tag{4}$$

$$\mathbf{H} = \frac{\partial \Psi}{\partial(\nabla\beta)} = k\nabla\beta \tag{5}$$

$$B^\beta = \frac{\partial \Psi}{\partial \beta} \tag{6}$$

$$B^\zeta = \frac{\partial \Psi}{\partial \zeta} \tag{7}$$

It can be observed that the state variables are taken as independent parameters in equations (4) - (7). To complete the constitutive equations additional information about the dissipative behavior must be obtained from the Fourier law and evolution laws for β and ζ.

Evolution laws

The evolution laws are obtained from a potential $\Phi(B^\beta, B^\zeta; \boldsymbol{\varepsilon}, \beta, \zeta)$:

$$\dot{\zeta} = -\frac{\partial \Phi}{\partial B^\zeta} \quad \text{and} \quad \dot{\beta} = -\left\langle \frac{\partial \Phi}{\partial B^\beta} - \frac{1}{c_0} F \right\rangle \tag{8}$$

where c_0 is a positive material constant, $\langle a \rangle = \max\{0, a\}$. An alternative set of constitutive equations is given by:

$$\dot{\zeta} = -\lambda \frac{\partial \Phi}{\partial B^{\zeta}} \quad \text{and} \quad \dot{\beta} = -\left\langle \lambda \frac{\partial \Phi}{\partial B^{\beta}} - \frac{1}{c_0} F \right\rangle, \quad \lambda \geq 0; \ \Phi \leq 0; \ \lambda \Phi = 0 \tag{9}$$

In Chimisso (1994) and Costa-Mattos *et al* (1995) is shown that the state laws, equations (4) - (7), and the evolution laws, equations (8) or (9), form a complete set of constitutive equations that always verify the second law restriction, regardless the geometry of the body, the external actions, the initial and boundary conditions, provided Φ is a convex and positive function of (B^{ζ}, B^{β}) such that:

$$\Phi(B^{\beta} = 0, B^{\zeta} = 0; \boldsymbol{\varepsilon}, \beta, \zeta) = 0 \quad \forall \ (\boldsymbol{\varepsilon}, \beta, \zeta) \tag{10}$$

3.1- Brittle-elastic constitutive model

The constitutive equations for brittle-elastic material are presented after some specific assumptions about the state variables and the state laws.

State variables: The local state of a brittle-elastic material is supposed to be a function of the total strain $\boldsymbol{\varepsilon}$, of the cohesion variable β and of its gradient $\nabla\beta$.

Free energy – state laws: In linear elasticity, it can be supposed that the free energy of the elastic material is composed by a mechanical term $\hat{\Psi}_e(\boldsymbol{\varepsilon})$. Thus, the proposed free energy Ψ can be represented by the following expression:

$$\Psi = \beta \ \hat{\Psi}_e(\boldsymbol{\varepsilon}) + \frac{1}{2} k \nabla\beta . \nabla\beta \tag{11}$$

$$\hat{\Psi}_e(\boldsymbol{\varepsilon}) = \mu \ \boldsymbol{\varepsilon}.\boldsymbol{\varepsilon} + \frac{1}{2} \lambda \ (\mathrm{tr}\boldsymbol{\varepsilon})^2 \tag{12}$$

where μ and λ are Lamé's constants. The here called thermodynamic forces $(\boldsymbol{\sigma}, B^{\beta}, \mathbf{H}, s)$ related to the state variables $(\boldsymbol{\varepsilon}, \beta, \nabla\beta)$ are defined from the free energy by the state laws. These equations have the same form presented in the equations (4) to (7).

Evolution laws: The evolution law for cohesion variable β can be derived from the evolution law adopted in the local damage theory:

$$\dot{\beta} = -\frac{1}{C} \left\langle (B^{\beta} - w) - F \right\rangle, \quad \text{if} \ \beta > 0 \quad \text{and} \quad \dot{\beta} = 0, \quad \text{if} \ \beta = 0 \tag{13}$$

where C and w are positive constants of the material, respectively, a coefficient related to the viscosity and the elastic strain energy (the area below the linear part of the curve stress versus strain). The constant C is associated to constant c_0 of (8).

