PARABOLIZED STABILITY EQUATIONS: A REVIEW

Márcio T. Mendonça
Centro Técnico Aeroespacial - Instituto de Atividades Espaciais
Pç Mal. Eduardo Gomes, 50 - 12228-904 - São José dos Campos - SP - Brazil
email: marcio_tm@yahoo.com

Summary.

The Parabolized Stability Equations developed at the Ohio State University have given scientist a new tool to investigate hydrodynamic stability. This formulation results in a set of parabolic equations that describe the downstream evolution of convected instabilities in slowly varying shear flows such as boundary layers, jets, and wakes. It is able to consider nonlinear and nonparallel effects in a spatial analysis resulting in a much better representation of the flow physics. The present paper presents a review of the PSE formulation with emphasis on the discussion about the choice for the normalization condition, which is necessary to close the system of equations. Results obtained by the author and co-workers with their own implementation of a PSE code are also reviewed. First, the importance of non-parallel effects is discussed. Then, results for the evolution of stationary and traveling disturbances are presented and compared to experimental results.

Keyword: *Laminar flow instability, Laminar-turbulent transition, Parabolized stability equations.*

1. INTRODUCTION

In a natural environment a laminar flow is always subject to disturbances such as freestream turbulence, rugosity or structural vibration. if these disturbances are amplified the laminar flow may become turbulent. The field of hydrodynamic stability is concerned with the study of how a given flow field may amplify or damp these initial disturbances and how the evolution of this disturbances is related to transition to turbulence.

The equations that describe an instability problem may be derived by assuming that the instantaneous flow is decomposed into a mean steady flow and a small perturbation, $v = V + v'$. Initial disturbances of small amplitude propagate in the flow as wave structures that may be represented by their frequency, wavenumber and growth rate. In this way a small perturbation has a general form $v' = v(y)\exp[i(\alpha x + \beta z - \omega t)]$. Where x, y and z are the streamwise, normal and spanwise coordinate directions, α and β are the streamwise and spanwise wavenumbers, and ω the disturbance frequency. This solution is called a 'normal mode solution'.

The early work on stability of shear layers neglected the growth of the shear layer such that the mean flow could be represented by $V = V(y)$ only, independent of the streamwise direction x. This assumption that the flow is 'locally parallel' greatly simplifies the resulting governing equation. The resulting ordinary differential equation is know as the 'Orr-Sommerfeld' equation (after the two scientist who derived the equation in the beginning of the century) and correspond to what is called a 'local analysis' since it does not depend explicitly on the streamwise coordinate. Another simplifying assumption was that the disturbances grow or decay in time (temporal analysis), such that $\omega = \omega_r + i\omega_i$. The imaginary part represents the disturbance growth rate. With the advance of mathematical and computational methods it was possible to represent spatially growing disturbances assuming that $\alpha = \alpha_r + i\alpha_i$, which results in a more complex equation.

Over the years the Orr-Sommerfeld equations have been extensively used and modified to account for nonparallel (Gaster, 1974; Saric and Nayfeh, 1975) and nonlinear effects (Eckhaus, 1965; Herbert, 1988). These studies were based on secondary instability theory and perturbation methods which involve a large amount of algebraic work. In the late eighties and early nighties, Herbert and Bertolotti (Herbert and Bertolotti, 1987; Bertolotti and Herbert, 1991) realized that using WKBJ approximations or multiple scales, it was possible to arrive at parabolized equations that could be marched downstream without further simplifications to arrive at ordinary differential equations and, consequently without further limiting assumptions. They called the resulting equations 'Parabolized Stability Equations' (PSE).

Since then, the PSE have been applied to a diverse range of laminar flow stability problems, including incompressible and compressible boundary layers, cross flow instability, centrifugal instability, wave interactions, receptivity and secondary instability analysis.

The PSE can take into account both nonlinear and nonparallel effects in a consistent way. It takes into account the history of the disturbance since it is a parabolic formulation. Unlike temporal analysis, in which it is assumed that the disturbances grow in time, it follows the spatial development of the flow. Compared to DNS solutions PSE solutions are much less computer intensive and can be run on desktop workstations.

