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SOME COMMENTS ON FRACTIONAL MODELING AND NUMERICAL SIMULATIONS OF OSCILLATORY SYSTEMS

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Abstract. *In this paper, we applied the Riemann-Liouville approach and the fractional Euler-Lagrange equations in order to obtain the fractional nonlinear dynamic equations involving two classical physical applications: "Simple Pendulum" and the "Spring-Mass-Dumper System" to both, integer order calculus (IOC) and fractional order calculus (FOC) approaches. The numerical simulations were realized and the time histories and pseudo phase portraits were investigated. In both systems, the one which already has a damping behavior (Spring-Mass-Dumper) and the system which doesn't present any sort of damping behavior (Simple Pendulum), showed signs indicating a possible better capacity of attenuation of their respective oscillation amplitudes. This implication could mean that if the selection of the order of the derivative is conveniently made, systems which need greater intensities of damping or vibrating absorbers may benefit from using fractional order in dynamics and possibly in control of the aforementioned systems. Thereafter, we believe that the results showed on this paper are pertinent enough to be thoroughly analyzed and maybe it can be the very beginning of a stride towards more realistic and more precise results about fractional-order models when compared to the integer order models in these applications.*

Keywords: *Fractional calculus, oscillatory systems, dynamic systems, modeling, simulation.*

1. INTRODUCTION

The fractional calculus is an extension of the traditional classical calculus. Although, it is a much lesser known part of mathematics. The deep understanding about the fractional order calculus (FOC) and its implications and applications in several areas seems to be tardy if compared to its older "brother". Perhaps, this backwardness is due to the FOC's inherent complexity and to the current lack of meaning regarding its physical and geometric interpretation. Oldham and Spanier (2006) outlined the basic aspects regarding the FOC core. It is passive of notice that FOC can count on an additional degree of freedom since the order of the derivatives can be arbitrary changed to match a specific behavior. This advantage may enable the FOC to represent systems with high order dynamics and complex nonlinear phenomena making use of only a few coefficients.

In fact, numerous mathematicians contributed to the history of fractional calculus (*Machado et. Al, 2010)* with every researcher using different approaches and solutions, which led the FOC to various definitions that are proven equivalent. Liouville, Laurent, Cauchy and Caputo are among those researchers.

The Laurent publication is now recognized as the definitive paper on the fundaments of fractional calculus (David *et al.,* 2011). In this definition, it is exposed that to differentiate in an arbitrary order is necessary to integrate it up to an order that is bigger than the desired one and then traditionally differentiate it to the aimed non-integer order. Gorenflo and Mainardi (2008) also compiled an impressive amount of content regarding integral and differential equations of fractional order in the framework of the Riemann-Liouville approach.

Nowadays, it is known that the fractional calculus may contribute in many research areas. One example of versatility and application of the FOC is on the viscoelastic damping. Using the fractional calculus (Craiem *et. al*, 2008) obtained a better model of a spring-dashpot system that could describe arterial viscoelasticity in-vitro under stress. Other applications can yet be related to the fractional order calculus, including control and robotics, as exposed by Machado (2002), and electric circuits (Oustaloup, 1981)

Thus, the objective of this paper is to analyze the results of numeric simulations of dynamic systems modelled by differential equations, both for the integer order calculus as for the fractional order calculus point of view. It is planned to compare the results between these simulations and investigate possible characteristics, utilities and eventual advantages and disadvantages of the use of FOC as a tool for mathematical modelling in some applications.

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2. METHODOLOGY

2.1 Applications

The simple pendulum and the spring-mass-dumper systems were investigated as applications involving modelling and simulation utilizing as a primary tool the fractional calculus. Furthermore, in this paper, we sought to establish comparisons between the results of the fractional order and the integer order systems.

