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FAST ALGORITHMS FOR ALGEBRAIC WAVELET MULTIGRID METHOD APLIED TO TWO-DIMENSIONAL HEAT EQUATION

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Abstract. *Fast algorithms for algebraic multigrid method are presented, in which discrete wavelet transform are used to construct the system matrices to be solved, obtained via mathematical model discretization in different multigrid meshes as well as the transfer vectors for the various levels of the V cycle, used in this article. Length 2 Haar filters and length 4 Daubechies filters with decimation by a factor of 2 and 4 are employed in this research. The efficiency of these algorithms is demonstrated by numerical resolution of the two-dimensional Laplace equation.*

Keywords: *Algebraic Multigrid, Wavelets, Filters of Daubechies*

1. INTRODUCTION

Mechanical Engineering problems can be solved experimentally or via mathematical models. Among the advantages of the mathematical models there are two major situations where it could be used: (i) the absence of laboratory tests that are usually costly in financial terms, and (ii) a great versatility in the simulations. Mathematical models can be solved by analytic or numerically. In general, for specific and complex problems that appear in the daily, the models are basically solved numerically via computer codes. The first step is to numerically solve a mathematical model is the discretization of the model, resulting in a linear system of equations that can be solved using direct or iterative methods. An iterative method is preferred for solving a linear system as shown in Eq. (1) when the matrix $\boldsymbol{A}$ is large or sparse, according to (Burden and Faires, 2003). In the Eq. (1), the vector $\boldsymbol{u}$ is the exact solution being found for the system, i.e., unknown term. The matrix $\boldsymbol{A}$ and the vector $\boldsymbol{f}$ are known.

$$\boldsymbol{Au} = \boldsymbol{f} \tag{1}$$

For a better understanding of the procedures for solving linear systems by means of stationary iterative methods, see the article (Fagundes *et al.*, 2009a). The question of iterative methods like Gauss-Seidel (GS) is that the convergence becomes very slow in that the error becomes smooth. An alternative to accelerate the convergence of the problem is the use of the multigrid techniques.

The multigrid is currently recognized as an efficient technique to accelerate the convergence of iterative methods. The basic idea is to work with different meshes. The argument is that classical iterative methods such as GS smoothing the error making slow its removal. And a smooth error in a refined mesh becomes oscillatory when transferred to a coarse mesh. This oscillatory error is easily removed with a few iterations.

There are two types of multigrid, the geometric multigrid (MG) indicated for problems in which it is possible to establish a sequence of meshes, and Algebraic multigrid (AMG) useful when it is difficult or impossible to establish a hierarchy of discretizations. In AMG all information is obtained from the matrix of the original system. The initial difficulty of AMG is the choice of the matrix elements of the system refined mesh which should represent the problem in a coarser mesh. This difficulty was overcome by AMG via wavelets (WAMG). For a proper understanding of multigrid, consult the authors (Wesseling, 1992; Mccormick, 198; Briggs and Henson , 2000; Trottenberg *et al.*, 2001). To meet WAMG suggests the following studies (De Leon, 2000; Pereira, 2007; Fagundes *et al*. 2009b).

This paper is organized as follows: section 2 presents an overview of multigrid, as well as new algebraic multigrid approach via wavelets (WAMG). Section 3 provides the mathematical model of the two-dimensional Laplace equation and the numerical results obtained via WAMG. Section 4 shows the conclusions of this work.

2. MULTIGRID

The first reference found in the literature concerned to accelerate the convergence of iterative processes dating from 1950. It is a text written in Russian (Abramov, 1950 *apud* Fedorenko, 1964). The own Fedorenko paper citing Abramov was also originally written in Russian and translated into English by D. E. Brown. The term referenced in its

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mathematical formulation is *'auxiliary net'* (Fedorenko, 1964). The MG was developed in the 1970s independently by Brandt and Hackbusch, when they published the first practical work (Brandt, 1973; Hackbusch, 1976).

Three elements are necessary for the implementation anyone multigrid: (i) restriction operator R which transfers the information from a fine mesh h to a coarse mesh H, (ii) prolongation operator P which transfers the information from a coarse mesh to a fine mesh; (iii) and from the matrix of the problem to be solved to the different levels of the mesh. Both the prolongation P and the restriction R are matrices. The matrices (of the problem to be solved) in the different levels can be obtained through problem discretization, if MG is being employed, or through Galerkin condition, (except for the AMG) which expression is (Briggs and Henson, 2000)

$$\boldsymbol{A}^H = \boldsymbol{R}\boldsymbol{A}^h\boldsymbol{P} \tag{2}$$

The computational cost for linear systems by direct methods is $O(n^3)$, while the classical iterative methods with Jacobi and GS is $O(n^2)$, where n is the number of unknowns. The computational cost of the multigrid technique is ideally $O(n)$, i.e. the cost is proportional to the number of unknowns (Falgout, 2006). Soon for large systems the multigrid is a powerful technique.