After some consideration, Costa-Mattos *et al* (1995) and Domingues (1996), the final relations are the following:

$$\boldsymbol{\sigma} = \left(\frac{\beta E}{1+\nu}\right)\left[\frac{\nu}{1-2\nu}\,\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} + \boldsymbol{\varepsilon}\right] \tag{14}$$

$$F = \left(\frac{E}{2(1+\nu)}\right)\left[\frac{\nu}{(1-2\nu)}\,(\mathrm{tr}(\boldsymbol{\varepsilon}))^2 + \boldsymbol{\varepsilon}.\boldsymbol{\varepsilon}\right] - w + \lambda_\beta + C\dot{\beta} + \lambda_{\dot{\beta}} \tag{15}$$

$$\mathbf{H} = k(\nabla\beta) \tag{16}$$

where E is the Young modulus, ν is the Poisson's ratio, C is a coefficient related to the viscosity and k is a diffusive constant. The terms λ_β and $\lambda_{\dot{\beta}}$ are Lagrange multipliers associated, respectively, to the constraints $\beta \geq 0$ and $\dot{\beta} \leq 0$, they are such that: $\lambda_\beta \leq 0$, $\beta\lambda_\beta = 0$ and $\lambda_{\dot{\beta}} \leq 0$, $\dot{\beta}\lambda_{\dot{\beta}} = 0$.

3.2- Elastic-plastic constitutive model

State Variables : Under the hypothesis of small deformations, the local state of a elastic-plastic material is supposed to be a function of the total strain $\boldsymbol{\varepsilon}$, of the plastic strain $\boldsymbol{\varepsilon}^p$, of the cohesion variable β, of its gradient β, and also of a scalar variable p associated with the isotropic hardening, and of a second order tensor variable **c** associated with the kinematic hardening .

Free Energy - State Laws: Following the classical assumption of the Thermodynamic of Irreversible Processes, the free energy is supposed to be a function of the state variables. Thus, the following expression is proposed for the free energy, considering a elastic-plastic behaviour:

$$\Psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, p, c, \beta, \nabla\beta) = \beta[\Psi_e(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) + \Psi_p(p) + \Psi_c(\mathbf{c})] + \frac{1}{2}k\nabla\beta.\nabla\beta \tag{17}$$

$$\Psi_p = v_1\left(p + \frac{e^{-v_2 p}}{v_2}\right) + p\sigma_y \tag{18}$$

$$\Psi_c = \frac{1}{2}\,a\,(\mathbf{c}\,.\,\mathbf{c}) \tag{19}$$

where $\Psi_e(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) = \hat{\Psi}_e(\boldsymbol{\varepsilon})$, equation (12), v_1, v_2, a, $\boldsymbol{\sigma}_y$ are non negative constants and $(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) = \boldsymbol{\varepsilon}^e$ is the elastic strain tensor. The here called thermodynamic forces ($\boldsymbol{\sigma}$, **x**, y, G, **H**), related to the state variables $(\boldsymbol{\varepsilon}^e, \mathbf{c}, p, \beta, \nabla\beta)$ are defined from the free energy by the state laws:

$$\boldsymbol{\sigma} = \frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}^e} = \frac{\beta\,E}{1+\nu}\left[\frac{\nu}{1-2\nu}\,\mathrm{tr}\left(\boldsymbol{\varepsilon}^e\right)\mathbf{I} + \boldsymbol{\varepsilon}^e\right] \tag{20}$$

$$\mathbf{x} = \frac{\partial\Psi}{\partial\mathbf{c}} = \beta\,(a\,\mathbf{c}) \tag{21}$$

$$y = \frac{\partial\Psi}{\partial p} = \beta\left[v_1\left(1 - e^{-v_2 p}\right) + \sigma_y\right] \tag{22}$$

$$G = \frac{\partial \Psi}{\partial \beta} = \Psi_e + \Psi_p + \Psi_c \tag{23}$$

where $\boldsymbol{\sigma}$ is the stress tensor and **H** was presented in last section.

