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SKIN BUCKLING OF FUSELAGES UNDER COMPRESSION

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Abstract. *A solution procedure for skin buckling of fuselages, based on Lévy method, is presented by employing the bvp4c routine of Matlab. The skin panel, considered as a portion between adjacent stringers and frames, is subjected to axial compressive load with boundary conditions chosen properly. The influence of both stringer torsional and warping rigidities is evaluated and the accuracy of the proposed procedure is verified by means of finite element results.*

Keywords: *buckling, fuselage, Lévy method*

1. INTRODUCTION

Stiffened shells are widely employed in aeronautical industry, when light weight aspects are essential, e.g., in an aircraft fuselage. Such structures are thin-walled and, therefore, susceptible to loss of equilibrium stability, i.e., the well-known buckling phenomenon. The fuselage load carry capacity depends on a large extent on the buckling strength of their individual panel components. The structural components of a stiffened shell are: circumferential stiffeners (or frames), longitudinal stiffeners (or stringers), and shell skin. Usually frames account for shear and pressure loading providing an effective end support to skin and stringers lying in between. There are various problems to be adressed in analyzing such a complex structure, where finding out the buckling loading have been far the most popular one.

There is one important point to be made at the outset, which is that there exist different distinct buckling modes for a stiffened panel. Depending on the spacing and the size of the frames and stringers relative to the shell skin, buckling deflections could be developed within each shell panel or encompass a number of stiffened panels. In the first case, commonly referred to as skin buckling, the line of connection between skin and frames remains virtually straight, with the stringers exhibiting only minor radial movement. The second case, known as overall buckling, involves a variable radial displacement of the stringers. These modes are neither mutually exclusive nor independent. For a stiffened panel subjected to compression, like the ones found on the lower fuselage belly of commercial transport aircraft, the overall (global) buckling and the skin (local) buckling modes are usually distinct from each other. As sudden global buckling is undesirable at design limit load levels, typical designs exhibit skin buckling first, followed by load re-distribution to the stiffeners. Therefore, an investigation of the local mode can be carried out by examining the behaviour of a curved panel between two successive frames and stringers. In such analysis the effect of stringers can only be implicitily accounted for in the boundary conditions along the straight edges of the panel. The standard buckling analysis methodologies use simplified assumptions and conservative boundary conditions, which can lead to over-designed structures. To remedy this, nowadays, the standard design proceedings focus on the finite element method that enables to access the buckling strenght of stiffened panels without having to place emphases on simplifying assumptions. However, the use of finite element method is not recommended for early design phases, where a large amount of preliminary optimization iterations are performed, due to computational and simulation time costs.

A comprehensive review on buckling of axially-compressed cylinders is that by Hoff (1966, 1967). The survey by Kollár and Dulácska (1984) may be recommended for further reading. The large number of publications, that have appeared over the last decades, dealing with the buckling behavior of axially-compressed stiffened cylindrical shells clearly demonstrates the importance to this particular problem. In general, the buckling problem is settled on Donnell set of equations for "shalow" shells due to the simplicity of the assumed kinematic relations, and also for the fact that an equivalent partially uncoupled set of equations can be derived. Engineering approach for computation of the local buckling strength is realized mainly assuming that the skin panel is simply supported or clamped along the panel edges. Nevertheless, the local buckling strength should increase acting upon the torsional rigidity of frames and stringers. Sthephens (1971) studied the effects of longitudinal edge stiffeners on the buckling and initial postbuckling behavior of axially compressed long cylindrical panels. The influence of edge restraint on the critical load of curved panels has also been examined by Rehfield and Hallauer Jr. (1968) assuming different sets of boundary conditions. Concerned only with critical load estimates, Wang and Lin (1973) present a theoretical investigation of stability of axially compressed simply supported thin cylindrical shells stiffened by longitudinal stringers, where the shell and stringers are treated as discrete components.

This earliest studies were carried out in the context of the Donnell stability equations in uncoupled form, and had resulted in cumbersome numerical solutions. More recently, Buermann *et al.* (2006) have developed a fast semi-analytical

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model for the post-buckling analysis of stiffened cylindrical panels which includes both stringer and frames as structural elements, and captures their instabilities. It is an excelent study that lead to a consistent post-buckling behavior within an inherently complicated formulation. In a different manner, following a simpler engineering approach, Pevzner *et al.* (2008) present some analytical formulae for calculating the collapse load of an axially compressed laminated curved panel where torsional buckling and combined bending and torsion buckling of the stringers are included, but the panel edges is assumed to be simply supported. Thus, their prediction of the buckling load could be somewhat mistaken. A simple and accurate formulation for the buckling analysis of stiffened cylindrical panels have recently been proposed by Monteiro *et al.* (2011), that extent the work of Bisagni and Vescovini (2009) to shell panels, where the stringer torsional rigidity is properly taken into account with unrestrained warping.