2.1.1 Euler-Lagrange Fractional Equations

Basing on the Riemann-Liouville approach and seeing, now, an action function in the form (Agrawal, 2006):

$$S=\frac{1}{\Gamma(\alpha)}\int_a^b L(t,\ {}_aD_t^{\beta}\,q,\ {}_tD_b^{\gamma}\,q)\,(t-\tau)^{\alpha-1}\,d\tau \tag{1}$$

Where $0 \leq \beta \leq 1$, $0 < \gamma < 1$, $0 \leq \alpha \leq 1$. If ε indicates the variation of the function S, then

$$\Delta_{\varepsilon}S=\int_a^b L(q+\varepsilon\,\delta q,\ {}_aD_t^{\beta}\,q+\varepsilon\,{}_aD_t^{\beta}\,\delta q,\ {}_tD_b^{\gamma}\,q+\varepsilon\ {}_tD_b^{\gamma}\,\delta q)\,(t-\tau)^{\alpha-1}\,d\tau \tag{2}$$

The Equation (2) may be rewritten as

$$\Delta_{\varepsilon}S=\int_a^b\left(\frac{\partial L}{\partial q}(t-\tau)^{\alpha-1}+\frac{\partial L}{\partial({}_aD_t^{\beta}\,q)}(t-\tau)^{\alpha-1}\,{}_aD_t^{\beta}\,\delta q+\frac{\partial L}{\partial({}_tD_b^{\gamma}\,q)}(t-\tau)^{\alpha-1}\,{}_tD_b^{\gamma}\,\delta q\right)\times\ \varepsilon\,d\tau+0(\varepsilon^2) \tag{3}$$

Or also

$$\Delta_{\varepsilon}S=\int_a^b\left(\frac{\partial L}{\partial q}(t-\tau)^{\alpha-1}+{}_aD_t^{\beta}\,[\frac{\partial L}{\partial({}_aD_t^{\beta}\,q)}(t-\tau)^{\alpha-1}]+{}_tD_b^{\gamma}[\frac{\partial L}{\partial({}_tD_b^{\gamma}\,q)}(t-\tau)^{\alpha-1}]\right)\times\ \delta q\,\varepsilon\,d\tau+0(\varepsilon^2) \tag{4}$$

Thus, the Euler-Lagrange equations are written with fractional derivatives, as the following

$$\frac{\partial L}{\partial q}-\frac{1}{(t-\tau)^{\alpha-1}}[{}_tD_b^{\beta}\,(\frac{\partial L}{\partial({}_aD_t^{\beta}\,q)}(t-\tau)^{\alpha-1})+{}_aD_t^{\gamma}\,(\frac{\partial L}{\partial({}_tD_b^{\gamma}\,q)}(t-\tau)^{\alpha-1})]=0 \tag{5}$$

For $\beta = \gamma = 1$ and assuming that the Lagrangian depends only on ${}_aD_t^{\beta}\,q$ or on ${}_tD_b^{\gamma}\,q$, the following is obtained:

$$\frac{d}{dt}(\frac{\partial L}{\partial\dot{q}})\ -\frac{\partial L}{\partial q}+\frac{(\alpha-1)}{(t-\tau)}\frac{\partial L}{\partial\dot{q}}=0 \tag{6}$$

2.1.2 The Simple Pendulum – Modelling

Iinitially, we considered a simple pendulum whose "classical" and known Lagrangian may be written as:

$$L=\frac{1}{2}ml^2\dot{\theta}^2+\frac{1}{2}ml\,\dot{\theta}+mgl\cos\theta \tag{7}$$

In this system, "L" is the Lagrangian, "m" is the mass of the pendulum, "l" represents the length of the wire and, finally, θ is the angle. The application of the Euler-Lagrange equation to this Lagrangian also provides the equation also known as the equation of movement

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$$m l^2\ddot{\theta} + mglsen\,\theta = 0 \tag{8}$$

Utilizing the Lagrangian from Eq. (7) and the Eq. (6), knowing that q ↔ θ it is possible to notice that

$$\frac{\partial L}{\partial \theta} = -mglsen\theta \quad ; \quad \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} + \frac{1}{2}ml\,; \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta} \tag{9}$$

It can be observed as well that, for analogy and assuming that the Lagrangian depends only on ${}_aD_t^{\beta}\,q$, it is written

$$L = \frac{1}{2}ml^2\left({}_aD_t^{\beta}\,\theta\right)^2 + \frac{1}{2}ml\;{}_aD_t^{\beta}\,\theta + mgl\cos\theta \tag{10}$$

