SOME RECENT ADVANCES ON ADAPTIVE PROCEDURES
IN NONLINEAR FE ANALYSIS

C. A. Almeida
Department of Mechanical Engineering
Pontifícia Universidade Católica/RJ
Rio de Janeiro - Brazil

ABSTRACT

A methodology initially proposed for authomatic mesh generation of triangular and quadrilateral finite element discretizations in linear two-dimension problems is now extended to material nonlinear analysis. The technique, which is based on a h-adaptive process, is capable of achieving a specified discretization density using a powerful mesh generator. The element solutions at the nodes are obtained through a general stress recovery procedure employing an *a posteriori* error estimator. The constitutive equation is approached in the formulation using a flow theory to describe the elasto-plastic material behavior. In this study the von Mises condition is employed for the state of multiaxial stress corresponding to the start of plastic flow, the normality condition furnishes a flow rule in the plastic strain increments subsequent to yielding and the kinematic hardening is assumed as hardening rule. The adaptive procedure is based on the complete mesh regeneration and specific mesh requirements (boundary conditions, geometry definitions and space node function), and aims for an optimality condition with the least number of elements that yields an uniform error distribution in all elements. In the stress recovery process the nodal values are assumed to belong to a polinomial expansion defined over patches of elements adjoining a particular assembly node considered. The nodal point parameters, at each element, are obtained using a least square fit of superconvergent sampling points existing in the patch. The material uniaxial elasto-plastic constitutive behavior is represented using overlays, defined over small strain increments, allowing for the representation of the material kinematic hardening behavior beyond the classical bilinear relation. The procedure error estimation is obtained from differences between the post-processed stress gradients and those from the finite element solutions. The energy error norm associated with stress field diferences and the finite element strain energy gives an effective error estimate, used for comparison with the process tolerance. Evaluation of the proposed technique is presented through numerical sampling analyses to illustrate its applicability in the improvement of the solution accurance of general two-dimension finite element model solutions.

INTRODUCTION

The success of the finite element method in numerical analysis is based largely on the basic finite element procedures used, namely, the formulation of the problem in variational or weighted residual form, the finite element discretization of this formulation, and the effective solution of the resulting finite element equations. These basic steps are the same whichever problem is considered and provide a general frame-work and a quite natural approach to engineering analysis. Besides being the most general analysis tool available today, the method may still require from the analyst a broad knowledge and some experience in using the numerical procedures employed, to perform a reliable modelling analysis of practical applications.

In this context, capabilities such solution error estimates, combined with an effective mesh adaptive technique, have been added to extend the method numerical efficiency. For a given mesh, error estimates current available in the literature are classified in two types: residual estimators and flux projection estimators. In the first, the solution error is evaluated over the elements or mesh subdomains by solving a local boundary value problem using samplings of the differential equation residuals in each element domain and the residual in the stress components sampled on the boundary of each element (Kelly, Graco, et. al.-83). In the second type, the error is evaluated from the stress component fields post-processed using some projection technique (Zhu, Zienkiewicz-87; Ladeveze, Coffignal & Pelle-86; Ortiz, Quigley-91), such the least square method. In Ref. (Tetambe, Saigal-94), a comparative study for five flux projection error estimators in elasto-plastic analysis of two-dimension plane strain and axisymmetric solids, undergoing large deformations, is presented. In this study no mesh refinement is employed and it is shown that error estimators based on the energy rate and on the L_2-norm of the incremental strains accurately predicts the region of maximum error. From all error estimators tested in the analyses considered, the L_2-norm of the incremented strains gave the most conservative estimate of error.

In this paper the h-adaptive and mesh generation procedures, presented in Ref. [Almeida-94] for the finite element analysis of two-dimension isotropic linear-elastic problems, are extended to account for material nonlinear effects. The material constitutive relation for the uniaxial stress-strain is represented by a multi-linear idealized model using the overlay model procedures reported in Refs. (Nayak, Zienkiewicz-72; Zienkiewicz, Villiapan, King-69; Owen, Prakash, Zienkiewicz-74). In this material modelling technique, the multi-linear relation is replaced by the superposition of a number of elasto-perfectly plastic material models. This technique has been proven attractive on representing the elasto-plastic behavior of materials undergoing cyclic-loadings. It is worthy a notice that that even under complex loading conditions, the material model represented by overlays gives accurate numerical solution responses in representing the Bauschinger and the strain softening effects. In the following sections a brief outline of the finite element steps in the elasto-plastic analysis, the details of the overlay model, a review of the adaptive procedure based on an *a posteriori* error estimator and the conclusions founded from sample analysis results are presented, indicating the applicability of the proposed methodology to general structure analyses.