An extension of our previous formulation is herein presented, where now the stringer torsional rigidity is fully exploited by inclusion of the warping contribution. The structure is analyzed considering the portion of a panel between two adjacent frames and stringers, such that along the unloaded edges the panel bending moment is set equal to the stiffener torsional moment, and along the loaded edges the frames are sought to give only simply support conditions to the shell skin. A boundary value problem for local skin buckling of stiffened circular cylindrical shells is derived by means of a simplified linearization of von Kármán-Donnell theory. Under these hypothesis the stringer Saint-Venant torsional rigidity and the stringer warping rigidity are easily introduced into the model. The calculation of the bifucation-point load are settled on the coupled form of Donnell stability equations. A numerical solution procedure for the skin buckling of fuselages under compression, based on Lévy method, is presented by employing the *bvp4c* routine of Matlab®. The influence of both stringer torsional and warping rigidities are evaluated and the accuracy of the proposed procedure is verified by means of finite element results. Towards this aims, the following section reviews the classical formulation setting it into perspective of the present study. This is followed by development of approximate solutions for determination of local skin buckling of circular cylindrical stiffened shells. Simulation methodology are presented before the validation results. Some conclusions about computational findings and recommendations for future works are presented at the end.

2. PROBLEM FORMULATION

Consider the circular cylindrical stiffened shell shown in Fig. 1, that represents a section of a traditional fuselage of an aircraft. It is assumed that: (i) stringers are equally spaced; (ii) axial compressive loading $\bar{N}$ is applied uniformly over the skin; (iii) skin buckling occurs first. The fuselage load carry capacity depends on the buckling strength of their stiffened panels, which by the way depend on the skin critical buckling load. Hence, to access the fuselage buckling strenght it suffices to determine the critical buckling load of an individual panel, like the one highlighted in the figure.

The analyzed panel has length a, width b, thickness h, radius of curvature R, and its middle surface is referred to a set of orthogonal curvilinear coordinates xyz. It is considered that frames give simply support condition to the panel, while stringers provide some rotational restraint along edges, offering a torsional moment along the panel edges immediately after the skin buckling takes place.

2.1 Kinematic relations

Consistent with the Kirchhoff-Love assumptions of a thin-shell theory, the following displacement field is assumed

$$u_x(x,y,z) = u(x,y) - zw_{,x} \qquad u_y(x,y,z) = v(x,y) - zw_{,y} \qquad u_z(x,y,z) = w(x,y) \tag{1}$$

in which u, v, w are the displacements of a point on the shell midsurface. The derivatives with respect to x and y coordinates are, respectively, denoted by $(\ )_{,x}$ and $(\ )_{,y}$.

Substituting the displacement field into the von Kármán-Donnell strain-displacement relations, we obtain the associated nonzero strains

$$\epsilon_x = \epsilon_x^m + z\kappa_x \qquad \epsilon_y = \epsilon_y^m + z\kappa_y \qquad \gamma_{xy} = \gamma_{xy}^m + z\kappa_{xy} \tag{2}$$

where

$$\begin{aligned} \epsilon_x^m &= u_{,x} + \frac{1}{2}w_{,x}^2 \qquad & \epsilon_y^m &= v_{,y} + \frac{w}{R} + \frac{1}{2}w_{,y}^2 \qquad & \gamma_{xy}^m &= v_{,x} + u_{,y} + w_{,x}w_{,y} \\ \kappa_x &= -w_{,xx} & \kappa_y &= -w_{,yy} & \kappa_{xy} &= -2w_{,xy}. \end{aligned} \tag{3}$$

2.2 Equilibrium equations

Use of the kinematic relations along with the principle of virtual work yields the following equilibrium equations

$$\begin{aligned} N_{x,x} + N_{xy,y} &= 0 \\ N_{xy,x} + N_{y,y} &= 0 \\ M_{x,xx} + 2M_{xy,xy} + M_{y,yy} + N_x w_{,xx} + 2N_{xy}w_{,xy} + N_y\left(w_{,yy} - \frac{1}{R}\right) &= 0, \end{aligned} \tag{4}$$

