A MODIFIED LOGARITHMIC LAW FOR FLOWS OVER LOW HILLS UNDER NEUTRAL ATMOSPHERE

Cláudio C. Pellegrini
FUNREI – Departamento de Ciências Térmicas e dos Fluidos
Praça Frei Orlando 170, São João del-Rei, MG, 36.300-000. E-mail: pelle@serv.com.ufrj.br

Gustavo C. R. Bodstein
COPPE/UFRJ – Departamento de Engenharia Mecânica
C.P. 68503, 21.945-970, Rio de Janeiro, RJ. E-mail: gustavo@serv.com.ufrj.br

Abstract
In this work, a new modified logarithmic law for flows over 2D hills under neutral atmosphere is proposed. A simplified form of the averaged x-momentum equation is solved using a mixing length turbulence closure. The solution is expressed in the form of a power series correction to the classic logarithmic law, valid for the flat terrain case. As a result of the use of streamline co-ordinates, a new flow parameter appears: the *radius length* of the hill, which is shown to be a function of the hill's geometry. Results for the speed-up are compared with a small set of 'control' field data and agreement is considered to be rather good.

Key-words: flow over hills, logarithmic law, atmospheric boundary layer.

1. INTRODUCTION

The ability to predict the atmospheric boundary layer (ABL) flow over hills has long been of great interest to meteorologists, environmentalists and engineers, among others. To meteorologists, the main interest is probably the development of precise models to forecast weather and climate in large scale. Those models depend critically on some parameterisations adopted for the ABL. In particular, expressions like the logarithmic law, valid for flow over flat terrain, are widely used as a lower boundary condition in those models. The reason is to avoid the integration to be carried out all the way down to the surface, where strong velocity gradients would demand great refining of the computational mesh, making computation slower.

The first description of the vertical wind profile at the ABL over a hill was probably due to Jackson and Hunt (1975). Their linear theory divides the flow field in three regions and provides specific wind profiles for each one of them. Their expressions are, however, not easy to use. The logarithmic law, on the other hand, was always known to represent well the atmospheric flow over flat terrain, under neutral atmosphere and ignoring changes in the wind direction. Panofsky (1973) shows that the logarithmic law remains valid through the lower 150 m of the atmosphere under neutral conditions. Even in the case of very rough surfaces (say for $z_0 > 0.1\,\mathrm{m}$), with individual roughness elements close enough, the velocity profile can still be correctly represented if we displace the origin by *d*. This success suggests that the logarithmic law may be extended to flows over hills.

Few attempts to extend the logarithmic law for flows over hills can be found in the literature. Finnigan (1992) assumes the existence of buoyancy-curvature and acceleration-curvature analogies and obtains a modified logarithmic law, which depends on the curvature Richardson number, R_c, and on the curvature of the *z*-axis (in streamline co-ordinates), L_a. His results were compared to a restricted set of observational data and were encouraging. The drawback of Finnigan's law is its dependence on L_a and R_c, which, in turn, depend on the horizontal velocity, *u*, forming thus an implicit equation. Another result was proposed by Taylor and Lee (1984) and revised by Walmsley et al. (1989) and by Weng et al. (to appear). It appears in the form of an empirical exponential damping of the maximum speed-up over the hilltop. The resulting expressions, however, depend crucially on the calculated value for the maximum speed-up.

In the present work, we propose a modified logarithmic law for the atmospheric flow over a low, 2D hill under neutral stratification conditions. As in the flat terrain case, our expression appears in the form of a flux-profile relationship, establishing a relation between the vertical velocity profile and the momentum flux at the surface. The associated flux-gradient relation and expressions for the velocity speed-up are also derived. The result for the vertical speed-up profile is compared with field data from the Askervein hill (Taylor and Teunissen, 1983 and 1985).

2. **DEFINITION OF THE PROBLEM**

Consider an isolated 2D hill in the middle of an otherwise flat terrain, of constant roughness and under a neutrally stratified atmosphere. For our purposes, we consider a hill to be a topographical variation with characteristic length about 5 Km and height less than 500m. A hill is called low when its slope never exceeds 20°. Fig. 1 illustrates the main features of a typical low hill. The vertical co-ordinate *z* is defined as the height above the local terrain, rather than the vertical height above sea level. For the cases of very large roughness elements, *z* is considered to be the displaced height above the local terrain.