Thus, the Euler-Lagrange fractional equation, in this case, will be

$$\frac{\partial L}{\partial \theta} - \frac{1}{(t-\tau)^{\alpha-1}}\left[{}_tD_b^{\beta}\left(\frac{\partial L}{\partial({}_aD_t^{\beta}\,\theta)}(t-\tau)^{\alpha-1}\right)\right] = 0 \tag{11}$$

With

$$\frac{\partial L}{\partial \theta} = -mglsen\theta \tag{12}$$

$$\frac{\partial L}{\partial({}_aD_t^{\beta}\,\theta)} = \frac{\partial}{\partial({}_aD_t^{\beta}\,\theta)}\left\{\frac{1}{2}ml^2\left({}_aD_t^{\beta}\,\theta\right)^2 + \frac{1}{2}ml\;{}_aD_t^{\beta}\,\theta + mgl\cos\theta\right\} = ml^2\,{}_aD_t^{\beta}\,\theta + \frac{1}{2}ml \tag{13}$$

Therewith, the fractional equation becomes:

$$-mglsen\,\theta - \frac{1}{(t-\tau)^{\alpha-1}}\left[{}_tD_b^{\beta}\left(\left(ml^2\,{}_aD_t^{\beta}\,\theta + \frac{1}{2}ml\right)(t-\tau)^{\alpha-1}\right)\right] = Q1 \tag{14}$$

Or also

$$\frac{1}{(t-\tau)^{\alpha-1}}\left[{}_tD_b^{\beta}\left(\left(ml^2\,{}_aD_t^{\beta}\,\theta + \frac{1}{2}ml\right)(t-\tau)^{\alpha-1}\right)\right] + mglsen\,\theta = Q1 \tag{15}$$

It is worth highlighting that in the Eq. (15), if α = 1 (integer order) we obtain back, as expected, the movement equation, Eq. (8).

2.1.3 Spring-Mass-Dumper System – Modelling

Now, we considered a dissipative system, regarding mass (represented by “m”), spring (stiffness constant “k”) and dumper (dumping constant “c”). The “classical” and known is given by

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{16}$$

Applying the Euler-Lagrange equation for non-conservative systems

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \tag{17}$$

Where Q_i represents the dissipative(s) force(s), obtains:

$$\frac{\partial L}{\partial x} = -k\,x \quad ; \quad \frac{\partial L}{\partial \dot{x}} = m\,\dot{x} \;; \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\,\ddot{x} \tag{18}$$

So that,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = Q_i \tag{19}$$

Will provide the known movement equation

$$m\,\ddot{x} + c\dot{x} + kx = Q1 \tag{20}$$

Where Q1 is the dissipative force. Applying the Lagrangian that depends only on ${}_aD_t^{\beta}\,q$, written as

$$L = \frac{1}{2}m\left({}_aD_t^{\beta}\,x\right)^2 - \frac{1}{2}k\,x^2 \tag{21}$$

The fractional Euler-Lagrange equation will be:

$$\frac{\partial L}{\partial x} - \frac{1}{(t-\tau)^{\alpha-1}}\left[{}_tD_b^{\beta}\left(\frac{\partial L}{\partial({}_aD_t^{\beta}\,x)}(t-\tau)^{\alpha-1}\right)\right] = Q1 \tag{22}$$

And now

$$\frac{\partial L}{\partial x} = -k\,x \tag{23}$$

$$\frac{\partial L}{\partial({}_aD_t^{\beta}\,x)} = \frac{\partial}{\partial({}_aD_t^{\beta}\,x)}\left\{\frac{1}{2}m\left({}_aD_t^{\beta}\,x\right)^2 - \frac{1}{2}kx^2\right\} = m\;{}_aD_t^{\beta}\,x \tag{24}$$

Therewith, the fractional equation becomes:

$$-kx - \frac{1}{(t-\tau)^{\alpha-1}}\left[{}_tD_b^{\beta}\left((m\,{}_aD_t^{\beta}\,x)(t-\tau)^{\alpha-1}\right)\right] = c\;{}_aD_t^{\beta}\,x \tag{25}$$

Or also

$$\frac{1}{(t-\tau)^{\alpha-1}}\left[{}_tD_b^{\beta}\left(({}_aD_t^{\beta}\,x\,)(t-\tau)^{\alpha-1}\right)\right] + c\;{}_aD_t^{\beta}\,x + kx = Q1 \tag{26}$$

Once again, on Eq. (26), if α =1 (integer order) we obtain back, as expected, the known movement equation – Eq. (20).