Plastic Potential - Evolution Laws: To complete the constitutive equations additional information about the dissipative behaviour must be given. This information can be obtained from a plastic potential F and are called evolution laws. The potential F is supposed to have the following form :

$$\chi = J\,(\boldsymbol{\sigma} - \mathbf{x}) - y + g(\mathbf{x}, G\,; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, p, c, \beta) \tag{24}$$

with $J(\boldsymbol{\sigma} - \mathbf{x})$ being the Von Mises equivalent stress, and

$$g = \frac{b}{2a}(\mathbf{x}\,.\,\mathbf{x}) - \frac{ab}{2}(\beta\,\mathbf{c}\,.\,\mathbf{c}) + \frac{G^2}{2S_0} - \frac{1}{2S_0}\left(\frac{\Psi_e + \Psi_p + \Psi_c}{\beta}\right)^2 \tag{25}$$

Besides the plastic potential χ, another potential $\hat{\chi}(\dot{\beta}) = \dot{\beta}$, is used to take into account the restriction $\dot{\beta} \leq 0$. Hence the following evolution laws are postulated:

$$\hat{\boldsymbol{\varepsilon}}^p = \lambda \frac{\partial \chi}{\partial \sigma} = \lambda\, \frac{3}{2}[(\boldsymbol{\sigma} - \mathbf{x}\,)_{dev}\, /\, J(\boldsymbol{\sigma} - \mathbf{x})] \tag{26}$$

$$\dot{\mathbf{c}} = -\lambda \frac{\partial \chi}{\partial \mathbf{x}} = \dot{\boldsymbol{\varepsilon}}^p - \frac{b}{a}\ \mathbf{x}\lambda \tag{27}$$

$$\dot{p} = -\lambda \frac{\partial \chi}{\partial y} = \lambda \tag{28}$$

$$\dot{\beta} = F - \lambda \frac{\partial \chi}{\partial G} - \hat{\lambda} \frac{\partial \hat{\chi}}{\partial \dot{\beta}} = F - \frac{\lambda\left(\psi_e + \psi_p + \psi_c\right)}{S_0} - \hat{\lambda} \tag{29}$$

where $\lambda \geq 0,\ \chi \leq 0,\ \lambda\chi = 0$ and $\hat{\lambda} \geq 0,\ \hat{\chi} \leq 0,\ \hat{\lambda}\hat{\chi} = 0$. Besides, F is a microscopic internal force associated with β, λ is the Lagrange multiplier related with the restriction $\chi \leq 0$, and $\hat{\lambda}$ is the Lagrange multiplier related with the restriction $\hat{\chi} \leq 0$.

It is possible to prove that the state laws (20) - (23) and the evolution laws (26) - (29) define a complete set of thermodynamically admissible constitutive equations.

Introducing equations (16) and (29) in the equation of the balance of microscopic forces (Chimisso & Costa Mattos, 1994) and using the change of variables:β = 1-D, the following balance equation is obtained:

$$\left\langle C\,\Delta D + \lambda(\psi_e + \psi_p + \psi_c)/S_0 \right\rangle = \dot{D} \tag{30}$$

4- EXAMPLES

The main goal of this work is to show a general idea of the theory which is presented and discussed in several papers, for example, Domingues (1996) and Chimisso (1994).

4.1- Brittle-elastic

The double edge cracked plate, which contains crack length a = 4 mm, length L = 25 mm and width W = 10 mm, is loaded with a prescribed displacement u(t) at the both sides, figure 1. The existent symmetry permits to analyze only a quarter of the plate, in that case the upper right quarter of the plate, figure 1.

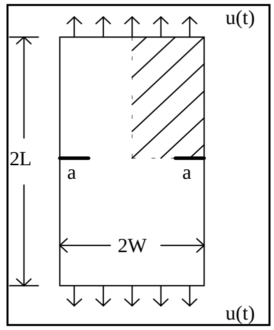

Figure 1. The cracked plate

The usual bilinear quadrilateral finite element mesh are used on the discretization of the problem. In the region where the highest levels of damage are expected the mesh has a great quantity of elements with small mesh parameter. In this study were considered plates of concrete and the following values: E=27.0GPa, $k = 0.2 \text{MPa.mm}^2$, $C = 1.0 \times 10^{-3}$ MPa.s and $w = 5.0 \times 10^{-5}$ MPa, Domingues (1996). The prescribed displacement and the adopted time step are given respectively by $u(L,t) = \alpha t$, ($\alpha = 5.0\text{x}10^{-3}$ mm / s).