2. PARABOLIZED STABILITY EQUATIONS

The formulation and numerical method presented in this paper was implemented in a code developed by the author (Mendonça, 1997). It is based on the original PSE formulation developed by Herbert and Bertolotti (Herbert and Bertolotti, 1987; Bertolotti and Herbert, 1991).

The Navier-Stokes equations for an incompressible flow of a Newtonian fluid are simplified by assuming that the dependent variables are decomposed into a mean component and a fluctuating component as: $\vec{u^*} = \vec{U^*} + \vec{u'^*}$, and $p^* = P^* + p'^*$, where $\vec{u^*} = [u^*, v^*, w^*]^T$ is the velocity vector and p^* is the pressure. The superscript '$*$' indicates dimensional variables.

The coordinate system is based on the streamlines (ψ^*) and potential lines (ϕ^*) of the inviscid flow over a curved plate. This choice of coordinate system is used to simplify the equations in curvilinear coordinate systems for the analysis of centrifugal instabilities.

The equations are nondimensionalized using δ_0^* and U_∞^* as the length and velocity scaling parameters, where $\delta_0^* = (\nu^* \phi_0^* / U_\infty^*)^{1/2}$ is the boundary layer thickness parameter, U_∞^* is the free stream velocity, ϕ_0^* is a reference length taken as the streamwise location where initial conditions are applied, and ν^* is the kinematic viscosity. The Reynolds number is defined as: $Re = U_\infty^* \delta_0^* / \nu^*$.

The mean flow is governed by Prandtl boundary layer equations for the flow over a flat plate. The resulting governing equations for the perturbations are elliptic and the perturbations propagate in the flow field as wave structures. The governing equations can be simplified if the wave like nature of the perturbations are represented by their frequency, wavenumber, and growth rate. The perturbation Φ' is assumed to be composed of a slowly varying shape function and an exponential oscillatory wave term. It is represented mathematically as a Fourier expansion truncated to a finite number of modes:

$$\Phi' = \sum_{n=-N}^{N} \sum_{m=-M}^{M} \Phi_{n,m}(\phi, \psi) \exp \left[\int_{\phi_0}^{\phi} a_{n,m}(\xi) d\xi + im\beta z - in\omega t \right]. \tag{1}$$

where $a_{n,m}(\phi) = \gamma_{n,m}(\phi) + in\alpha(\phi)$, and $\Phi_{n,m}(\phi, \psi) = [u_{n,m}, v_{n,m}, w_{n,m}, p_{n,m}]^T$ is the complex shape function vector. This procedure is similar to a normal mode analysis, but, in this case, the shape function $\Phi_{n,m}$ is a function of both ϕ and ψ.

The streamwise growth rate $\gamma_{n,m}$, the streamwise wavenumber α, and the spanwise wavenumber β were nondimensionalized using the boundary layer thickness parameter δ_0^*. The frequency ω was nondimensionalized using the free stream velocity U_∞^* and the boundary layer thickness parameter δ_0^* .

The perturbation variable Φ', as defined in Eq. (1), is substituted in the governing equations which are then simplified by assuming that the shape function, wavelength, and growth rate vary slowly in the streamwise direction. Second order derivatives and products of first order derivatives can, therefore, be neglected. After performing a harmonic balance in the frequency, a set of coupled nonlinear equations is obtained. These resulting equations are known as the Parabolized Stability Equations (PSE). For each mode (n, m) the equation in vector form results:

$$\overline{A}_{n,m}\Phi_{n,m} + \overline{B}_{n,m}\frac{\partial \Phi_{n,m}}{\partial \phi} + \overline{C}_{n,m}\frac{\partial \Phi_{n,m}}{\partial \psi} + \overline{D}_{n,m}\frac{\partial^2 \Phi_{n,m}}{\partial \psi^2} = \frac{\overline{E}_{n,m}}{e^{\int_{\phi_0}^{\phi} a_{n,m}(\xi) d\xi}}, \tag{2}$$

where the coefficient matrices can be found in Mendonça (1997).

The resulting equations are parabolic in ϕ and the solution can be marched downstream given initial conditions at a starting position ϕ_0. The approach is correct as long as the instabilities are convective and propagate in the direction of the mean flow, not affecting the flow field upstream.