THE ELASTO-PLASTIC ANALYSIS

In finite element analysis the basic step is the unique representation, within an element (i), of the unknown displacement vector

$$\mathrm{v}^{(i)} = \mathrm{N}^{(i)}\ \hat{\mathrm{v}}^{(i)} \tag{1}$$

in terms of the element nodal displacement vector $\hat{\mathbf{v}}^{(i)}$ and the displacement transformation matix $\mathbf{N}^{(i)}$ [Bathe-82], which depends on the spatial coordinates and the interpolation function used in the problem discretization. At any point of the problem domain, represented by **n** elements, the strains can be then obtained from

$$\boldsymbol{\varepsilon} = \sum_{i=1}^{n} \mathbf{B}^{(i)}\, \hat{\mathbf{v}} = \mathbf{B}\hat{\mathbf{v}} \tag{2}$$

where matrices $\mathbf{B}^{(i)}$ are generally obtained from the shape functions and their derivatives. From linear elasticity the proportional constitutive law for an initially strained material, with initial strains $\boldsymbol{\varepsilon}_o$ and initial stresses $\boldsymbol{\sigma}_0$, gives the relation

$$\boldsymbol{\sigma} = \mathbf{D}\,(\varepsilon - \boldsymbol{\varepsilon}_o) + \boldsymbol{\sigma}_o \tag{3}$$

in which **D** is the elasticity matrix. If nodal forces acting on the structure are listed in a vector **R** and the stresses at any point are as in eq. (3), then for equilibrium it is required that

$$\int_V \mathbf{B}^T \mathbf{D}\mathbf{B} dV\, \hat{\mathrm{v}} = \mathbf{R} - \int_V (\mathbf{B}^T \boldsymbol{\sigma}_o - \mathbf{B}^T \mathbf{D}\boldsymbol{\varepsilon}_o) dV. \tag{4}$$

In elasto-plasticity eq. (3) does not hold to represented the full constitutive law. However, for the case of small strain analysis, the strain-displacement relationship remaining linear, the problem can be solved without complete reformulation.

Plastic deformations are characterized by an irreversible straining and begins once certain level of stresses has occured. This level is governed by a yield condition function

$$\mathrm{F}(\boldsymbol{\sigma}, \kappa) = 0 \tag{5}$$

where κ is a state variable which depends on the plastic strain vector $\boldsymbol{\varepsilon}_p$. Strain state increments in the material may be decomposed into elastic and plastic components, in the form

$$\delta\, \boldsymbol{\varepsilon} = \delta\, \boldsymbol{\varepsilon}_e + \delta\, \boldsymbol{\varepsilon}_p. \tag{6}$$

In equation (6) we use the following definition

$$\delta\, \boldsymbol{\varepsilon}_e = \mathbf{D}^{-1}\, \delta\, \boldsymbol{\sigma} \tag{7}$$

according to eq. (3), and the plastic strain increments being associated gradients of the yielding potential by to the following flow rule

$$\delta\, \varepsilon_{pij} = \lambda \partial \mathrm{F} / \partial \sigma_{ij} = \lambda \mathrm{q}_{ij} \tag{8}$$

in which λ is a scalar to be determined. Since during plastic deformations the stresses should remain on the yield surface, i.e. δF= 0, we also have

$$\mathbf{q}^T\ \delta\boldsymbol{\sigma} - \mathbf{p}^T\ \delta\boldsymbol{\varepsilon}_p = 0 \tag{9}$$

where $\mathrm{p}_{ij} = \partial \varepsilon_{pij}$. Using eqs. (7) to (9), the scalar λ is evaluated,

$$\lambda = \mathbf{q}^T\, \mathbf{D}\, \delta\, \varepsilon / (\mathbf{p}^T\, \mathbf{q} + \mathbf{q}^T\, \mathbf{D}\mathbf{q}). \tag{10}$$