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2.2 Conditions and parameters of the simulations

The previously mentioned equations were numerically simulated with the intention of studying its dynamic behavior and establishing comparisons between the results of the fractional order systems and the integer order systems. These equations were simulated through the Matlab Simulink®. The solver (method/algorithm of solution) utilized in all the cases was the "ode113 (Adams)". The Table 1 illustrates the three different cases simulated, which involve the absence and presence of external forces acting in the system.

Table 1. Cases simulated

	Simple Pendulum	Spring-Mass-Dumper System
Case	External force	External force
Case A	Q1 = 0	Q1 = 0
Case B	Q1 = A cos(wt)	Q1 = A cos(wt)
Case C	Q1 = A cos (wt) l_1 sin (θ)	Q1 = Impulsive function

On the other hand, the Table 2 describes the pre-established values of parameters and coefficients utilized in the simulations.

Table 2. Simulations parameters for all the three cases

Simple Pendulum		Spring-Mass-Dumper System	
Mass	m = 1kg	Mass	m = 1kg
Acceleration of gravity	g = 9.81m/s²	Acceleration of gravity	g = 9.81m/s²
Length of the string	l_1 = 1m	Stiffness and damping constants	k = 5; c = 0.1
Coefficient tau	τ=1	Coefficient tau	τ=1
Coefficient α (with β = α)	τ=1; α = 0.4; α = 0.6; α = 0.9; α = 1.0; α = 1.1; α = 1.2	Coefficient α (with β = α)	τ=1; α = 0.4; α = 0.6; α = 0.9; α = 1.0; α = 1.1; α = 1.2

3. SIMULATIONS RESULTS

3.1 Results regarding the simple pendulum

The results of the numerical simulations of the Eq. (15) regarding the simple pendulum are presented below, with different values of α and β (α = β), according to Table 2 and different external excitations crescent in intention of the case A to the case C (aforementioned in Table 1). The first graph of each case compares the curves obtained with α smaller or equal to 1.0 and the second one compares the curves obtained with α bigger or equal to 1.0.

Case A:

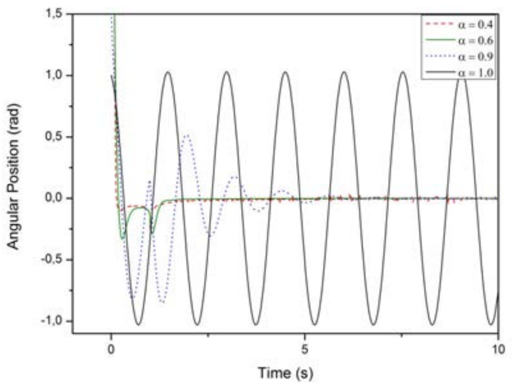

Figure 1. Time history (values of α ≤ 1)

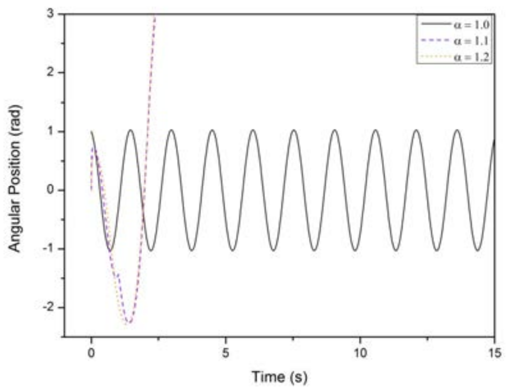

Figure 2. Time history (values of α ≥ 1)

Case B:

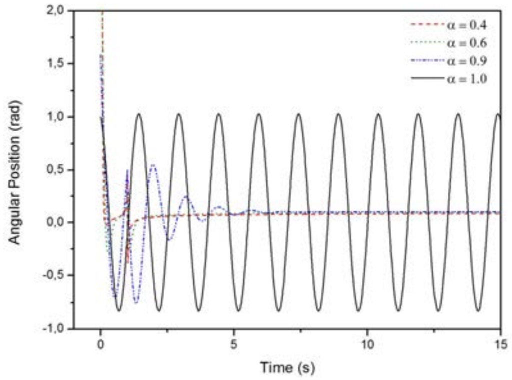

Figure 3. Time history (values of α ≤ 1)

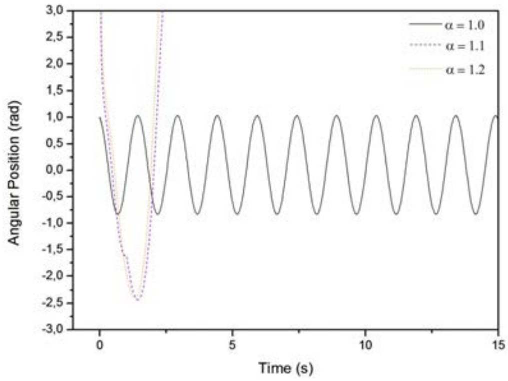

Figure 4. Time history (values of α ≥ 1)

Case C:

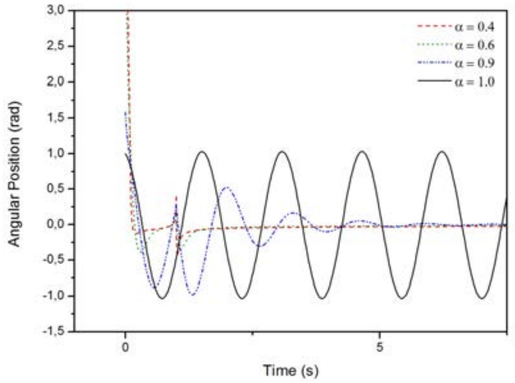

Figure 5. Time history (values of α ≤ 1)

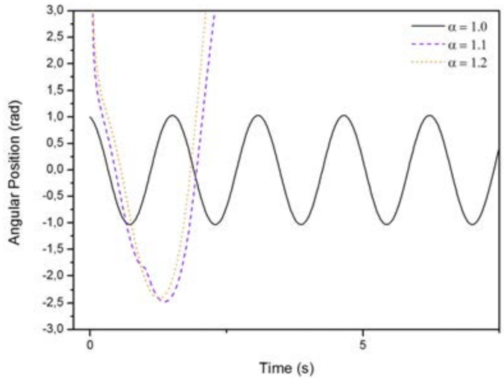

Figure 6. Time history (values of α ≥ 1)

Besides the graphs containing the simulations of the angular positions for all the three cases of the simple pendulum, it is presented in the Figures 7, 8 and 9 the pseudo-portraits-phase for the Case B, with differents values of α.

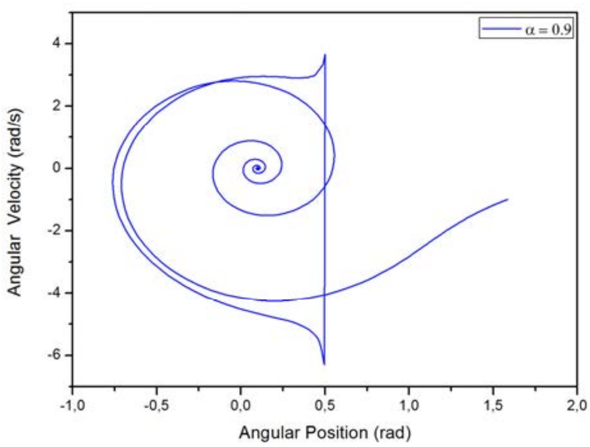

Figure 7. Pseudo phase-portrait (Case B for α = 0.9)