The boundary conditions for Eq. (2) are given by homogeneous Dirichlet no-slip conditions at the wall, Neumann boundary conditions for the velocity components in the far field, and homogeneous Dirichlet condition for pressure in the far field. Non-homogeneous boundary conditions for the normal velocity components are also possible, allowing suction and blowing at the wall to be introduced. For the parabolic formulation, it is necessary to specify initial conditions at a starting position ϕ_0 downstream of the stagnation point at the leading edge of the curved plate. The initial conditions are obtained from Orr-Sommerfeld solutions for Tollmien-Schlichting waves and similar local solutions for centrifugal instability problems (Görtler or Dean problems).

2.1 Normalization condition

The splitting of the perturbation $\Phi'(\phi, \psi, z, t)$ in Eq. (1) into two functions, $\Phi_{n,m}(\phi, \psi)$ and $a_{n,m}(\phi)$, is ambiguous, since both are functions of the streamwise coordinate ϕ. It is necessary to define how much variation will be represented by the shape function

$\Phi_{n,m}(\phi,\psi)$, and how much will be represented by the complex wavenumber $a_{n,m}(\phi)$. This definition has to guarantee that rapid changes in the streamwise direction are avoided so that the hypothesis of slowly changing variables is not violated. The objective is to transfer fast variations of $\Phi_{n,m}(\phi,\psi)$ in the streamwise direction to the streamwise complex wavenumber $a_{n,m}(\phi) = \gamma_{n,m}(\phi) + in\alpha(\phi)$. If this variation is represented by $b_{n,m}$, for each step in the streamwise direction it is necessary to iterate on $a_{n,m}(\phi)$ until $b_{n,m}$ is smaller than a given threshold. At each iteration k, $a_{n,m}(\phi)$ is updated according to $(a_{n,m})_{k+1} = (a_{n,m})_k + (b_{n,m})_k$. The variation $b_{n,m}$ of the shape function can be monitored in different ways. Possible choices are presented below.

$$b_{n,m} = \frac{1}{\int_0^\infty \|\vec{u}_{n,m}\|^2 d\psi} \int_0^\infty \left(\vec{u}^{\,\dagger}_{n,m} \cdot \frac{\partial \vec{u}_{n,m}}{\partial \phi} \right) d\psi, \tag{3}$$

$$b_{n,m} = \frac{1}{\int_0^\infty E d\psi} \int_0^\infty \frac{\partial E}{\partial \phi} d\psi. \tag{4}$$

$$b_{n,m} = \frac{1}{|u_{n,m}(x, y_{max})|} \frac{\partial u_{n,m}(x, y_{max})}{\partial \phi}, \tag{5}$$

In Eq. (3), $\vec{u}^{\dagger}_{n,m}$ is the complex conjugate of $\vec{u}_{n,m}$. The integral of $\|\vec{u}_{n,m}\|^2$ was used to assure that the variation is independent from the magnitude of $\vec{u}_{n,m}$. Equation (4) monitors the variation of the Kinetic energy $E = u^2_{n,m} + v^2_{n,m} + w^2_{n,m}$. In Eq. (5) $u_{n,m}(x, y_{max})$ is the streamwise velocity component measured at the location away from the wall where it reaches a maximum.

2.2 Numerical method

The system of parabolic nonlinear coupled equations given by Eq. (2) is solved numerically using finite differences. The partial differential equation is discretized implicitly using a second order backward differencing in the streamwise direction, and fourth order central differencing in the normal direction. The resulting coupled algebraic equations form a block pentadiagonal system which is solved by LU decomposition.

To start the computation a first order backward differencing is used. The first order approximation is used also in a few subsequent steps downstream in order to damp numerical transients more efficiently. For the points neighboring the boundaries, second order central differencing in the normal direction is used.

The nonlinear terms are evaluated iteratively at each step in the streamwise direction. The iterative process is used to enforce both the normalization condition and the convergence of the nonlinear terms. A Gauss-Siedel iteration with successive overrelaxation is used. The nonlinear products are evaluated in the time domain. The dependent variables in the frequency domain are converted to the time domain by an inverse Fast Fourier Transform subroutine. The nonlinear products are evaluated and the results are transformed back to the frequency domain. The complex wavenumber is updated at each iteration and the variation in the shape function is monitored through Eq. (3). The iteration is considered converged when the normalization condition is no larger than a given small threshold.