Then substituting from (8) and (10) into (9) one obtains

$$\delta\, \boldsymbol{\sigma} = \mathbf{D}_{ep} \delta\, \boldsymbol{\varepsilon} \tag{11}$$

where the matrix

$$\mathbf{D}_{ep} = \mathbf{D} - (\mathbf{D}\,\mathbf{q})\,(\mathbf{D}\mathbf{q})^T / (\,\mathbf{p}^T \mathbf{q} + \mathbf{q}^T\ \mathbf{D}\mathbf{q}) \tag{12}$$

represents the instantaneous elastic-plastic stress-strain law. This constitutive law depends on the yield function F used. In this work the von Mises yield criterion with isotropic hardening was employed.

To solve the nonlinear equilibrium equation resulting from (4), an iterative procedure must be employed with a series of elastic solutions being performed until all the problem nonlinear conditions are satisfied. In a linear analysis where the resulting stresses are constrained to satisfy the yield criterion, the equilibrium equation in (4) will not be satisfied and the residual forces

$$\mathbf{R}^* = \mathbf{R} - \int_V \mathbf{B}^T \boldsymbol{\sigma} dV \neq 0 \tag{13}$$

will arise. In eq. (13) $\boldsymbol{\sigma}$ is a vector containing the actual stress components obtained from the stress level reached as governed by the yield criterion. Variations of the out-of-balance force, $\delta \mathbf{R}^*$, due to changes $\delta \hat{\mathbf{v}}$ in the displacement vector leads to the problem tangent stiffness matrix

$$\mathbf{K}_T = \int_V \mathbf{B}^T \mathbf{D}_{ep} \mathbf{B} dV. \tag{14}$$

The solution is then obtained by starting from a trial solution, calculating the residuals by means of (13) and then obtaining a corrected solution displacement vector, at iteration (n + 1), in the form

$$\hat{\mathbf{v}}_{n+1} = \hat{\mathbf{v}}_n + \mathbf{K}_T^{-1} \mathbf{R}^*. \tag{15}$$

Since the constitutive law in (11) is expressed incrementally the process is repeated for small increments, starting from previously established conditions until further changes in displacements are sufficiently small.

THE OVERLAY MODEL

The elasto-plastic solution method outlined in the previous section provides adequate numerical results if no load reversal occurs or, for increasing loads, the uniform expansion of the yielding surface is assumed to occur in the analysis. Nevertheless, it does not simulate the Bauschinger effect on reverse loading, and the initial yield value on reversal increases with the amount of straining on the previous loading. As indicated by analytical modelling approaches available in the literature, Refs. (Kröener-61; Hutchinson-64), the Bauschinger effect can be simulated by considering the continuum composed of individual grains each possessing individual properties. Thus, it is quite natural to conceive the structure as an assembly of element models with different material properties assigned to each, or to the Gaussian points, as in the isoparametric element discretization. Although convenient, this procedure would not be practical to use because the internal stress and displacement distribution would be meaningless, being dependent on the prescribed distribution of the material properties. To overcome this difficulty the overlay model has been proposed [Owen, et. al-74]. By this modelling technique the material is assumed to be composed of several layers of overlays. Each overlay may have different material properties and thickness, with an elastic-perfectly plastic behavior being assigned. Nodes in each overlay are coincidental and the same strain pattern is produced in each overlay. As a result, different stress fields are produced in each overlay, which contribute to the total stress distribution

$$\boldsymbol{\sigma} = \sum_{i=1}^{\ell} \boldsymbol{\sigma}_i \xi_i \tag{16}$$

suitable weighted by the overlay thickness ξ_i. in eq. (16) the condition $\sum_{i=1}^{\ell} \xi_i = 1$ must hold. This material model has the advantage of providing stress and displacement distributions that

are physically meaningful. These stresses may be recognized as the continuum average of the stresses in the material individual grains.
Considering the constitutive linear elastic relation in each overlay to be equal to the material Young's Module E_1 and the slopes in the multi-linear model as E_2, E_3, ..., the overlay thicknesses are evaluated by improsing the conditions in eq. (16) to ℓ sampling points from the uniaxial stress-strain relation, resulting in

$$\xi_i = (E_i - E_{i+1}) / E_1, \quad \text{for} \quad i = 1 \quad \text{to} \quad \ell \tag{17}$$

Although being originally developed for plane stress state representations, the overlay technique is readily extensible to three-dimension stress states and, in this case, the overlay thicknesses become weight parameters. Negative values for these parameters allows the technique to simulate materials with strain-softening effects, Ref. (Owen, Prakash, Zienkiewicz-74).