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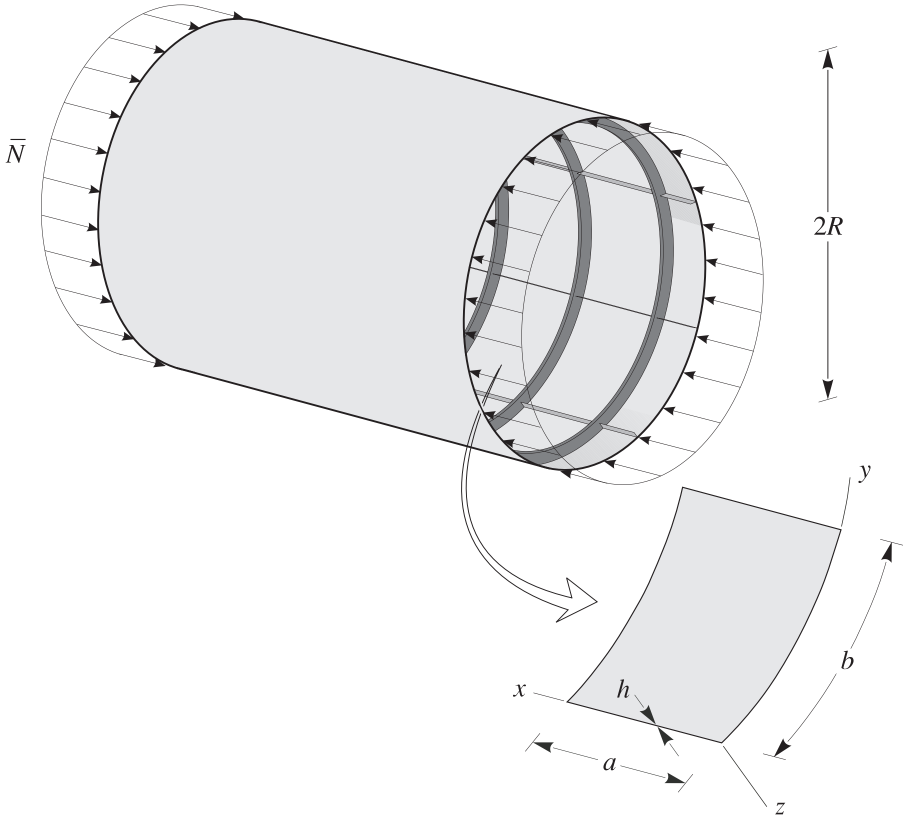

Figure 1. Circular cylindrical fuselage

and the panel boundary conditions

$$\begin{array}{rclllll} N_x & = & \bar{N} & v = 0 & w = 0 & M_x = 0 & : x = 0, a \\ u & = & 0 & N_y = 0 & w = 0 & M_y = \bar{M} & : y = 0, b. \end{array} \tag{5}$$

Since the stringers are equally spaced and have all the same geometry,

$$\bar{M}\left(x,0\right) = \frac{G_r J_r}{K} w_{,xxy}\left(x,0\right) - \frac{E_r \Gamma_r}{K} w_{,xxxxy}\left(x,0\right) \qquad \bar{M}\left(x,b\right) = -\frac{G_r J_r}{K} w_{,xxy}\left(x,b\right) + \frac{E_r \Gamma_r}{K} w_{,xxxxy}\left(x,b\right) \tag{6}$$

where $G_r J_r$ represents the stringer torsional rigidity, $E_r \Gamma_r$ represents the stringer warping rigidity and the distribution factor K is set to equal to two. A schematic view of the assumed boundary conditions are depicted in Fig. 2.

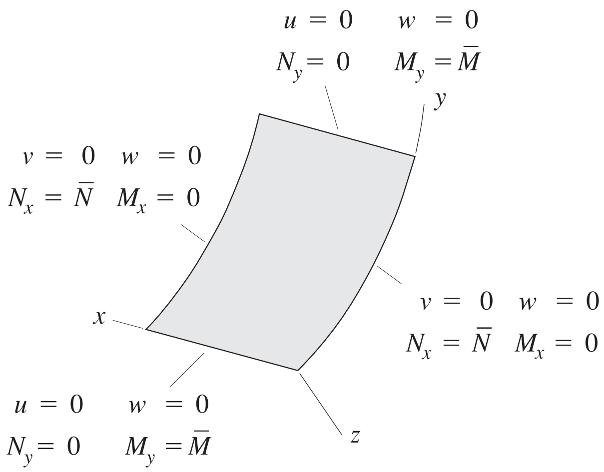

Figure 2. Panel boundary conditions

2.3 Linearized buckling equations

In the solution of equilibrium equations of a perfect shell panel subject to a compressive stress state, it is observed the existence of an equilibrium path bifurcation in the vicinity of the undeformed configuration. To investigate the possible existence of adjacent-equilibrium configurations, it is common to give an arbitrarily small increments u_1, v_1, w_1 to the displacement variables and, then, examine equilibrium configuration represented by the displacements after the increment (Brush and Almorth, 1975):

$$u \to u_0 + u_1 \qquad v \to v_0 + v_1 \qquad w \to w_0 + w_1. \tag{7}$$

Fields (u_0, v_0, w_0) and (u, v, w) describe adjacent-equilibrium configurations associated with the primary (before increment) and secondary (after increment) equilibrium paths, respectively. In stability applications the displacement (u_0, v_0, w_0) is called the prebuckling deformation and (u_1, v_1, w_1) is called the buckling mode.