Results from the present numerical implementation of the PSE have been compared to experimental and numerical results for K-type breakdown, H-type breakdown and for the nonlinear development of Görtler Vortices. The code was able to reproduce the nonlinear development of interacting disturbances with good accuracy.

3. ON THE NORMALIZATION CONDITION

Using the code developed based on the above formulation, we may now investigate the effect of normalization condition on the results obtained for the evolution of Tollmien-Schlichting waves. Figure 1 presents a comparison between the three different normalization conditions presented above for the variation of the amplitude of a two-dimensional (2D) TS wave of frequency $F = \omega/Re10^6 = 86$. The computation starts at a stable position close to the lower branch of the neutral curve and continues up to a position past the upper branch of the neutral curve. It can be seen that the results are not dependent on the choice of the normalization condition. Figure 2 and 3 present the variation of the growth rate and streamwise wavenumber for the three different choices of the normalization condition presented above. It can be seen that the choice of the normalization condition changes slightly the value of these variables.

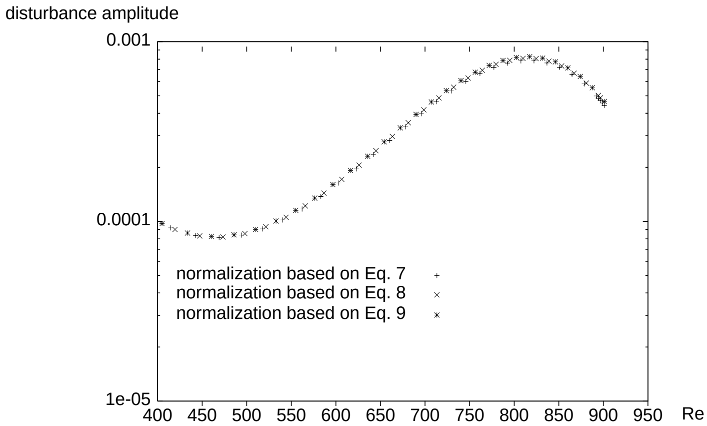


Figure 1: Growth and decay of the maximum amplitude of a 2D TS wave for different choices of normalization condition.

The real (physical) wavenumber and growth rate should take into account the variations left in the shape function. This is done by computing:

$$\overline{\gamma}(x) = \gamma(x) + \left[\frac{1}{u_{n,m}(x, y_{max})}\frac{\partial u_{n,m}(x, y_{max})}{\partial \phi}\right]_r,$$

$$\overline{\alpha}(x) = \alpha(x) + \left[\frac{1}{u_{n,m}(x, y_{max})}\frac{\partial u_{n,m}(x, y_{max})}{\partial \phi}\right]_i,$$

where the subscripts r and i represent the real and imaginary parts respectively.

It can be seen in figure 4 and 5 that the physical quantities are independent of the choice for the normalization condition. In fact, since the normalization condition transfers fast variations of the shape function $\Phi_{n,m}$ to the exponential function $\exp\left[\int a_{n,m}(\xi)d\xi\right]$, as long as the hypothesis of slow variation of the shape function in the streamwise direction is not violated, small variations in the normalization condition does not change the results.

4. NONPARALLEL EFFECTS

Nonparallel effects are significant mainly for the evolution of three-dimensional (3D) disturbances. Using PSE to compute the growth rate of 2D TS waves Bertolotti (Bertolotti

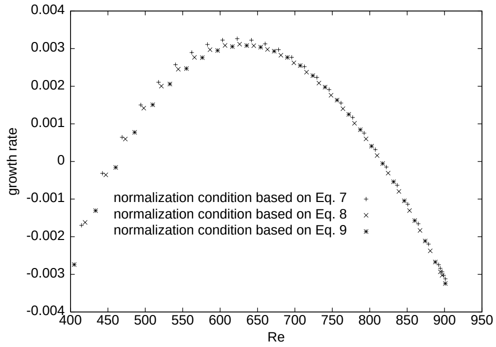


Figure 2: Variation of the growth rate for different choices of the normalization condition.