ADAPTIVITY AND MESH GENERATION

In elasto-plastic analyses, lacking an exact solution for comparison with a particular finite element solution requires, at a certain loading, approximation of the numerical results to evaluate the solution error. Numerical experiments with various solution error procedures indicates the energy norm error measure as one of the simplest but reliable to use error estimate, in the evaluation of a particular problem discretization performance, Ref. [5]. For a certain mesh, the energy error norm is obtained from approximations of the stress field in the error evaluation at each element level, based on differences between numerical results and the approximation fields, as detailed in Refs. (Zhu, Zienkiewicz-87; Zienkiewicz, Zhu-91; Zienkiewicz, Zhu-92). This technique has been implemented and tested in two-dimension applications presenting severe stress singularities (Almeida-94; Almeida & Santana-93). It was shown that with the use of triangular and quadrilateral isoparametric finite element discretizations a great improvement in the solution efficiency is obtained when quadratic elements were used versus linear interpolation elements. Thus, the methodology initially presented for linear applications, and detailed in Ref. (Almeida-94), was extended for material nonlinear analyses, and is based on the following steps:

- after numerical convergence has been achieved, at a certain load step, an *a posteriori* solution error estimation is evaluated from differences between post-processed stress gradients and those from the finite element solutions. The energy error norm associated with stress field differences and the finite element predicted strain energy gives an effective error estimate used for comparison with the process tolerance;
- if the convergence condition is not satisfied, an h-adaptive process is employed, based on a complete mesh regeneration, guided by specified mesh requirements such as boundary conditions, geometry definitions, and space node functions to achieve an optimal refinement. The optimality condition used requires the mesh refinement with the least number of elements that yields a uniform strain energy norm error distribution in all elements. This is generally referred as Zienkiewicz-Zhu condition, described in Ref. (Zienkiewicz, Zhu-91);
- to proceed with the analysis, a general stress recovery is required to obtain the element solution at the new node positions. In this procedure, the nodal values are assumed to belong to a polynomial expansion of the same complete order in the interpolation function basis used, which is valid over all elements adjoining a node. A least-square fit of superconvergent sampling points existing in the path is used to obtain the recovered nodal point parameters for

each element. These parameters, stresses and displacements, are averaged to all elements adjoining the node of interest.

The numerical solution continues for increasing load steps, using the new discretization mesh and solutions for evaluation of the equilibrium condition in (4).

SAMPLE ANALYSIS

The foregoing enhancements for the material nonlinear representation in two-dimension continuum mechanics problems have been implemented to the adaptivity procedures presented earlier. Although the prime motivation for these enhancements is the better description of the material behavior under cyclic loading, the analysis that follows provides an assessment of the model for results in conventional plasticity analysis with monotonically inceasing loads. In such situations isotropic hardening plasticity gives accurate results.