Accordingly, let

$$\begin{aligned} N_x &\to N_x^0 + N_x^1 \qquad N_y \to N_y^0 + N_y^1 \qquad N_{xy} \to N_{xy}^0 + N_{xy}^1 \\ M_x &\to M_x^0 + M_x^1 \qquad M_y \to M_y^0 + M_y^1 \qquad M_{xy} \to M_{xy}^0 + M_{xy}^1 \end{aligned} \tag{8}$$

where the terms with 0 superscripts are associated to u_0, v_0, w_0 displacements, and the terms with 1 superscripts represent a portion of the incremental generalized stress that are linear in u_1, v_1, w_1.

Substitution of Eq. (8) into Eq. (4) yields two sets of equilibrium expressions. One of them is the nonlinear prebuckling problem:

$$\begin{aligned} N_{x,x}^0 + N_{xy,y}^0 &= 0 \\ N_{xy,x}^0 + N_{y,y}^0 &= 0 \\ M_{x,xx}^0 + 2M_{xy,xy}^0 + M_{y,yy}^0 + N_x^0 w_{0,xx} + 2N_{xy}^0 w_{0,xy} + N_y^0 \left(w_{0,yy} - \frac{1}{R} \right) &= 0 \end{aligned} \tag{9}$$

and

$$\begin{aligned} N_x^0 &= \bar{N} \qquad v_0 = 0 \qquad w_0 = 0 \qquad M_x^0 = 0 \qquad : x = 0, a \\ u_0 &= 0 \qquad N_y^0 = 0 \qquad w_0 = 0 \qquad M_y^0 = 0 \qquad : y = 0, b. \end{aligned} \tag{10}$$

The other defines the so-called linearized buckling problem:

$$\begin{aligned} N_{x,x}^1 + N_{xy,y}^1 &= 0 \\ N_{xy,x}^1 + N_{y,y}^1 &= 0 \\ M_{x,xx}^1 + 2M_{xy,xy}^1 + M_{y,yy}^1 - \frac{N_y^1}{R} + F_0(w_1) + F_1(w_0) &= 0 \end{aligned} \tag{11}$$

and

$$\begin{aligned} N_x^1 &= 0 \qquad v_1 = 0 \qquad w_1 = 0 \qquad M_x^1 = 0 \qquad : x = 0, a \\ u_1 &= 0 \qquad N_y^1 = 0 \qquad w_1 = 0 \qquad M_y^1 = \bar{M} \qquad : y = 0, b \end{aligned} \tag{12}$$

where F_0 and F_1 are given by

$$\begin{aligned} F_0(w_1) &= N_x^0 w_{1,xx} + 2N_{xy}^0 w_{1,xy} + N_y^0 w_{1,yy} \\ F_1(w_0) &= N_x^1 w_{0,xx} + 2N_{xy}^1 w_{0,xy} + N_y^1 w_{0,yy}. \end{aligned} \tag{13}$$

According to the boundary conditions of the linearized problem, Eq. (12), the stringers apply a torsional moment $\bar{M}$ to panel edges immediately after the onset of skin buckling.

2.4 Constitutive relations

Let the skin be defined by an isotropic material with Young's modulus E and Poisson's ratio ν. The generalized stress-strain equations of the skin panel is defined by

$$\begin{aligned} N_x &= A_{11}\epsilon_x^m + A_{12}\epsilon_y^m \qquad N_y = A_{12}\epsilon_x^m + A_{22}\epsilon_y^m \qquad N_{xy} = A_{66}\gamma_{xy}^m \\ M_x &= D_{11}\kappa_x + D_{12}\kappa_y \qquad M_y = D_{12}\kappa_x + D_{22}\kappa_y \qquad M_{xy} = D_{66}\kappa_{xy} \end{aligned} \tag{14}$$

where

$$\begin{aligned} A_{11} &= \frac{Eh}{1-\nu^2} \qquad A_{22} = A_{11} \qquad A_{12} = \nu A_{11} \qquad A_{66} = \frac{1-\nu}{2} A_{11} \\ D_{11} &= \frac{Eh^3}{12\left(1-\nu^2\right)} \qquad D_{22} = D_{11} \qquad D_{12} = \nu D_{11} \qquad D_{66} = \frac{1-\nu}{2} D_{11}. \end{aligned}$$