2.1 Algebraic multigrid via wavelest

An important characteristic of the algebraic multigrid (AMG) is the strong influence concept. This enables to select only the most significant elements from the $\boldsymbol{A}$ coefficients matrix in the fine mesh, h, to represent the problem in the coarse mesh, H. However, it is known that the computational implementation is not a simple one, which allows new approaches to the AMG, called AMG via wavelets (WAMG). (Mallat, 1989; Chui, 1992; Briggs and Henson, 1993; Moretin, 1999; Wang, 2000; Avudainayagam, 2004).

In WAMG the prolongation operators P and restriction R that appear in Eq. (2) can be constructed via filters bank (Pereira, 2007; Garcia *et al*., 2008;Fagundes *et al*., 2013). In this study Haar and Daubechies filters are used (Daubechies, 1988).

The multigrid, geometric (MG), algebraic (AMG), or algebraic via wavelets (WAMG), can be operationalized through the following algorithm.

Algorithm 1. Correction scheme with two mesh levels

1) Apply υ relaxation steps in $\mathbf{A}^h\boldsymbol{u}^h = \boldsymbol{f}^h$ with an initial estimate $\boldsymbol{v}^h$
2) Calculate $\boldsymbol{r}^h = \boldsymbol{f}^h - \boldsymbol{A}\boldsymbol{v}^h$
3) Calculate $\boldsymbol{r}^H = \boldsymbol{R}\boldsymbol{r}^h$
4) Apply υ relaxation steps in $\boldsymbol{A}^H\boldsymbol{e}^H = \mathbf{r}^H$, with null initial $\boldsymbol{e}^H$.
5) Calculate $\boldsymbol{e}^h = \boldsymbol{P}\boldsymbol{e}^H$
6) Do $\boldsymbol{v}^h \leftarrow \boldsymbol{v}^h + \boldsymbol{e}^h$
7) Apply υ relaxation steps in $\boldsymbol{A}^h\boldsymbol{u}^h = \boldsymbol{f}^h$ with an initial estimate $\boldsymbol{v}^h$

Algorithm 1 can be recursively implemented until the coarsest mesh possible, when step (4) must then be solved through direct methods, and then return to the finest mesh. This algorithm is called V-cycle. The number of cycles is defined as the number of times that the cycle is carried out in order to reach the tolerance required for the experiment. In the correction scheme (CS), the solution of the problem is solved (iteratively) in just more refined mesh. The other levels only the error is determined. The idea is to apply some iterations in step (1) until the error becomes smooth. In practice a small number of iterations is fixed, usually around 3.

For implementation steps (3) and (5) of the algorithm 1, to WAMG, recent work (Fagundes *et al., 2013)* proposed fast algorithms without cost matrix operations.

3. NUMERICAL RESULTS

In this section some numerical results for two-dimensional Laplace Equation as Eq. (3) is presented. The tests were carried out through a computer with Intel processor (R) Core(TM) i7-2600 CPU@, 3.40GHz, 16.0 GB of RAM memory. The computational code was written in a MATLAB© compiler version 7.8. The correction scheme (CS), and V-cycle are used. The initial solution $\boldsymbol{v}$ is zero, and the method to solving linear equations system is Gauss-Seidel (GS), with and without over relaxation. Numerical results consider following nomenclature. WAMG via length 2 Haar filter (WAMG_hD2(2)), and WAMG via length 4 Daubechies filter WAMG_hD4(2) and WAMG_hD4(4), with decimation by a factor of 2 and 4, respectively. The tolerance for convergence with maximum error for the lower residue is 10^{-10}.

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$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0\ ,\ 0 < x, y < 1. \tag{3}$$

The Eq. (3) is solve with the Dirichlet's boundary conditions, i.e., $T(x, 0) = T(0, y) = T(1, y) = 0$; $T(x, 1) = \mathrm{sen}(\pi x)$. The analytical solution for Eq. (3) for specified boundary conditions is

$$T(x, y) = \sin(\pi x)\frac{\sinh(\pi y)}{\sinh(\pi)} \tag{4}$$

Figure 1 show figures obtained from the solution of the Eq. (3) analytical and numerically for 1024 mesh points. Numerical solution was obtained via WAMG_hD2(2) algorithm.