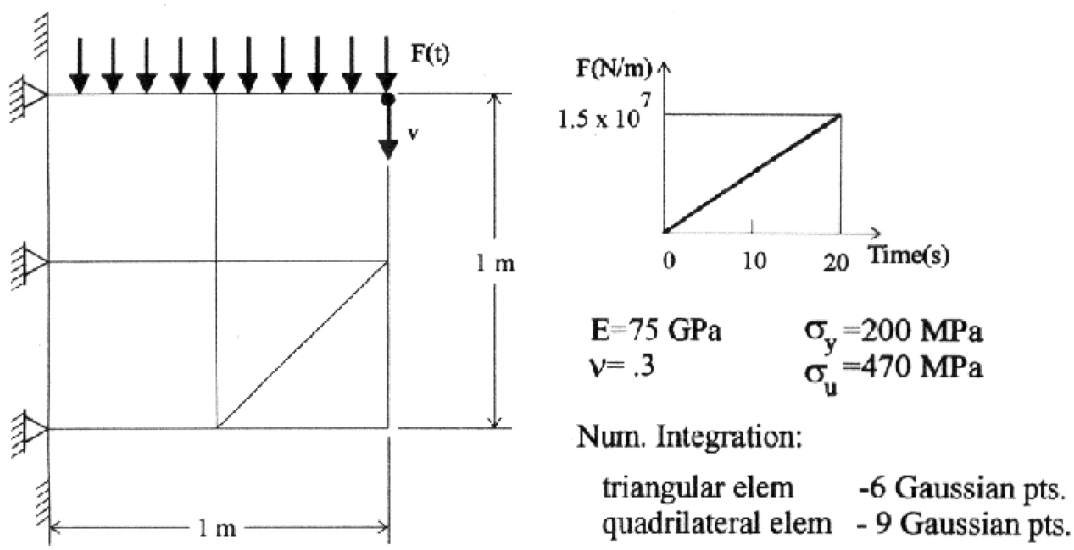
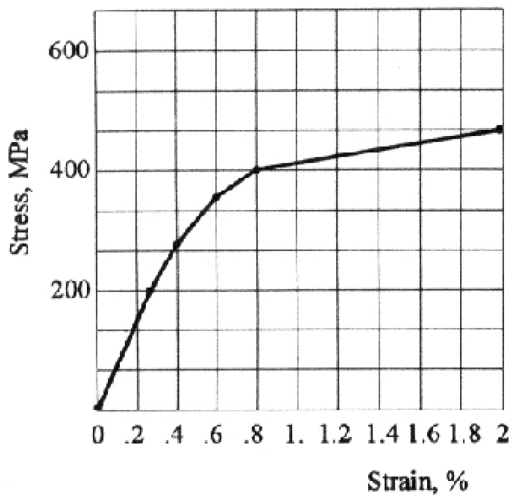


Figure 1 - Short Cantilever Beam Under Transverse Uniform Loading.
Initial Finite Element Meshes Used.

The first problem concerns the plane strain behavior of a short cantilever beam, subjected to uniform transverse loading, linearly incremented in 20 time steps, as shown in Fig. 1. Figure 1 also presents the initial triangular (6 nodes) and quadrilateral (9 nodes) quadratic isoparametric element meshes employed. Five elastic perfectly-plastic overlays were used to represent the material stress-strain relationship. Computed weighting parameters and yielding stress values are indicated in Fig. 2. In both finite element discretization analyses a 6% energy error criteria for mesh reformation was required and a .1% converge rate was used for the iterative procedure during evaluation of the out-of-balance load **R***, defined in eq. (13). The energy error norms obtained from the finite element solution and the number of degrees-of-freedom required in each solution step are shown in Tables 1a and 1b, for both discretization models. In the first load step of each analysis, two mesh refinement steps were required due to the very crude discretization initially used, yielding to large energy error norm solutions. These results are, in essence, the same obtained in previous linear analyses in Ref. (Almeida-94), because at this load step the material is still



Overlay	t_i	σ_{y_i} (MPa)
1	.180	200
2	.353	300
3	.133	450
4	.256	600
5	.078	1500

Figure 2 - Material Stress-strain Curve Approximation.
Parameters Used in the Overlay Representation.

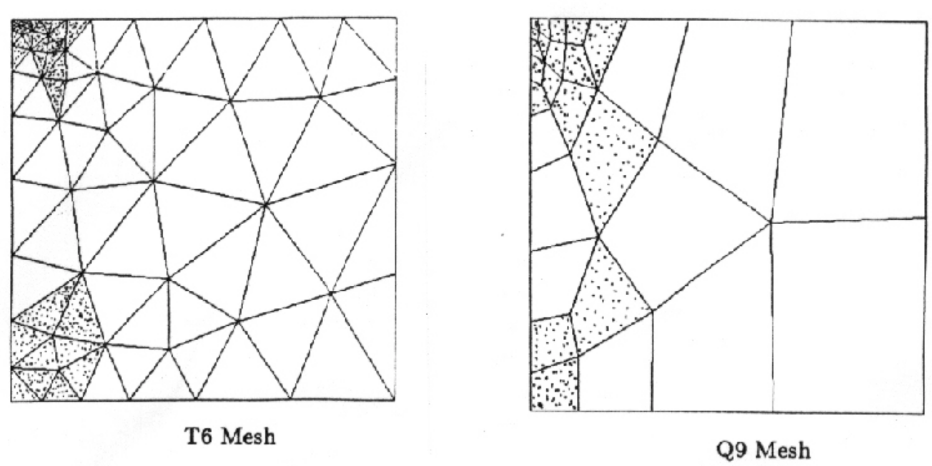


Figure 3 - Final Obtained Meshes with Triangular and Quadrilateral
Finite Element Models Used.