Figure 2 shows the damage evolution on the plate. To simulate the damage evolution on the plate, the damage field $(D = 1 - \beta)$ at four different time steps are presented. After the instant 1.577s, the plate is broken in two parts undergoing a rigid body motion.

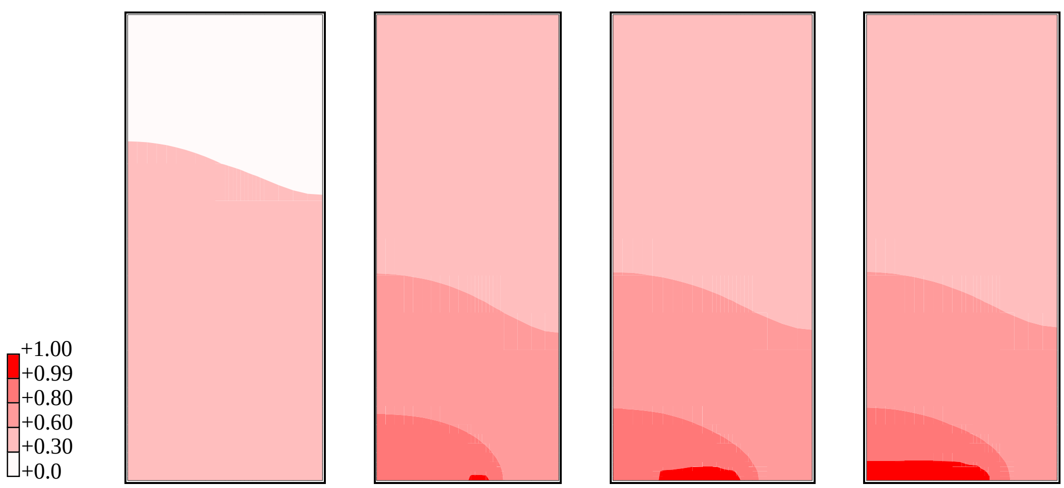

Figure 2. Damage levels at the instants t=1.3s, t=1.573s, t=1.575s e t=1.577s.

In order to complete the study a curve of the external force versus the displacement u(t) are presented in the figure 3. The curve permit to observe the expected softening behavior.

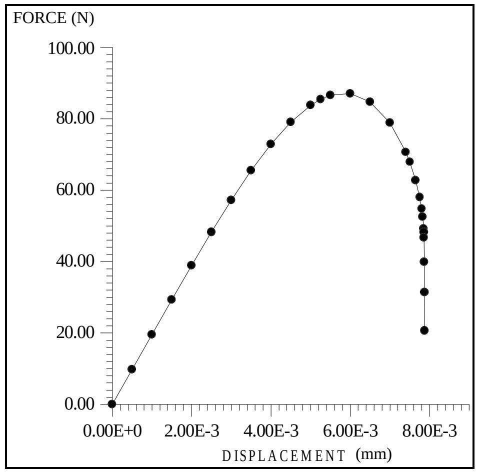

Figure 3. Force x displacement u(t).

4.2- Elastic-plastic

A near rupture behaviour example

To exemplify the use of the theoretical model, we simulate a monotonic test in an ASTM 6351 aluminium alloy and its behaviour near the final rupture. The test was simulated with controlled strain amplitude where the displacement initial conditions was u(t = 0, z =L)=0 and u(t = T, Z = L) = 0,7 L. It is considered length L = 1,0, k = 0,01 and S_O = 56,0.