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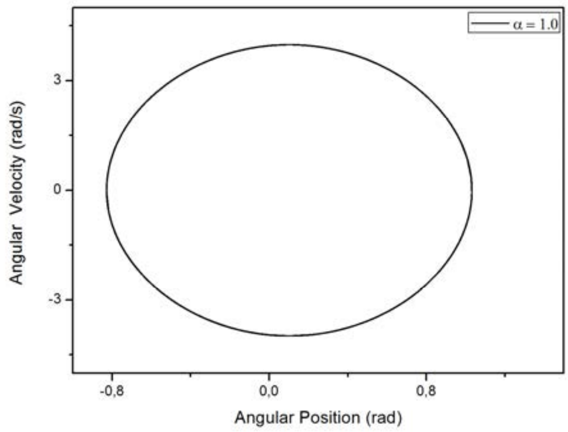

Figure 8. Pseudo phase-portrait (Case B for α = 1.0)

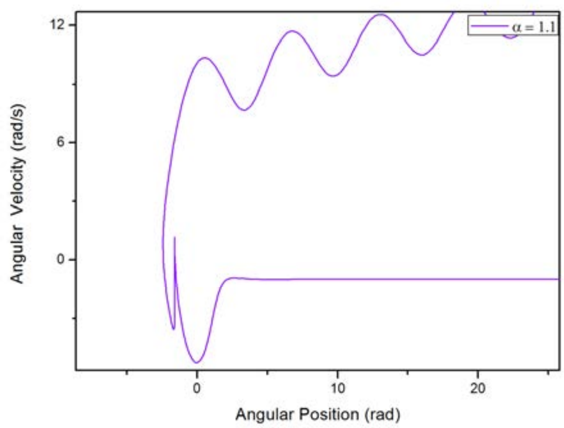

Figure 9. Pseudo phase-portrait (Case B for α = 1.1)

3.2 Results regarding the spring-mass-dumper system

The results of the numerical simulations of the Eq. (26) regarding the spring-mass-dumper system are presented below, with different values of α and β (α = β), according to Table 2 and different external excitations crescent in intention of the case A to the case C (aforementioned in Table 1). Once more, the first graph of each case compares the curves obtained with α smaller or equal to 1.0 and the second one compares the curves obtained with α bigger or equal to 1.0.

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Case A:

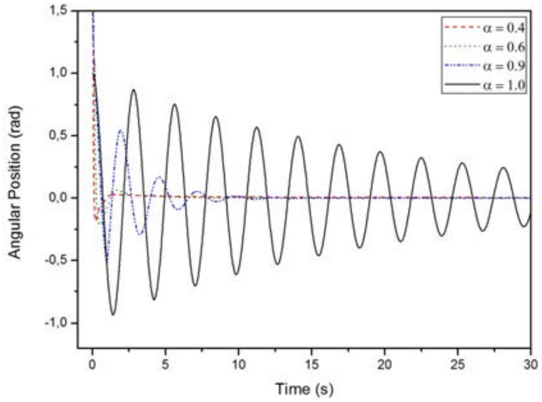

Figure 10. Time history (values of α ≤ 1)

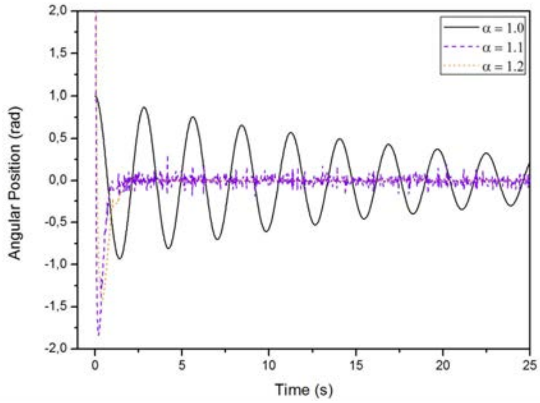

Figure 11. Time history (values of α ≥ 1)

Case B:

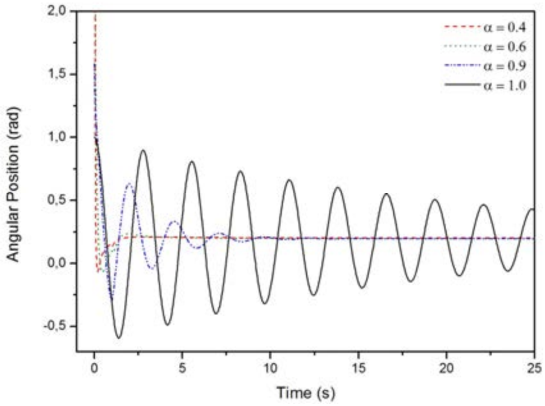

Figure 12. Time history (values of α ≤ 1)