2.5 Simplified linearized buckling equations

The extraction of the portion of the incremental generalized stress that are linear in u_1, v_1, w_1 could be obtained from the associated generalized stress increment as follows. Substituting Eq. (7) into Eq. (3) and considering the constitutive relations (14) results in

$$\begin{aligned} N_x &= A_{11}\left(u_{0,x}+\frac{1}{2}w_{0,x}^2\right)+A_{12}\left(v_{0,y}+\frac{w_0}{R}+\frac{1}{2}w_{0,y}^2\right) \\ &\quad +A_{11}\left(u_{1,x}+w_{0,x}w_{1,x}+\frac{1}{2}w_{1,x}^2\right)+A_{12}\left(v_{1,y}+\frac{w_1}{R}+w_{0,y}w_{1,y}+\frac{1}{2}w_{1,y}^2\right) \end{aligned} \tag{15}$$

from which

$$N_x^1 = A_{11}\left(u_{1,x}+w_{0,x}w_{1,x}\right)+A_{12}\left(v_{1,y}+\frac{w_1}{R}+w_{0,y}w_{1,y}\right). \tag{16}$$

In a similar manner, it could be derived

$$\begin{aligned} N_y^1 &= A_{12}\left(u_{1,x}+w_{0,x}w_{1,x}\right)+A_{22}\left(v_{1,y}+\frac{w_1}{R}+w_{0,y}w_{1,y}\right) \\ N_{xy}^1 &= A_{66}\left(v_{1,x}+u_{1,y}+w_{0,y}w_{1,x}+w_{0,x}w_{1,y}\right) \\ M_x^1 &= -D_{11}w_{1,xx}-D_{12}w_{1,yy} \\ M_y^1 &= -D_{12}w_{1,xx}-D_{22}w_{1,yy} \\ M_{xy}^1 &= -2D_{66}w_{1,xy}. \end{aligned} \tag{17}$$

Substituting Eqs. (16) and (17) into Eq. (11) and considering small increment displacements,

$$\begin{aligned} A_{11}u_{1,xx}+A_{12}\left(v_{1,xy}+\frac{w_{1,x}}{R}\right)+A_{66}\left(v_{1,xy}+u_{1,yy}\right)+\theta_u\left(w_0,w_1\right) &= 0 \\ A_{66}\left(v_{1,xx}+u_{1,xy}\right)+A_{12}u_{1,xy}+A_{22}\left(v_{1,yy}+\frac{w_{1,y}}{R}\right)+\theta_v\left(w_0,w_1\right) &= 0 \\ D_0\left(w_1\right)+\frac{A_{12}}{R}u_{1,x}+\frac{A_{22}}{R}\left(v_{1,y}+\frac{w_1}{R}\right)-F_0\left(w_1\right)+\theta_w\left(w_0,w_1\right) &= 0 \end{aligned} \tag{18}$$

where D_0, θ_u, θ_v and θ_w are given by

$$\begin{aligned} D_0\left(w_1\right) &= D_{11}w_{1,xxxx}+2\left(D_{12}+2D_{66}\right)w_{1,xxyy}+D_{22}w_{1,yyyy} \\ \theta_u\left(w_0,w_1\right) &= A_{11}\left(w_{0,x}w_{1,x}\right)_{,x}+A_{12}\left(w_{0,y}w_{1,y}\right)_{,x}+A_{66}\left(w_{0,y}w_{1,x}+w_{0,x}w_{1,y}\right)_{,y} \\ \theta_v\left(w_0,w_1\right) &= A_{66}\left(w_{0,y}w_{1,x}+w_{0,x}w_{1,y}\right)_{,x}+A_{12}\left(w_{0,x}w_{1,x}\right)_{,y}+A_{22}\left(w_{0,y}w_{1,y}\right)_{,y} \\ \theta_w\left(w_0,w_1\right) &= \frac{1}{R}\left(A_{12}w_{0,x}w_{1,x}+A_{22}w_{0,y}w_{1,y}\right)-F_1\left(w_0\right). \end{aligned} \tag{19}$$

The quantities θ_u, θ_v, θ_w are prebuckling rotation terms.