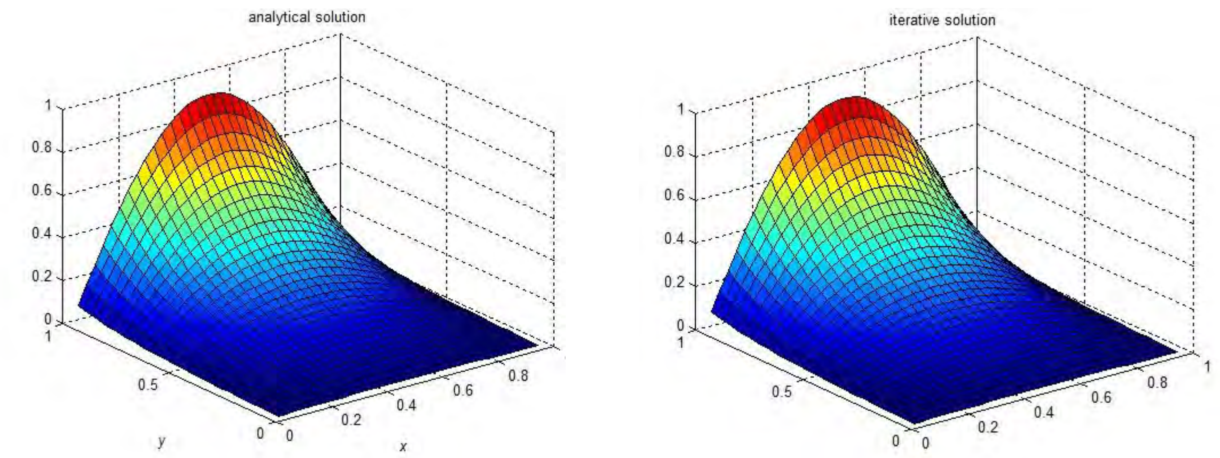
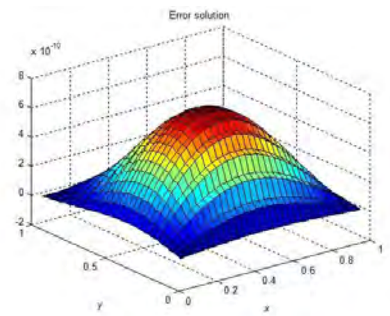
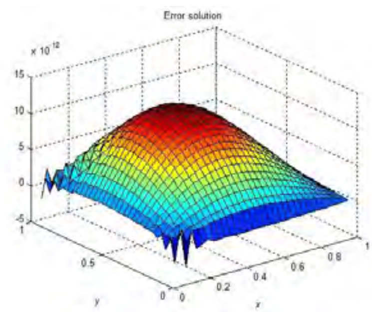


Figure 1. Numerical and analytical solution for two-dimensional Laplace Equation

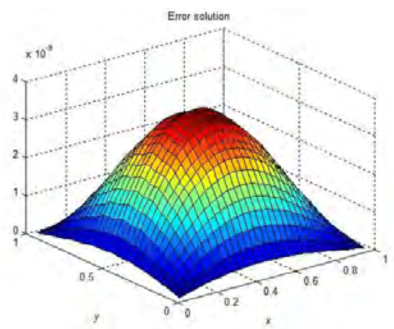
Figure 2 shows the difference between analytic and iterative solutions for different algorithms. It is observed, in the solution terms that the three algorithms are efficient.



(a) By algorithm WAMG_hD2(2)



(b) By algorithm WAMG_hD4(2)



(c) Algorithm AMG_hD4(4)

Figure 2. Error solutions for different algorithms

The eq. (3) is solved numerically with the specified boundary conditions. The results are presented in Tab. 1 for 4096 mesh points and three internal iterations for each algorithm above mentioned. Table 1 shows that the algorithm WAMG_hD4(2) obtained the best result for the investigated case.

Table 1. Computational effort for different algorithms

Algorithm	WAMG_hD2(2)	WAMG_hD4(2)	WAMG_hD4(4)
Time (*s*)	17,9	12,3	111,6
Cycle	58	39	717

Following the same case is investigated, only changing the smoothing for SOR with ω = 1.6. Table 2 shows results. Table shows the best results for all algorithms evaluated.

Table 2. Computational effort for different algorithms, GS with ω = 1.6

Algorithm	WAMG_hD2(2)	WAMG_hD4(2)	WAMG_hD4(4)
Time (*s*)	15,8	10,1	87,7
Cycle	51	32	556

4. CONCLUSIONS

Observing the numerical results obtained with this study note that the WAMG is a good alternative to numerical solution. Among the surveyed algorithms, the surprise is for the WAMG_hD4(2) algorithm which performed better compared with WAMG_hD2(2), contradicting the literature stating that shorter filters are more efficient because they avoid filling the matrices for implementation of the WAMG levels. On the other hand it enhances the performance of the new algorithms used in this work.

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