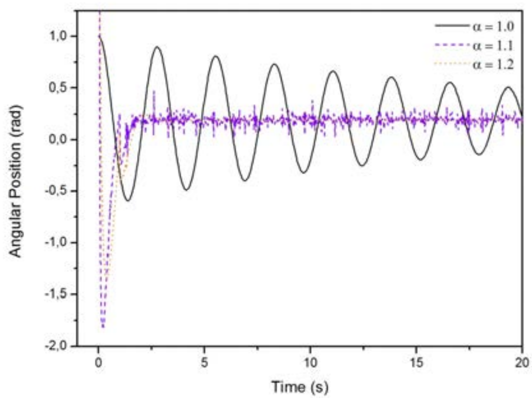

Figure 13. Time history (values of α ≥ 1)

Case C:

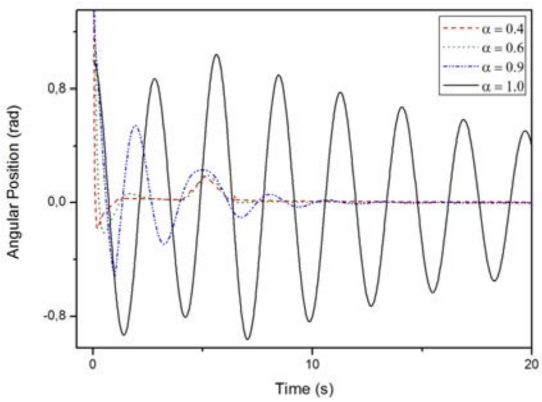

Figure 14. Time history (values of α ≤ 1)

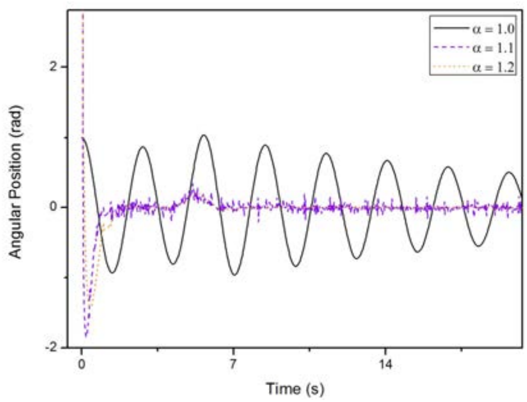

Figure 15. Time history (values of α ≥ 1)

Besides the graphs containing the simulations of the angular positions for all the three cases of the spring-mass-dumper system, it is presented in the Figures 16, 17 and 18 the pseudo-portraits-phase for the Case B, with differents values of α.

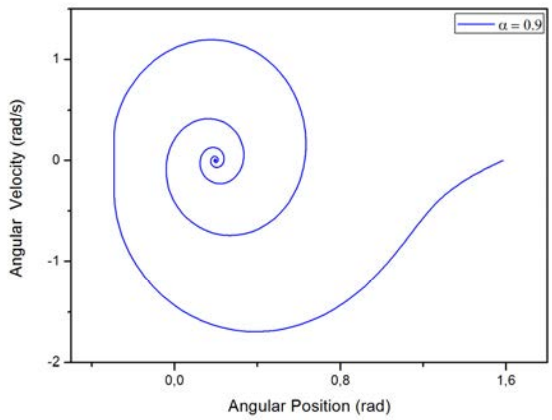

Figure 16. Pseudo phase-portrait (Case B for α = 0.9)