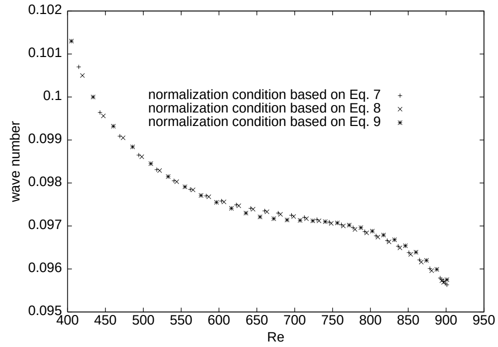


Figure 3: Variation of the streamwise wavenumber for different choices of the normalization condition.

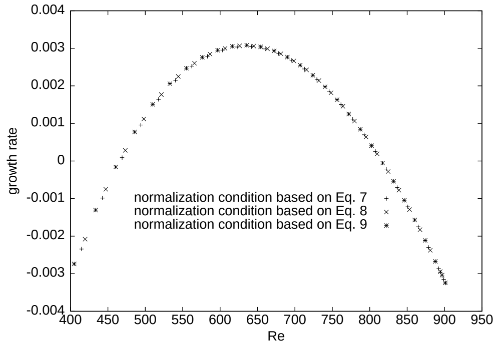


Figure 4: Variation of the physical growth rate.

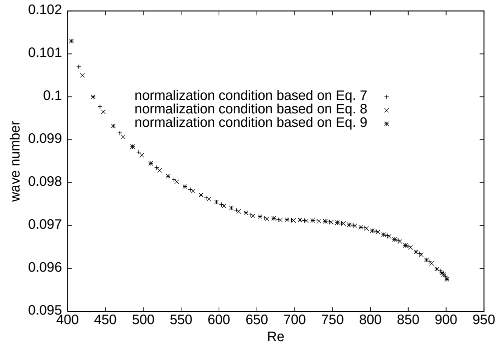


Figure 5: Variation of the physical streamwise wavenumber.

et al. , 1992) showed that the large discrepancies observed between experimental results and normal modes computations could not be attributed to the small effect of nonparallelism. These discrepancies are due to the choice of the way the disturbance growth is monitored, either based on amplitude, total kinetic energy or at a fixed distance from the wall.

For 3D TS waves nonparallel effects increase with the increase of the angle between the mean flow direction and the wave propagation direction. The present PSE implementation is now used to show this effect. Figure presents comparisons between local parallel computations and nonparallel PSE computations for different wavenumbers $b = \beta 10^3/Re$. The TS wave frequency is $F = \omega 10^6/Re = 86$. It can be seen that as b increases the oblique waves are destabilized by nonparallel effects. For $b = .3$ the parallel theory predicts stable waves while nonparallel theory still predicts unstable disturbances for a range of Reynolds numbers.

5. SOME NONLINEAR RESULTS

In this section some computational results for the nonlinear evolution of 3D disturbances are presented and compared to experimental results to illustrate the capabilities of the PSE. Figure 7 presents results for the nonlinear evolution of Görtler vortices. The computational results are compared to experimental results (Swearingen and Blackwelder, 1987) for two different streamwise positions. AT $x = 60$ cm the spanwise periodic structure is already visible. At $x = 80$ cm the vortices are forming the characteristic mushroom

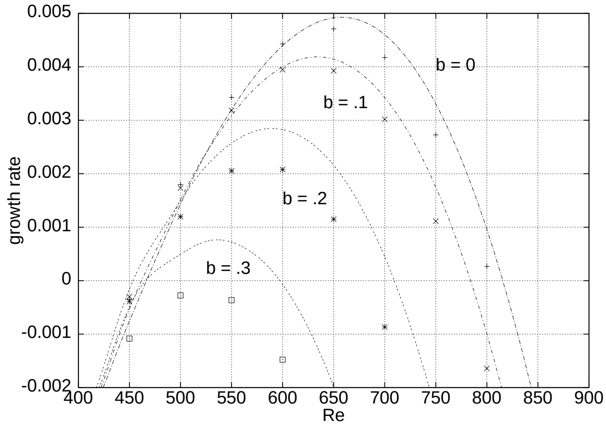


Figure 6: Growth rate based on maximum amplitude. Comparison between parallel (symbols) and nonparallel computations (lines).

type structures. The computational results compare well with the experimental results.