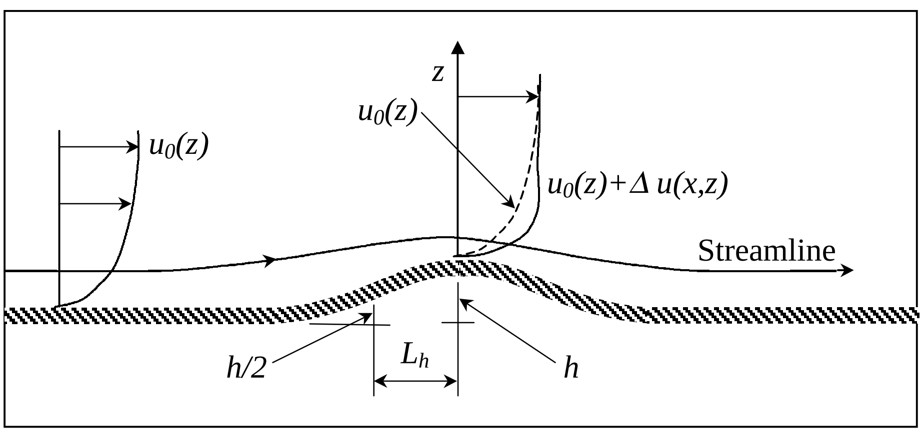

Fig.1. Definitions of h, L_h, Δu, u_0 and z.

In the case under study, we assume that the vertical profile of the mean horizontal wind is essentially logarithmic far from the hill. Hereafter, we refer to this profile as $\overline{u}_0(z)$, and the location upwind of the hilltop (HT) where it is found as the reference site (RS). The RS profile suffers the influence of the hill in such a way that it is modified by a speed-up quantity $\Delta\overline{u}(x,z)$ and becomes $\overline{u}(x,z)$ at a given point over the hill. Thus, we have

$\overline{u}(x,z) \equiv \overline{u}_0(z) + \Delta\overline{u}(x,z)$, .where $\Delta\overline{u}$ is positive at HT, because the flow is accelerated to satisfy the continuity equation. If we divide the speed-up by the RS velocity, we have the relative speed-up, $\Delta S(x,z) \equiv \overline{u}(x,z)/\overline{u}_0(x,z) - 1$.

Many researchers over the last two decades focused their attention on obtaining the vertical profiles of $\Delta\overline{u}$ and ΔS and on calculating the maximum value of ΔS, called $\Delta S_{\max}$, and the height of maximum $\Delta\overline{u}$, called *l*. The series of publications known as the *Guidelines project,* (Taylor and Lee, 1984; Walmsley et al., 1989 and Weng et al., to appear) present a very successful way of estimating $\Delta S_{\max}$.

3. SIMPLIFIED SOLUTION FOR THE GOVERNING EQUATIONS

First, we shall choose an appropriate co-ordinate system. Kaimal and Finnigan (1994) recommend the use of streamline co-ordinates for distorted flows of this type. The most convenient consequence of adopting streamline co-ordinates is the fact that the wind components in the directions perpendicular to the streamline vanish. Finnigan (1983) gives the *x*-momentum turbulent equation, for 2D stationary flow in this system as

$$\overline{u}\frac{\partial\overline{u}}{\partial x} = -\frac{1}{\overline{\rho}}\frac{\partial\overline{p}}{\partial x} - \frac{\partial\overline{u'^2}}{\partial x} - \frac{\partial\overline{u'w'}}{\partial z} + \frac{\overline{u'^2} - \overline{w'^2}}{L_a} + 2\frac{\overline{u'w'}}{R} - g_x\frac{\overline{T}}{\overline{T}_0} + V_x . \tag{1}$$

In eq. (1), *x* is the direction parallel to the streamlines and $\overline{u}$ and *u'* are the mean and turbulent velocities in this direction, respectively. The direction normal to the streamlines is *z*, and $\overline{w}$ and *w'* are the corresponding velocities. The thermodynamic mean pressure is denoted by $\overline{p}$, the mean density by $\overline{\rho}$, the mean temperature by $\overline{T}$, the reference mean temperature by $\overline{T}_0$, the *x*-component gravity acceleration by g_x and the *x*-component mean viscous force by V_x. *R* and L_a are flow length scales and they are related to the mean variables through $R = \overline{u}/(\Omega + \partial\overline{u}/\partial z)$ and $L_a = \overline{u}/(\partial\overline{u}/\partial z)$, where *Ω* is the mean vorticity component in the *z* direction, expressed in the original Cartesian co-ordinate system.