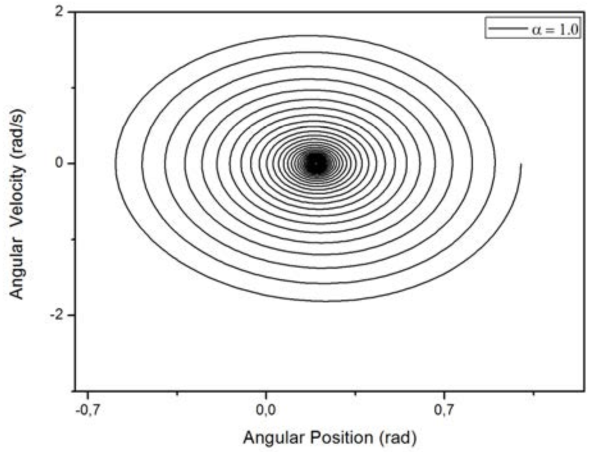

Figure 17. . Pseudo phase-portrait (Case B for α = 1.0)

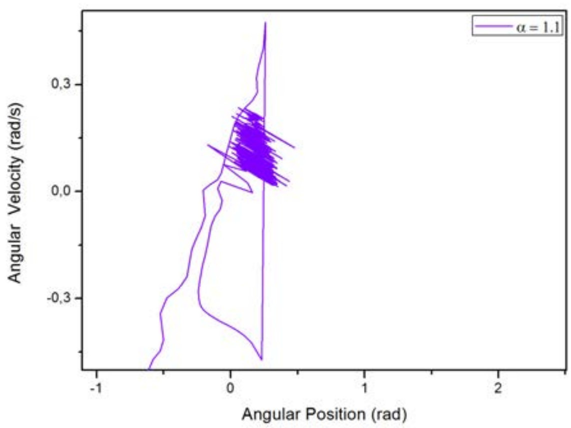

Figure 18. . Pseudo phase-portrait (Case B for α = 1.1)

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4. DISCUSSIONS AND COMMENTS

The results of several numerical simulations run during the development of this research project are, partly, synthetized in the previous section. The results point to a curious aspect and, at the same time, interesting and instigating of the effects arising from the using of fractional orders in the respective differential equations, which represent the dynamics of these systems.

It is passive of notice in the first case (simple pendulum) that, in first instance, the problem deals with a free oscillatory system (vibratory) and without the presence of damping. When the referent movement equation is simulated, taking into account the general form of the dynamic movement equation which involves derivatives of arbitrary order (integer or fractional), the results reunited in the Figures (1~ 9) briefly show that: a) For α=1 (integer order), the system displays the predicted behavior (expected), that is, an oscillation which amplitude is maintained due to the absence of damping. b) For values of $\alpha > 1$, the system loses the oscillatory characteristics and the amplitudes seem to grow very fast and indefinitely – as shown in the Figures 2, 4, and 6. c) However, for values of $\alpha < 1$, the Figures 1, 3 and 5, imply that the system acquires a "dumping capacity" which is indirectly proportional to the value of α. For example, the smaller the value of α, the bigger the "dumping capacity". Thus, the system that is not dumped will display characteristics that go from "under damping" (α=0.9) to "super damping" (α=0.4).

On the other hand, observing the second case (spring-mass-dumper system), it is possible to notice that: a) for α = 1, the dynamic movement equation of arbitrary order (integer or fractional), represented by the Figures 10 to 15, presents again an expected behavior for integer order behavior. The Figures 10, 12 and 14 also display a greater "damping capacity" in the system for values of $\alpha < 1$ – as already happened in the first case. It is noticed, in this case, that the Figures 11, 13 and 15, which involve values of $\alpha < 1$, not only seem to decrease even more drastically the amplitudes of the oscillations (vibrations), but it can present signs of nonlinear phenomena which can be object of future investigations. Another particularity gathered from the results of the simulations is the presence, in some cases, of certain instability in the behavior of the system (Figs. 7, 9 16 and 18) and even an indicative of possible occurrence of chaos under some conditions. In a future work, we would like investigate about the chaos in these fractional dynamical systems using the maximal Lyapunov exponent based on the Wolf's algorithm.

Thus, in this paper, the numeric simulations realized for the fractional order systems seems to show a possible better "capacity" of attenuation of the respective amplitudes of the oscillations. That means that if the choice of the order α of the derivative is conveniently made, systems that need bigger intensities of dumping or vibrating absorbers may benefit from the use of the fractional order in dynamics and possibly in control of the aforementioned systems.

5. ACKNOWLEDGEMENTS

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7. RESPONSIBILITY NOTICE

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