elastic. As the analysis proceed, a gradual increasing in the energy norm error is observed but no need of refinements is required until the 7th and the 6th load steps were reached for the triangular e quadrilateral meshes, respectively. At these load steps, only one cycle of refinement was then required for convergence. Further, as plasticity region advances over the elements, larger values for the energy error norm requires frequent cycles of refinements (steps 11 to 20). Figure 3 despicts the obtained final meshes in both analyses; the indicated areas correspond to the elements in which the material yielding conditions have been satisfied in at least one Graussian integration point. The displacement-load curves displayed in Fig. 4 are concerned with four quatrilateral finite element analyses. The vertical displacement solution v for the node located at the structure upper right corner is obtained from the full mesh refinement procedure proposed in this work and compared to the solutions given by discretizations obtained at the first step of mesh refinement. These mesh discretizations were kept constant throughout the analyses with energy error norms equal to 19.9%, 11.1% and 5.62%, see Table 1b step 1. From the results it may be observed that obtained errors in the displacement solutions at the latest steps are larger than at the begining of analysis when almost the entire structure is still under the linear elastic behavior. Moreover, for the entire range of loading considered, the results obtained with the initially proposed mesh are inadequate.

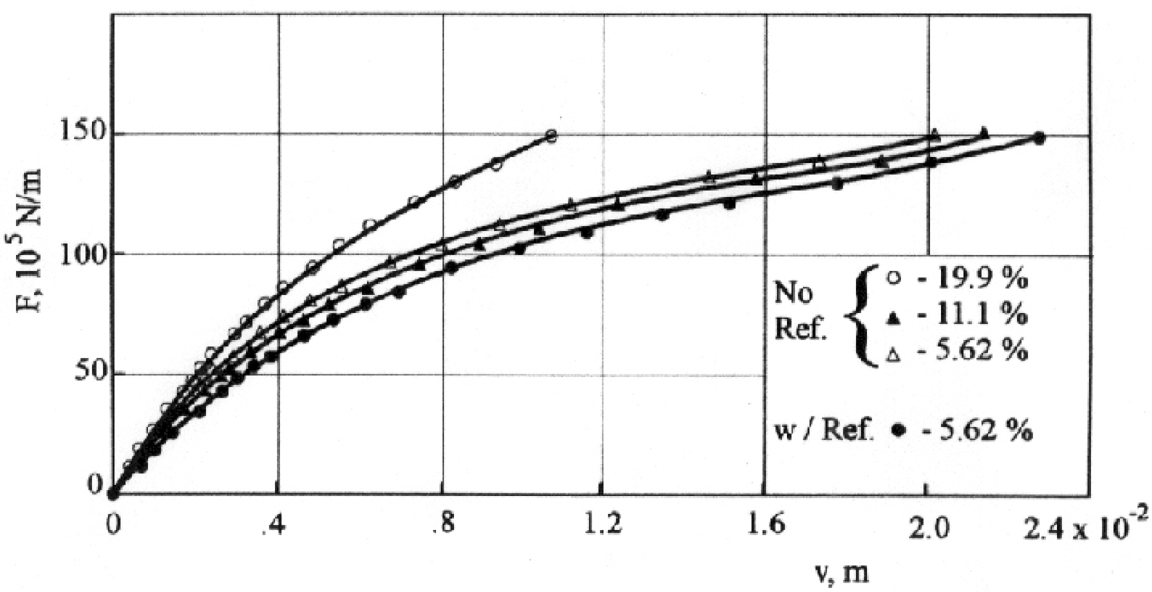


Figure 4 - Displacement-load Relations for the Analyses with Quadrilateral Elements, Using Different Mesh Discretizations, Obtained in the First Step of the h-Adaptive Procedure