The first simplification of interest to be done on eq. (1) is a boundary layer order-of-magnitude analysis. The classic assumptions are: $u' \sim w'$, $\overline{u} >> u'$, $\overline{w} >> w'$ and $\partial(\,)/\partial x << \partial(\,)/\partial z$. As a result, $(\overline{u'^2} - \overline{w'^2})/L_a = (\overline{u'^2} - \overline{w'^2})(1/\overline{u})\partial\overline{u}/\partial x << \overline{u}\partial\overline{u}/\partial x$ and $\partial\overline{u'w'}/\partial z >> \partial\overline{u'^2}/\partial x$ .If we neglect buoyancy and viscous effects and assume a statically neutral atmosphere, eq. (1) reduces to

$$\overline{u}\frac{\partial\overline{u}}{\partial x} = -\frac{1}{\overline{\rho}}\frac{\partial\overline{p}}{\partial x} - \frac{\partial\overline{u'w'}}{\partial z} + 2\frac{\overline{u'w'}}{R} . \tag{2}$$

The second step in the simplification of the *x*-momentum equation comes from an order of magnitude analysis of the remaining terms as we approach the surface. Initially, turbulence and curvature are smaller than inertia and pressure because of the presence of the turbulence quantities. *R*, also makes the curvature term vanishingly small as we move away from the surface to regions of undisturbed flow. As $z \to 0$, the inertia and pressure terms keep their orders, but turbulence and curvature grow larger because of ∂z and *R* at the denominators, respectively. Therefore, we assume that there exists a region close to the surface where inertia and pressure can be neglected, and eq. (2) becomes

$$\frac{\partial \overline{u'w'}}{\partial z} = 2\frac{\overline{u'w'}}{R}. \tag{3}$$

We observe that as $z \to 0$, the turbulent term grows unbounded whereas the curvature tends to a limit as the local radius of curvature, R, tends to its surface value, R_h. We can infer that close enough to the ground turbulence must dominate curvature. Therefore, there seems to be a region where turbulence and curvature effects balance.

Let us now suppose that the region we are studying is close enough to the surface so that $R(x,z) \approx R_h(x)$. Equation (3) can then be rewritten as

$$\partial\phi/\partial z - p(x)\phi = 0, \tag{4}$$

where $\phi(x,z) = -\overline{u'w'}$ and $p(x) = 2/R_h$. In this form, we notice it is the classic linear first-order differential equation. Multiplying both sides by the integrating factor $\exp(-\int p dz)$ and integrating in z, we obtain

$$\phi = C_1(x)e^{2z/R_h}. \tag{5}$$

If we assume that turbulence can be appropriately represented by the mixing length theory with the mixing length $l_m = kz$, where k is von Karman's constant, we have

$$\left(kz\frac{\partial \bar{u}}{\partial z}\right)^2 = C_1(x)e^{2z/R_h}, \tag{6}$$

Equation (6) is separable in $\bar{u}$, and can thus be integrated. The resulting integral can be found in integral tables and reads

$$\bar{u} = \frac{\sqrt{C_1}}{k}\left[\ln z + \sum_{n=1}^{\infty}\frac{(z/R_h)^n}{n \cdot n!}\right] + C_2(x), \tag{7}$$

where the power series converges for any z in the interval $[-\infty,+\infty]$. C_2 may be obtained assuming that $u=0$ at $z=z_0$, as in the flat terrain case. In order to obtain C_1, we rewrite eq. (6) in the form $-u'w' = C_1(x)e^{2z/R_h}$, and suppose that the turbulent flux of momentum does not vary much in the region. So, $-\overline{u'w'}(x,z) = -\overline{u'w'}(z_0,x) = {u_*}^2$, where the friction velocity, u_*, is defined as $u_* = (\tau_s/\rho)^{1/2}$, with τ_s the surface stress. Therefore, we have ${u_*}^2 = C_1(x)e^{2z/R_h}$ in the region and, supposing that this expression is valid at $z=z_0$, we have ${u_*}^2 = C_1(x)e^{2z_0/R_h} \approx C_1(x)$, because $z_0 << R_h$. Equation (7) can now be written

$$\bar{u} \approx \frac{u_*}{k}\left[\ln\frac{z}{z_0} + \sum_{n=1}^{\infty}\frac{(z/R_h)^n - (z_0/R_h)^n}{n \cdot n!}\right]. \tag{8}$$