In order to solve the linearized buckling problem Eq. (18), we make the following simplifications. Firstly, the prebuckling problem defined by Eqs. (9) and (10) is solved disregarding all flexural effects (linear membrane solution):

$$N_x^0 = \bar{N} \qquad N_{xy}^0 = 0 \qquad N_y^0 = 0. \tag{20}$$

Thus,

$$F_0\left(w_1\right) = \bar{N}w_{1,xx}. \tag{21}$$

Secondly, the influence of rotation terms are thought negligible:

$$\theta_u \to 0 \qquad \theta_v \to 0 \qquad \theta_w \to 0. \tag{22}$$

After having applied these assumptions the simplified linearized buckling equations, also known as Donnell stability equations in coupled form, is finally obtained

$$\begin{aligned} A_{11}u_{1,xx}+A_{12}\left(v_{1,xy}+\frac{w_{1,x}}{R}\right)+A_{66}\left(v_{1,xy}+u_{1,yy}\right) &= 0 \\ A_{66}\left(v_{1,xx}+u_{1,xy}\right)+A_{12}u_{1,xy}+A_{22}\left(v_{1,yy}+\frac{w_{1,y}}{R}\right) &= 0 \\ D_0\left(w_1\right)+\frac{A_{12}}{R}u_{1,x}+\frac{A_{22}}{R}\left(v_{1,y}+\frac{w_1}{R}\right)-\bar{N}w_{1,xx} &= 0 \end{aligned} \tag{23}$$

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or

$$\left([E]-\bar{N}\left[E_0\right]\right)\{u\}=\{0\} \tag{24}$$

where

$$[E]=\left[\begin{array}{ccc} E_{11}(\cdot) & E_{12}(\cdot) & E_{13}(\cdot) \\ E_{12}(\cdot) & E_{22}(\cdot) & E_{23}(\cdot) \\ E_{13}(\cdot) & E_{23}(\cdot) & E_{33}(\cdot) \end{array}\right] \qquad \left[E_0\right]=\left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & E_{00}(\cdot) \end{array}\right] \qquad \{u\}=\left\{\begin{array}{c} u_1 \\ v_1 \\ w_1 \end{array}\right\} \tag{25}$$

with

$$\begin{aligned} E_{11}(\cdot) &= A_{11}(\cdot)_{,xx}+A_{66}(\cdot)_{,yy} \qquad E_{12}(\cdot)=\left(A_{12}+A_{66}\right)(\cdot)_{,xy} \qquad E_{13}(\cdot)=\frac{A_{12}}{R}(\cdot)_{,x} \\ E_{22}(\cdot) &= A_{22}(\cdot)_{,yy}+A_{66}(\cdot)_{,xx} \qquad E_{23}(\cdot)=\frac{A_{22}}{R}(\cdot)_{,y} \qquad E_{33}(\cdot)=D_0(\cdot)+\frac{A_{22}}{R^2} \\ E_{00}(\cdot) &= (\cdot)_{,xx}. \end{aligned} \tag{26}$$

3. BUCKLING SOLUTION

In view of the boundary conditions Eq. (12) and Donnell stability equations Eq. (23), we apply the Lévy method by substituting

$$\{u\}=\lfloor\, u_1 \quad v_1 \quad w_1 \,\rfloor^T=\lfloor\, U(y)\cos\lambda_m x \quad V(y)\sin\lambda_m x \quad W(y)\sin\lambda_m x \,\rfloor^T \qquad \lambda_m=\frac{m\pi}{a} \tag{27}$$

into Eq. (23) to obtain

$$\left\{\begin{array}{c} E_{11}\left(X_u U\right)+E_{12}\left(X_v V\right)+E_{13}\left(X_w W\right) \\ E_{12}\left(X_u U\right)+E_{22}\left(X_v V\right)+E_{23}\left(X_w W\right) \\ E_{13}\left(X_u U\right)+E_{23}\left(X_v V\right)+E_{33}\left(X_w W\right)-\bar{N} E_{00}\left(X_w W\right) \end{array}\right\}=\{0\}. \tag{28}$$

Introducing the change of variables

$$\{Z\}=\lfloor\, Z_1 \quad Z_2 \quad Z_3 \quad Z_4 \quad Z_5 \quad Z_6 \quad Z_7 \quad Z_8 \,\rfloor^T=\lfloor\, U \quad U' \quad V \quad V' \quad W \quad W' \quad W'' \quad W''' \,\rfloor^T, \tag{29}$$

where a prime $(\ )'$ indicates the derivative with respect to y, the buckling equations can be reduced to the following differential first order system

$$\{Z\}'=[T]\{Z\} \tag{30}$$

with

$$[T]=\left[\begin{array}{cccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_1 & 0 & 0 & C_2 & C_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & C_4 & C_5 & 0 & 0 & C_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ C_7 & 0 & 0 & C_8 & C_9 & 0 & C_{10} & 0 \end{array}\right] \tag{31}$$