Table I - Energy Error Norms and Number of Degrees-of-freedom Required at Each Load Step

(a) the triangular element model (6 node)

STEP	Energy Error Norm (%)	Deg-of-Freed	STEP	Energy Error Norm (%)	Deg-of-Free
1	27.0 - 8.36 - 4.84	312	11	6.57 - 5.14	430
2	4.84	312	12	5.77	430
3	4.97	312	13	6.43 - 5.03	490
4	5.11	312	14	5.78	490
5	5.38	312	15	6.86 - 5.11	568
6	5.87	312	16	5.89	568
7	6.43 - 4.68	384	17	6.99 - 5.27	666
8	4.95	384	18	6.01 - 5.08	772
9	5.27	384	19	5.93	772
10	5.86	384	20	7.22 - 5.45	926

(b) the quadrilateral element model (9 node)

STEP	Energy Error Norm (%)	Deg-of-Freed	STEP	Energy Error Norm (%)	Deg-of-Free
1	19.9 - 11.1 - 5.62	230	11	6.26 - 4.73	306
2	5.62	230	12	5.21	306
3	5.73	230	13	5.77	306
4	5.88	230	14	6.41 - 4.93	342
5	5.97	230	15	5.59	342
6	6.24 - 4.57	272	16	6.23 - 4.90	394
7	4.89	272	17	5.63	394
8	5.16	272	18	6.37 - 5.19	464
9	5.48	272	19	5.97	464
10	5.83	272	20	6.94 - 5.76	558

In the second problem the doubly-cracked square plate reported in Ref. (Almeida–94) and shown in Fig. 5 was considered. The uniform loading was linearly increased in 12 time steps, bringing the material to the behavior beyond its elastic limit. As in linear analysis, only one fourth of the plate was represented in the model analyses with four and eight element discretizations in the initial quadrilateral and triangular meshes, respectively. In both analyses only quadratic elements were considered with nine nodes for the quadrilateral mesh and six nodes for the triangular. As in the first problem, the material constitutive law employed is as shown in Fig. 2, using the overlay modelling technique. A 6% energy error norm criteria for mesh updating was required with 0.1% convergence rate for evaluation of the out-of-balance loading, during iterations. Obtained error norms from the finite element solutions are presented in Tables 2a and 2b for the element discretizations employed. As in the short cantilever beam problem, two mesh refinement steps were required due to the proposed crude initial meshes. The energy error norms gradually increase with loading increments, not requiring requirements until steps 5 and 6 for the triangular and quadrilateral meshes, respectively. At these steps one cycles of refinement was required for convergence. Larger values of the energy error norms require frequent cycles of refinements, as the plastic region advances over the elements. Final obtained meshes are shown in Figure 5; in dashed area is represented the elements with the stress state fulfilling the yielding condition, in eq. (5).

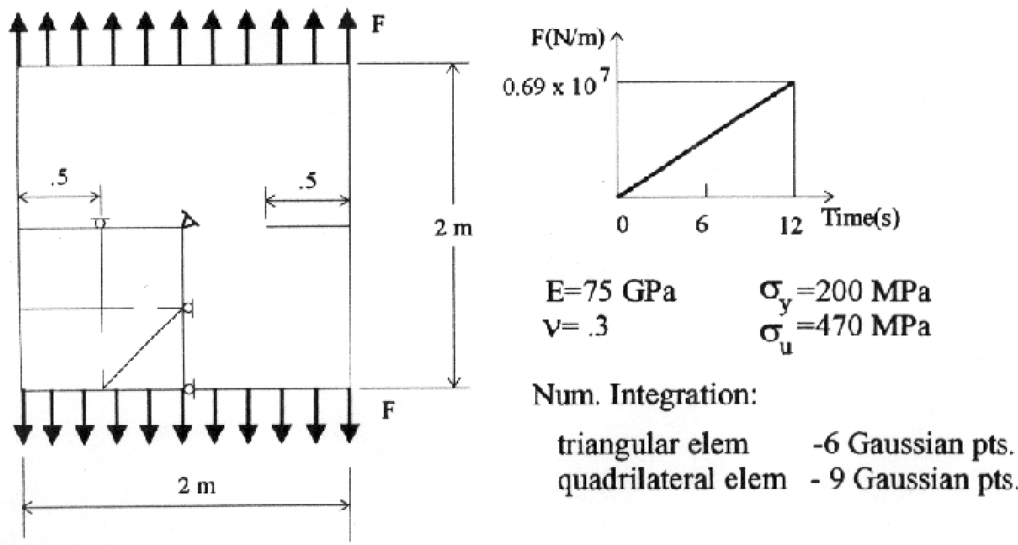


Figure 5 - Doubly-notched Plate Under Uniaxial Loading
Initial Finite Element Meshes Used.