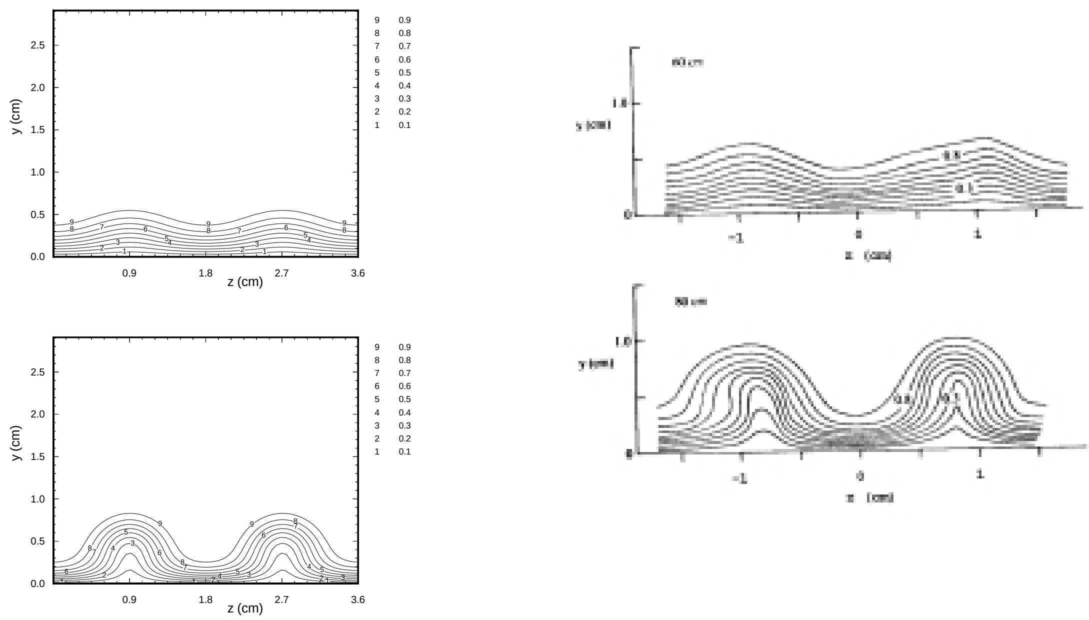
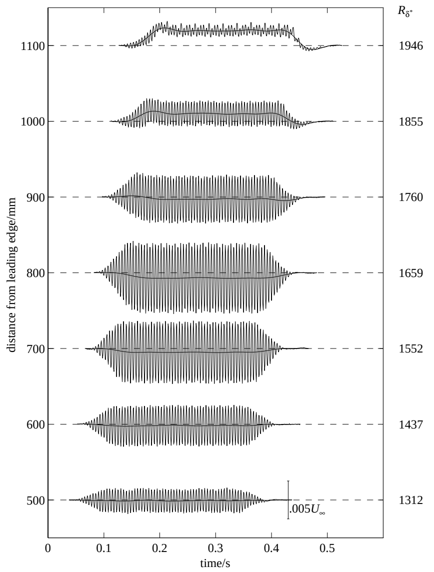


Figure 7: Nonlinear evolution of Görtler vortices. Comparison between PSE results (left) and experimental results (right) (Swearingen and Blackwelder, 1987) for the streamwise velocity distribution in the spanwise plane.

Figures 8 presents experimental results for the evolution of a 3D wave-train emanating from a harmonic point source. The nonlinear evolution is responsible for the development of streamwise streaks. These streaks evolve downstream and, at the centerline a negative streak splits into two. A positive streak grows between the two resulting streaks. A model problem consisting of a pair of oblique waves was used to represent the evolution of a wave-train. The computational result for this model problem shows the growth of longitudinal vortical structures given by Fourier modes (0,2) and (0,4). The corresponding, spanwise periodic, longitudinal streaks are shown in Fig. 9. The figure shows a positive streak forming at the centerline, growing close to the wall and splitting a negative streak in a way consistent with experimental results.