This is the modified logarithmic law for flows over hills. We call it, hereafter, the *logarithmic-polynomial* law in analogy to the logarithmic linear law of the non-neutral flat case. Substituting equation (7) in the definitions for $\Delta\overline{u}$ and ΔS, we have

$$\Delta\overline{u} \approx \frac{u_* - u_{*0}}{k} \ln\frac{z}{z_0} + \frac{u_*}{k}\sum_{n=1}^{\infty}\frac{\left(z/R_h\right)^n - \left(z_0/R_h\right)^n}{n \cdot n!}, \tag{9}$$

$$\Delta S \approx \frac{u_*}{u_{*0}}\left[1 + \frac{1}{\ln\left(z/z_0\right)}\sum_{n=1}^{\infty}\frac{\left(z/R_h\right)^n - \left(z_0/R_h\right)^n}{n \cdot n!}\right] - 1. \tag{10}$$

It is easy to verify that expressions (8)—(10) reduce to their flat terrain forms when $R_h \to \infty$. In equations (9) and (10) we used the symbol u_{*0} to represent the friction velocity at RS. This calls attention to the fact that $u_{*0} \neq u_*$ and suggests that equations (8)—(10) may be valid for the up and down-slopes as well as for the HT.

4. COMPARISON WITH OBSERVATIONAL DATA

Comparison of eq. (8), (9) or (10) with field data depends on the parameters z_0, κ, u_* and R_h, which are not know *a priori* for the HT and the rest of the hill. As in the flat terrain case, they must be estimated before the logarithmic-polynomial law can be applied. The value of z_0 is considered to be the same observed at the RS. A great number of cases investigated along the years suggest κ to lie between 0.35 and 0.41. A recent work by Frenzen and Voguel (1995) proposes that κ, in fact, varies with a roughness Reynolds number, defined as $\mathrm{Re}^* = u_* z_0/\nu$, with ν the kinematic viscosity. They suggest an average value of $\kappa = 0.39$ for common applications, valid for $0.007 < z_0 < 0.087$ m, which we shall adopt here. The third parameter, u_* must be calculated from the observational data. Theoretically, R_h could be estimated from the topographical maps of the hill. We did this tentatively but the results were inconsistent. A different method was then employed. The procedure we used is described bellow. Following the method for the flat case, we define

$$Z = \ln(z) + \textstyle\sum_{n=1}^{\infty}(z/R_h)^n \big/ n \cdot n! \quad \text{and} \quad \mathrm{Z}_0 = \ln(\mathrm{z}) + \textstyle\sum_{\mathrm{n}=1}^{\infty}(\mathrm{z}_0/\mathrm{R}_\mathrm{h})^\mathrm{n} \big/ \mathrm{n} \cdot \mathrm{n}!, \tag{11}$$

so that the logarithmic-polynomial law, can be rewritten as

$$u \approx \frac{u_*}{\kappa}(Z - Z_0). \tag{12}$$

Equation (12) is linear in Z, with linear coefficient $b = -(\kappa/u_*)Z_0$ and angular coefficient $m = \kappa/u_*$. Using the observational data available, a best fit could be performed to provide the values of b and m, if the value of R_h could be determined in advance. z_0 could then be calculated through $Z_0 = -b(u_*/\kappa)$ and eqs. (11), and u_* through $u_* = \kappa/m$, assuming $\kappa = 0.39$. Since R_h was still unknown, we carried out the best fit on the RS velocity profiles, where $R_h \to \infty$ and, therefore, $Z = \ln(z)$ and $Z_0 = \ln(z_0)$ on eqs. (11). This analysis provided values for $z_0 = \exp(Z_0) = \exp(-bu_*/\kappa)$ and for $u_{*0} = \kappa/m$. After that, we obtained, by trail and error, a value for R_h that guaranteed at the same time the best fit to the HT velocity profiles and a value of z_0 identical to the one at RS. With the calculated value of R_h we finally obtained $u_* = \kappa / m$.

The analysis we have just described was applied to seven wind profiles over the hill of Askervein. The profiles were directly scanned from the reference work of Mickle et al. (1988) (all available profiles were scanned). The Askervein results were chosen because they are believed to be the most complete field experiment to date (Kaimal and Finnigan, 1994) and they still represent a benchmark for such studies (Walmsley and Taylor, 1996).