and

$$\begin{aligned} C_1 &= \frac{A_{11}}{A_{66}}\lambda_m^2 \qquad C_2=-\frac{A_{12}+A_{66}}{A_{66}}\lambda_m \qquad C_3=-\frac{A_{12}}{A_{66}R}\lambda_m \\ C_4 &= \frac{A_{12}+A_{66}}{A_{22}}\lambda_m \qquad C_5=\frac{A_{66}}{A_{22}}\lambda_m^2 \qquad C_6=-\frac{1}{R} \\ C_7 &= \frac{A_{12}}{D_{22}R}\lambda_m \qquad C_8=-\frac{A_{22}}{D_{22}R} \qquad C_9=-\frac{1}{D_{22}}\left(D_{11}\lambda_m^4+\frac{A_{22}}{R^2}+\bar{N}\lambda_m^2\right) \\ C_{10} &= \frac{2\left(D_{12}+2D_{66}\right)}{D_{22}}\lambda_m^2. \end{aligned} \tag{32}$$

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After the change of variables defined by Eq. (29), the boundary conditions along the edges $y = 0, b$ are

$$\begin{aligned} Z_1(0) &= Z_5(0) = Z_4(0) = -D_{22}Z_7(0) + \left(\frac{G_rJ_r}{K}\lambda_m^2 + \frac{E_r\Gamma_r}{K}\lambda_m^4\right)Z_6(0) = 0 \\ Z_1(b) &= Z_5(b) = Z_4(b) = -D_{22}Z_7(b) - \left(\frac{G_rJ_r}{K}\lambda_m^2 + \frac{E_r\Gamma_r}{K}\lambda_m^4\right)Z_6(b) = 0. \end{aligned} \quad (33)$$

Since $[T]$ is not a function of y, the formal solution of Eq. (30) is

$$\{Z\} = [e^{[T]y}]\{C\} \quad (34)$$

where $\{C\}$ is a constant vector to be determined from boundary conditions, Eq. (33), for details see (Franklin, 2000; Ogata, 2009). Now, we have to evaluate the matrix exponential $\left[e^{[T]y}\right]$ to assess the problem solution. Numerous methods are available for determining this matrix exponential. However regardless of any method used:

- consideration of computational stability and efficiency does not indicate a method which is completely satisfactory (Moler and Loan, 2003);
- it is found that incorrect solutions could be generated for very thin shells or long shells, where computer overflow and underflow may occur when the elements of the coefficient matrix $[T]$ are evaluated (Reddy, 2004).

Due to the aforementioned numerical troublesomeness, we have chosen an alternative solution way: the use of the *bvp4c* routine of Matlab®, which solution framework is a residual control based, adaptative mesh solver. The routine procedure is a finite difference code that employs an appropriated collocation formula to provide a C^1 continuous solution that is fourth order accurate uniformly in the solution interval, and seems to be an efficient tool in solving boundary value problems for ordinary differential equations (Shampine *et al.*, 2003). In what concern the solution of eigenvalue problems using the *bvp4c* routine, the presence of the unknown buckling load parameter $\bar{N}$ compels the specification of an additional boundary condition, and the following troublesome question arises: how to identify this extra boundary condition conveniently? It is a question that doesn't seem to have an answer or at least it doesn't have a single answer. For instance, we suppose that N^1_{xy} is practically constant along the x axis, i.e.,

$$N^1_{xy,x} \approx 0 \Rightarrow N^1_{xy,x} = \epsilon \sin\lambda_m x \qquad \epsilon < 1 \quad (35)$$

where ϵ is an adjustable parameter (the magnitude of ϵ is not critical, but it should be "small"). The specification of Eq. (35) at $y = 0$ results

$$A_{66}\lambda_m\left(\lambda_m V(0) + U'(0)\right) + \epsilon = A_{66}\lambda_m\left(\lambda_m Z_3(0) + Z_2(0)\right) + \epsilon = 0 \quad (36)$$

that gives us an assumed-extra boundary condition to be used within *bvp4c* routine. It is worth to note that stability and convergence of the numerical solution are highly dependent from the choice of the additional equation for the boundary conditions. The routine also requires a guess for the unknown parameter $\bar{N}$; however, the solution is usually sensitive to this value. If it is too far from the desired region, one may get an answer not viable. In addition, it is necessary to provide a guess that reasonably approximates the expected spatial shapes of the solutions.

4. RESULTS AND DISCUSSIONS

The proposed numerical procedure for the skin buckling of fuselages under compression was implemented in Matlab® computing program, and the obtained results are in this section compared with those provided by a more rigorous finite element model (FEM). The finite element analysis is based on modelling:

- an entire fuselage barrel section between two adjacent frames, where it is supposed that boundary frames provide a simply support condition to skin panel;
- stringers as beam elements, and skin barrel as an assemblage of shell elements.