Detailed discussions for these results are presented elsewhere (Mendonça, 1997; Mendonça *et al.* , 2000b; Mendonça *et al.* , 2000a; Mendonça and Medeiros, 1999; Medeiros and Mendonça, 1999).



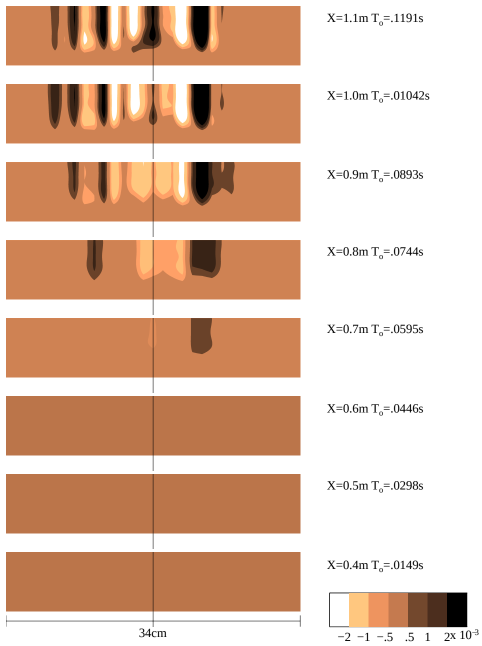


Figure 8: Experimental results for the evolution of a three-dimensional wave-train emanating from a point source. Left – the centerline evolution showing the growth of a mean flow distortion that changes from negative to positive. Right – downstream evolution of longitudinal streaks showing the spanwise structure with the splitting of a negative streak and the growth of a positive streak at the centerline.

6. CONCLUSIONS

The Parabolized Stability Equations can be used to study the propagation of traveling and stationary convected disturbances in slowly varying shear layers. They are able to take into account nonlinear, nonparallel effects in a consistent way. The PSE involves less analytical work than a multiple scales analysis and is much less computationally intensive than a direct numerical simulation.

A PSE code has been successfully implemented to study the nonlinear interaction of three-dimensional disturbances. This code has been used by the author and co-workers to investigate the evolution of longitudinal stationary structures due to centrifugal and nonlinear effects. The present review shows that the choice of the normalization condition does not affect the physical growth rate and wavenumbers. It also shows that nonparallel effects are significant for three-dimensional disturbances and should be taken into account in a stability analysis.

5. ACKNOWLEDGMENT

The author would like to acknowledge the financial support received from FAPESP for the participation in the CONEM 2000.

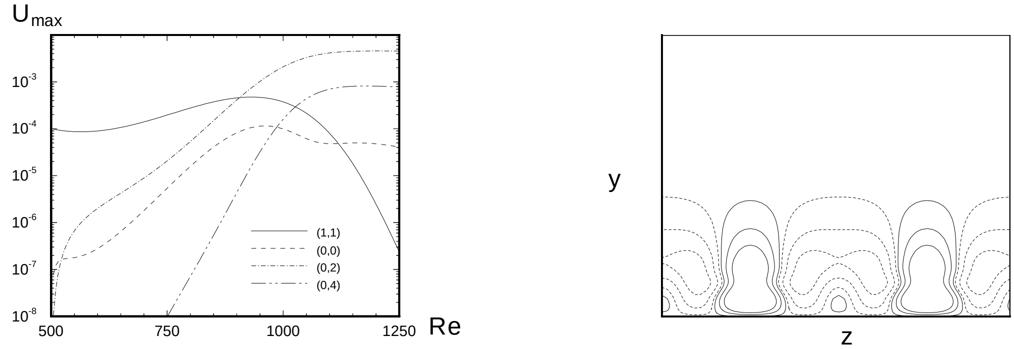


Figure 9: Evolution of a par of oblique waves as a model problem. left – Amplitude variation of different Fourier modes showing the evolution of a mean flow distortion with spanwise periodicity given by Fourier modes (0,0), (0,2) and (0,4). right – Spanwise structure showing the splitting of a streamwise streak in a way consistent with the experimental results (solid lines - positive velocity, dashed lines - negative velocity).

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