Fig. 2 shows the results obtained for R_h.as a function of L_h. over the Askervein hill. As the incident wind direction varies, the half-length of the hill varies too. Therefore, different values of L_h are possible for the same hill. The first remarkable fact about R_h is that it is *at least one order of magnitude smaller than the radius of curvature of the hill, R_{h0}*. This is exactly what happens with z_0 and h_0. For this reason, we shall refer to R_h hereafter as the *radius length of the hill*, in analogy to the roughness length, z_0.

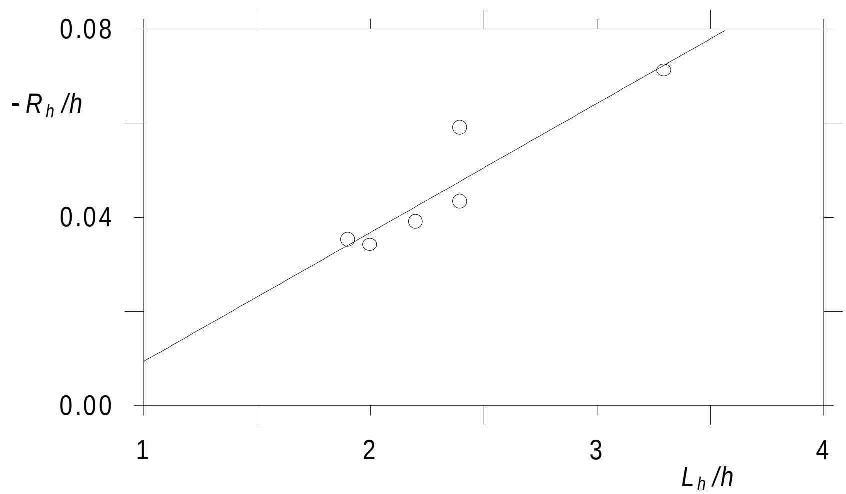

Fig.2. Non-dimensional radius length for the Askervein hill: °°°°: calculated; ——: best fit

An initial analysis of the Askervein data showed it was possible to establish a one-to-one correspondence between R_h and R_{h0} (as with z_0 and h_0). Because R_{h0} is a function of the hill's geometry, it should thus be possible to obtain a correspondence between R_h and the hill's geometry through some appropriate set of non-dimensional parameters. In fact, a simplified analysis through the Buckingham Π theorem shows possible scaling variables to be R_h/h and L_h/h, for a given hill (where R_{h0} is fixed). Fig. 2 shows a plot of the relation between them, confirming (up to the range of data used) that $R_h/h = f(L_h/h)$ indeed. Only one case was excluded from the plot of fig.2: run TU30-B, for wind direction 130^o. The result seemed far out of the trend, indicated by the best-fit line, and we suspect this case to be anomalous due to the incident wind direction nearly parallel to the major axis of the hill. Therefore, no simple 2D modelling could appropriately describe the real flow for run TU30-B. At the subsequent comparisons, this case also showed poor agreement with observational data and a different trend from the remaining six cases.

Figs. (3.a) to (3.g) show a comparison between expression (10) and the Askervein data for the relative speed-up. On average, good agreement is observed. In all cases, rather good agreement is observed below 15 m. Cases (3.a), (3.e), (3.f) and (3.g) deviated from the field results less than 15% almost up to their higher measurement points. Cases (3.b), (3.c) and (3.d) showed poor agreement on higher altitudes. In cases (3.b) and (3.c) the problem appears to be the non-logarithmic behaviour of the wind velocity at RS, as pointed out by the Mickle et al. (1988) who measured the profiles. In case (3.d), the discrepancy may be attributed to the occurrence of 3D effects in the flow, which are not taken into account in our 2D analysis.

The maximum height where predictions could be considered good seems to depend directly on the maximum height where the RS profile could still be considered logarithmic, as one would expect. All results showed some sensitivity to the calculated values of u_* and R_h, but almost no sensitivity to the choice of z_0. In all cases, the power series converged rather slowly. It was also observed that the number of terms necessary for convergence increased as z/R_h increased.

Good predictions were obtained for the maximum value of the speed-up. A detailed analysis of expression (10) has also shown that the maximum value of ΔS occurs at $z=z_0$, as suspected by many field studies (e.g. Mickle et al., 1988).

