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ON THE ACCURACY OF RELIABILITY INDEX BASED APPROACHES AND ITS APPLICATION TO RBDO PROBLEMS

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Abstract. *The approximate evaluation of failure probabilities using reliability index based approaches, such as First Order Reliability Methods (FORM), are among the most widespread techniques used for reliability assessment. For this reason the vast majority of papers addressing Reliability Based Design Optimization (RBDO) use some method based on the concept of reliability indexes. In the last years, it seems that most part of the effort of the scientific community was dedicated to the development of more efficient and robust RBDO methods based on reliability indexes, but alternatives not based on reliability indexes are rarely presented. Unfortunately, it is well known that the approximation given by reliability indexes can be poor in many cases of practical interest, thus preventing the use of RBDO methods based on this concept. In this paper we prove that reliability index based approaches may overestimate the true reliability index of the problem when several limit state functions are defined. We also present a very simple example where reliability index based RBDO methods provide a very poor result, putting in evidence the need for RBDO methods based on more accurate reliability analysis techniques.*

Keywords: *reliability based design optimization, reliability analysis, reliability index approach, performance measure approach, accuracy*

1. INTRODUCTION

In a Reliability Based Design Optimization (RBDO) problem one seeks an optimum solution that respect constraints on some minimum reliability level, assuming that some parameters of the problem are random variables. In the last years several authors pointed out the importance of taking into account randomness and several RBDO methods and approaches were presented in literature (Lopez and Beck, 2012).

However, reliability assessment can be very demanding from the computational point of view. For this reason, approximations of failure probabilities based on reliability indexes are frequently used in practice, such as the one made in the First Order Reliability Method (FORM) (Madsen *et al.*, 1986; Melchers, 1999). Consequently the vast majority of papers addressing Reliability Based Design Optimization (RBDO) use some method based on FORM reliability indexes, as pointed out by Lopez and Beck (2012).

In the last years, it seems that most part of the effort of the scientific community was dedicated to the development of more efficient and robust RBDO methods based on FORM, but alternatives not based on reliability indexes are rarely presented. Unfortunately, it is well known that the approximation given by the FORM can be poor in many cases of practical interest. In particular, it is well known that FORM can give poor approximations when (Melchers, 1999; Haldar and Mahadevan, 2000; Madsen *et al.*, 1986): random variables have non gaussian distributions and the limit state functions are highly non linear.

In this paper we show that problems were several limit state functions are defined present an additional case were FORM can give poor approximations. We prove this result and present a very simple example where RBDO methods based on FORM find a poor optimum solution, that do not respect the reliability level defined. This puts in evidence the need for RBDO methods based on more accurate reliability analysis techniques.

We highlight the fact that the main result of this paper holds for a large class of problems, were several limit state function are defined, even if the example presented addresses truss sizing optimization.

2. PROBLEM STATEMENT

The problem of truss sizing optimization can be stated as

$$\text{Find} \quad \mathbf{A} = \{A_1, A_2, ..., A_n\} \tag{1}$$

that gives

$$\min V(\mathbf{A}) = \mathbf{L}^T \mathbf{A} \tag{2}$$

subject to

$$g_{3i-2} = +\sigma_i - \overline{\sigma}_t \leq 0 \quad (i = 1, 2, ..., n), \tag{3}$$

$$g_{3i-1} = -\sigma_i - \overline{\sigma}_c \leq 0 \quad (i = 1, 2, ..., n), \tag{4}$$

$$g_{3i} = -\sigma_i - \overline{\sigma}_{b,i} \leq 0 \quad (i = 1, 2, ..., n), \tag{5}$$

$$0 < A_{\min} \leq A_i \leq A_{\max} \quad (i = 1, 2, ..., n), \tag{6}$$

where n is the number of bars, A_i are cross sectional areas, $\mathbf{L}$ is a vector of bar lengths, V is the volume of the structure, σ_i are the stresses inside the bars, $\overline{\sigma}_t$ is the allowable stress in tension, $\overline{\sigma}_c$ is the allowable stress in compression, $\overline{\sigma}_{b,i}$ is the Euler buckling stress of bar i and $A_{\min}$ and $A_{\max}$ are lower and upper bounds on the cross sectional areas.