For practical purpose it is assumed that buckling occurs at a bifurcation point in the vicinity of the undeformed configuration, that enable evaluating the panel critical load by a linear buckling analysis of the perfect structure.

A series of one-bay stiffened cylindrical shells with different number of stringers are analyzed using the solution SOL 105 (eigenvalue buckling analysis) of the finite element commercial code NASTRAN (MSC, 2008). The computations are carried out for 12 stiffened cylindrical shells reported in Tab. 1, where it was considered different geometries in order to evaluate the effect of different number of stringers n_r and different aspect ratios a/b, $b = 2\pi R/n_r$.

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Table 1. Panel geometry.

Panel ID	a (mm)	R (mm)	h (mm)	n_r
1	400	2000	1	24
2	400	2000	1	12
3	400	1000	1	24
4	400	1000	1	12
5	600	2000	1	24
6	600	2000	1	12
7	600	1000	1	24
8	600	1000	1	12
9	800	2000	1	24
10	800	2000	1	12
11	800	1000	1	24
12	800	1000	1	12

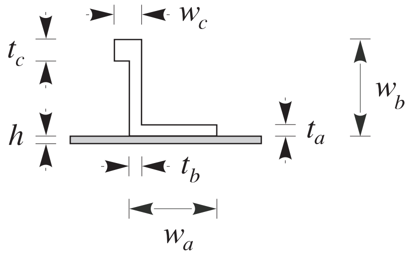

Figure 3. Stiffener section

It was considered stiffeners with Z cross-section type as depicted in Fig. 3, with the cross-section dimensions defined in Tab. 2. The stringer torsional constant is $J_r = 57.412$ mm^4 and its warping constant is $\Gamma_r = 16.857 \times 10^3$ mm^6. Aluminum alloys commonly used in aeronautical applications are employed, whose mechanical properties are described in Tab. 3 according to DOD U.S. (1998). Finally, the stiffened panels are discretized into sufficient number of elements to catch buckling modes correctly.

Table 4 compares the results obtained from Lévy procedure with those obtained from FEM. The initial guess chosen for the parameter $\bar{N}$ is the classical buckling load of axially compressed unstiffened circular cylindrical shells

$$\bar{N}^0 = Eh^2/R\sqrt{3\left(1-\nu^2\right)}, \tag{37}$$

where the superscript ^{0} denotes an initial guess value. We have also adopted the following displacement functions

$$\left(U^0, V^0, W^0\right) = \left(\sin \pi y/b, \cos \pi y/b, \sin \pi y/b\right). \tag{38}$$

From them, the guess for the spatial distribution of the solutions Z_i $(i = 1, \ldots, 8)$ are determined by straightforward differentiation. Finally, the critical buckling load $\bar{N}_{cr}$ is obtained by selecting the minimum value of $\bar{N} = \bar{N}\left(m\right)$ in the form

$$\bar{N}_{cr} = \min_m \bar{N}. \tag{39}$$

Despite some unlike numerical features of *bvp4c* (for instance, the need to provide a initial guess to the solution and also an additional equation to the boundary conditions), it can be observed that the mean difference from finite element results are lower than 4.0%, and in most cases the prediction of the number of half waves is correct.

Table 2. Stiffener cross-section dimensions.

Width (mm)		Thickness (mm)	
w_a	19.05	t_a	1.27
w_b	19.05	t_b	1.27
w_c	5.50	t_c	3.00

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Table 3. Material properties.

Material	Type	E (N/mm^2)	ν
Skin	AL 2024-T3	72400	0.33
Stiffener	AL 7050-T3511	71020	0.33

Table 4. Comparison between Lévy formulation and FEM.

Panel ID	m		$\bar{N}_{cr}$		$\lvert$Diff.$\rvert$ %
	Lévy	FEM	Lévy	FEM	
1	5	5	23.3	23.0	1.3
2	5	5	23.3	22.1	5.4
3	7	7	46.9	50.1	6.4
4	7	7	46.7	45.3	3.1
5†	8	8	23.4	23.7	1.3
6	8	7	23.3	22.1	5.4
7†	10	10	47.0	50.3	6.6
8†	11	10	46.7	46.3	0.9
9†	10	10	23.3	23.7	1.7
10†	10	10	23.3	22.0	5.9
11†	14	14	46.9	50.4	6.9
12†	14	14	46.7	46.6	0.2

† Panels for which the first buckling mode is of global type: tabled results are concerned to the first local mode.

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