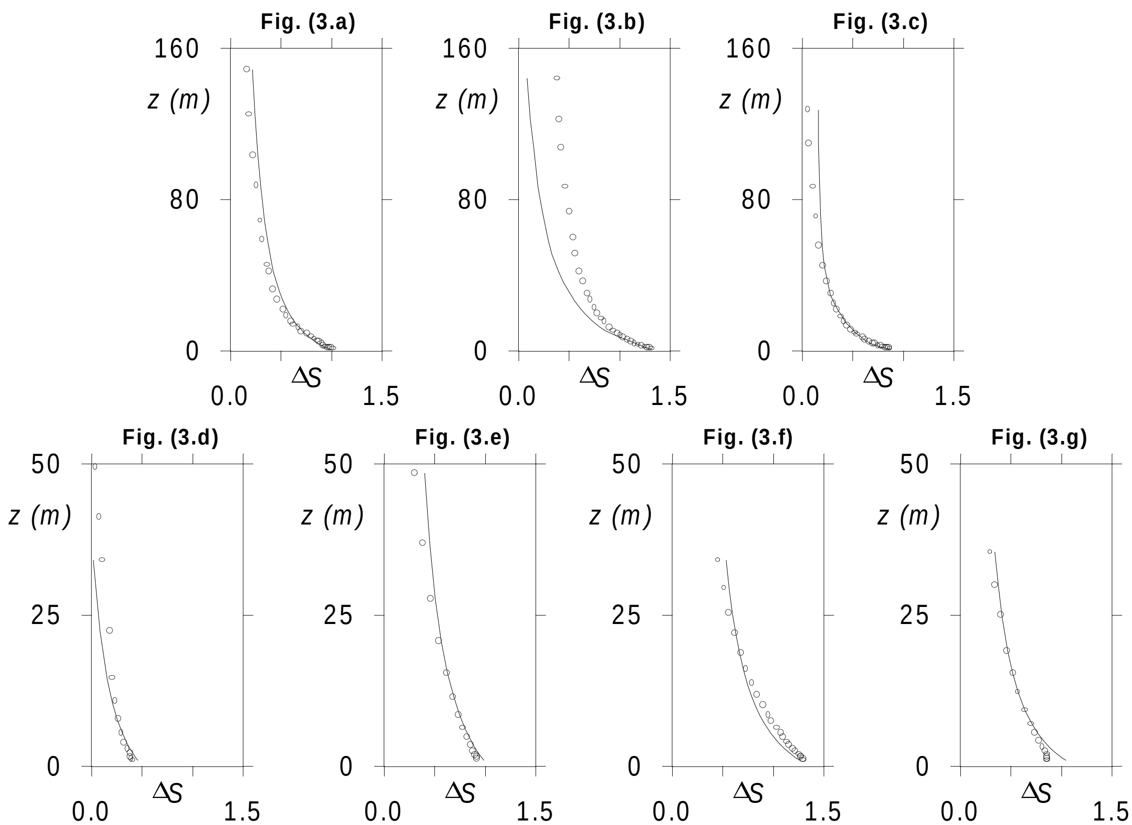

Fig. 3. Relative speed-up. Observation (Askervein hill): °°°°; (2.a): run TK02-A, direction, $\varphi=165^o$; (2.b): TK01-B, $\varphi=180^o$; (2.c): TK07-B, $\varphi=260^o$; (2.d): TU30-B, $\varphi=130^o$; (2.e): TU01-B, $\varphi=180^o$; (2.f): TU07-A, $\varphi=210^o$; (2.g): TU05-B, $\varphi=305^o$; Eq. (9): ——.

5. SUMMARY AND CONCLUSIONS

In this work, we propose a new modified logarithmic law for flows over 2D low hills under neutral atmosphere. The results for the velocity profiles were found to depend on a new parameter: *the radius length*, R_h. In fact, R_h is expected to depend on the flow characteristics and on the hill's geometry, through parameters such as R_{h0} and L_h, for example. For a given fully-developed turbulent flow over a fixed hill, where there is no Reynolds number dependence and R_{h0} is fixed, we have shown that R_h is strongly dependent on L_h. However, more data analysis is needed to provide a general relationship among all these parameters. In the future, we expect to be able to show that in the turbulent ABL, R_h can, in fact, be univocally related to the hill's geometry. This suggests that R_h. might come to be tabulated once and for all as z_0 is.

Comparison of expression (10) with observational data was performed and showed overall good agreement. More data analysis is needed to explain the poor agreement in cases (3.b), (3.c) and (3.d). In case (3.d), we believe that 3D affects may be important, since Askervein is an elliptical hill and in this run the flow is nearly aligned with its major axis. In cases (3.b) and (3.c) the problem appears to be the non-logarithmic behaviour of the RS profiles. The results are encouraging, but a more thorough analysis is needed.

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