Table II - Energy Error Norms and Number of Degrees-of-freedom Required at each Step for the Doubly-cracked Plate Analyses.

a)triangular element model (6 node)

STEP	Energy Error Norm(%)	Deg-of-Freed	STEP	Energy Error Norm(%)	Deg-of-Freed
1	33.8 - 16.4 - 7.31 - 2.49	2524	7	6.98 - 5.02	3048
2	2.57	2524	8	6.18 - 4.11	3392
3	2.96	2524	9	5.79	3392
4	3.61	2524	10	6.61 - 5.37	3858
5	4.57	2524	11	6.21 - 4.46	4276
6	5.72	2524	12	5.85	4276

b) quadrilateral element model (9 node)

STEP	Energy Error Norm(%)	Deg-of-Freed	STEP	Energy Error Norm(%)	Deg-of-Freed
1	27. - 10.8 - 3.16	1252	7	5.77	1666
2	3.51	1252	8	6.63 - 5.17	1836
3	4.02	1252	9	5.94	1836
4	4.87	1252	10	7.03 - 5.28	2018
S	5.67	1252	11	6.09 - 4.67	2204
6	6.65 - 4.91	1480	12	5.23	2204

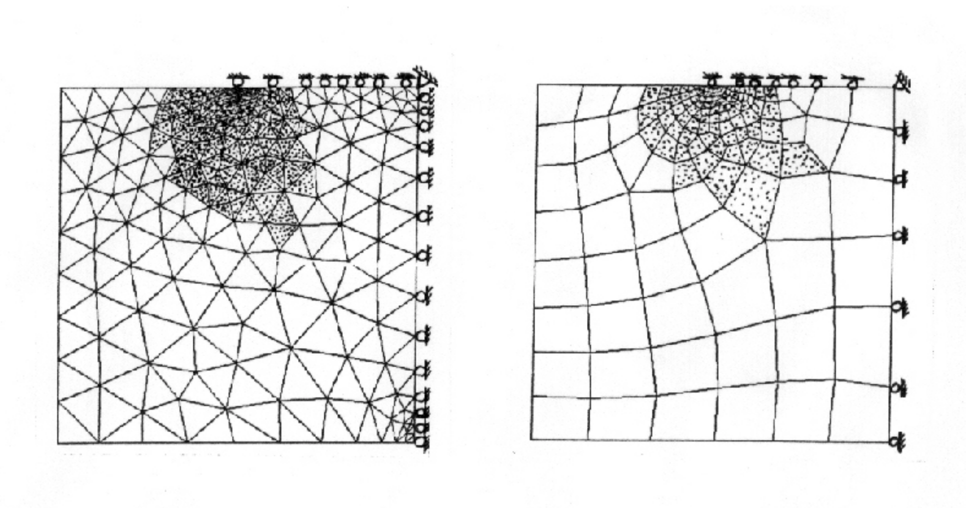


Figure 6 - Final Obtained Meshes with Triangular and Quadrilateral Finite Element Models Used in the Second Exemple.

CONCLUSIONS

Some recent developments with a fully automated h-adaptive strategy for an efficient two-dimensional finite element analysis has been presented and demonstrated with applications in two sampling problems presenting stress singularities. The use of the overlay concept to represent element constitutive behavior beyond yielding in a multi-linear relationship is physically considered as representing the action of individual grains. Moreover, the procedure can be treated as a mathematical artifice and, as stated, extended to general three dimension stress state representations. Moreover, the use of negative overlay weights may allow to simulate material strain-softening effects. As implemented the overall strategy provides an adequate basis for the formulation of a general h-p adaptive procedure.

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