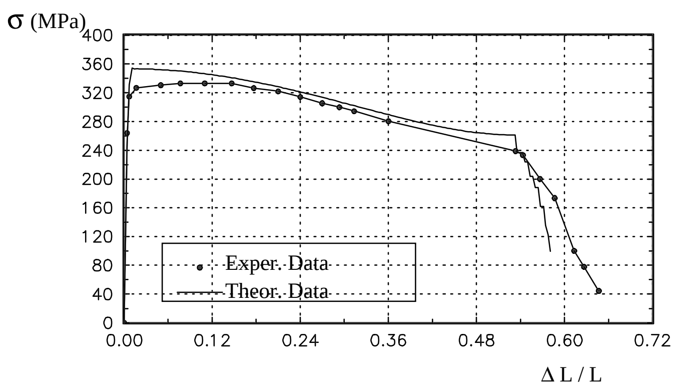

Figure 4. Curve σ x ΔL/L.

Figure 4 shows the axial stress component for different elongations, where dot line is the experimental data and hairline is the theoretical results. The theoretical results are in agreement with the experimental data.

5- CONCLUSIONS

The study of different problems demonstrate that the gradient-enhanced damage theories allows a correct qualitative description of the strain-softening phenomena. Also, the proposed damage theory can describe the evolution of the damage, the stress and the displacement fields for structure formed by different materials (concrete, glass and ceramic, metallic alloy, for instance), Domingues (1996) and Chimisso (1994).

6- ACKNOWLEDGMENTS

The authors would like to thanks: **FAPERJ** - Fundação de Amparo à Pesquisa do Estado do Rio de Janeiro and **CNPq** - Conselho Nacional de Desenv. Científico e Tecnológico.

7- REFERENCES

- Bazant, Z. P., Pijandier Cabot, G., Nonlocal Continuum Damage, Localisation, Instability and Convergence, ASME J. Appl. Mech.,Vol.55, pp. 287-293, 1988.
- Chimisso, F.E.G., Um Modelo de Dano Contínuo para Materiais com Comportamento Elasto-Plástico, Doctor Thesis, Pontifícia Univ. Católica do Rio de Janeiro - PUC-Rio, 1994.
- Chimisso, F.E.G. and Costa-Mattos, H. (1994). Modeling the softening behavior of damageable elasto-plastic bars. In Localized Damage III. Computer Aided Assesment and Control. M. H. Aliabadi, A. Carpinteri, S. Kaliszky and D. J. Cartwrigth eds. Computational Mechanics Publication.
- Costa Mattos, H. & Sampaio, R., Analysis of the Fracture of Brittle Elastic Materials Using a Continuum Damage Model, Structural Engineering and Mechanics, Vol. 3, N°5, pp. 411-428, 1995.
- Costa Mattos, H., Fremond, M. & Mamiya, E.N., A Simple Model of the Mechanical Behavior of Ceramic-Like Material, Int. J. for Solids and Structures, Vol. 20, N° 24, pp. 3185-3200, 1992.
- Domingues, S.M.P., Análise de Materiais Elásticos Frágeis através de um Modelo de Dano Contínuo, Doctor Thesis, Pontifícia Univ. Católica do Rio de Janeiro - PUC-Rio, 1996.
- Domingues, S.M.P., Costa-Mattos, H. & Rochinha, F.A., Continuum Damage Problems Solved by the Splitting technique in Connection with Finite element method, Anais do COBEM97, 1997.
- Frémond, M. & Nedjar, B., "Endommagement et Principle des Puissances Virtuelles", C. R. Acad. Sci., tome 317, serie II, n° 7, pp. 857-864, Paris, 1993.
- Frémond, M. & Nedjar, B., Damage, Gradient of Damage and Principle of Virtual Power, Int. J. Solids Structures, Vol. 33, No. 8, pp. 1083-1103, 1996.
- Kachanov, L. M., Introd. to Continuum Damage Mech., Martinus Nijhoff Publishers, Dordrecht, Netherland, 1986.
- Knowles, J.K. & Sternberg, E., "On the Failure of Ellipticity and the Emergence of Discontinuous deformation Gradients in Plane Finite Elastostatics", J. Elast., 8, 329-378, 1978.
- Lemaitre, J. & Chaboche, J.L., Mech. of Solid Materials, Cambridge Univ. Press, 1990.
- Vree, J. H. P. de, *et al*., "Comparison of Nonlocal Approaches in Continuum Damage Mechanics", Computer and Structure, Vol. 55, No. 4, pp. 581-588, 1995.