Assuming that some parameters of the problem are uncertain, the constraints from Eq. (3)-(5) become random variables. The RBDO problem is then given by Eqs. (1),(2) and (6) with constraints

$$P(g_i > 0) \leq P_{\max} \quad (i = 1, 2, ..., 3n), \tag{7}$$

where $P(\cdot)$ is the probability of a given event occurring and $P_{\max}$ is the maximum allowable failure probability. That is, we now ask for a solution with failure probabilities $P(g_i > 0)$ smaller than a given limit.

In almost every case, the failure probabilities $P(g_i > 0)$ cannot be evaluated efficiently and the the constraints are approximated by

$$\beta_i \geq \beta_{\min} \quad (i = 1, 2, ..., 3n), \tag{8}$$

where β_i is the Hassofer-Lind reliability index (Madsen *et al.*, 1986; Melchers, 1999; Haldar and Mahadevan, 2000) related to each limit state function $g_i = 0$ and $\beta_{\min}$ is a minimum allowable reliability index.

Several strategies have been proposed in order to solve RBDO problems similar to the one given by Eqs. (1),(2) and (8) (Lopez and Beck, 2012). Here we use the sequential optimization and reliability assessment (SORA) approach (Du and Chen, 2004), that is based on the performance measure approach (PMA) (Tu *et al.*, 1999).

2.1 Global limit state function

In practice, one expects that all constraints from Eqs. (3)-(5) to be satisfied simultaneously. In this case we have

$$g_i \leq 0, \ \forall i = 1, 2, ..., 3n. \tag{9}$$

Besides, if all constraints are smaller than zero, the maximum value between then is also smaller than zero and we have

$$G \leq 0, \tag{10}$$

where

$$G = \max_{i=1,..,3n} g_i. \tag{11}$$

This defines a new limit state function G, considering that all constraints must be respect at the same time. We call this the global limit state function, in order to put in evidence the fact that it takes into account all limit state functions of the original problem. It can also be noted that Eq. (11) implies Eq. (9). That is, if the global limit state function is respected then all the limit state functions from Eq. (9) are automatically respected.

A RBDO constraint defined according to the global limit state function from Eq. (11) can be written as

$$P(G > 0) = P\left(\max_{i=1,..,3n} g_i > 0\right) \leq P_{\max} \tag{12}$$

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and approximated by

$$\beta \geq \beta_{\min}, \tag{13}$$

where β is the reliability index obtained using the global limit state function $G = 0$.

In the case of a deterministic optimization problem, the constraint from Eq. (10) is equivalent to the ones from Eq. (9). However, in the case of a probabilistic approach, this is not necessarily true. In order to address the problem from a mathematical point of view, we can write the failure probability from Eq. (7) as

$$P(g_i > 0) = \int_{\Omega_i} \phi(\mathbf{x}) d\mathbf{x}, \quad \Omega_i = \{\mathbf{x} \in R^n : g_i(\mathbf{x}) > 0\}, \tag{14}$$

where $\phi(\mathbf{x})$ is the probability density function (PDF) of the random vector $\mathbf{x}$ (Haldar and Mahadevan, 2000; Melchers, 1999; Madsen *et al.*, 1986).

The failure probability defined using the global limit state function, on the other hand, can be written as

$$P(G > 0) = \int_{\Omega_{\max}} \phi(\mathbf{x}) d\mathbf{x}, \quad \Omega_{\max} = \{\mathbf{x} \in R^n : G(\mathbf{x}) > 0\}. \tag{15}$$

The sets

$$\Omega_i = \{\mathbf{x} \in R^n : g_i(\mathbf{x}) > 0\} \tag{16}$$

and

$$\Omega_{\max} = \{\mathbf{x} \in R^n : G(\mathbf{x}) > 0\} \tag{17}$$

then define the failure regions for the limit state functions $g_i = 0$ and $G = 0$, respectively.

However,

$$\mathbf{x} \in \Omega_i \rightarrow \mathbf{x} \in \Omega_{\max} \tag{18}$$

since

$$g_i(\mathbf{x}) > 0 \rightarrow \max_{i=1,..,3n} g_i(\mathbf{x}) = G(\mathbf{x}) > 0 \tag{19}$$

and we have

$$\Omega_i \subset \Omega_{\max}, \ \forall i = 1, 2, ..., 3n. \tag{20}$$

This fact can also be noted from the example presented in Fig. 1. We note that the failure regions defined by the limit state functions g_1 and g_2 are subsets of the failure domain defined by the global limit state function G.

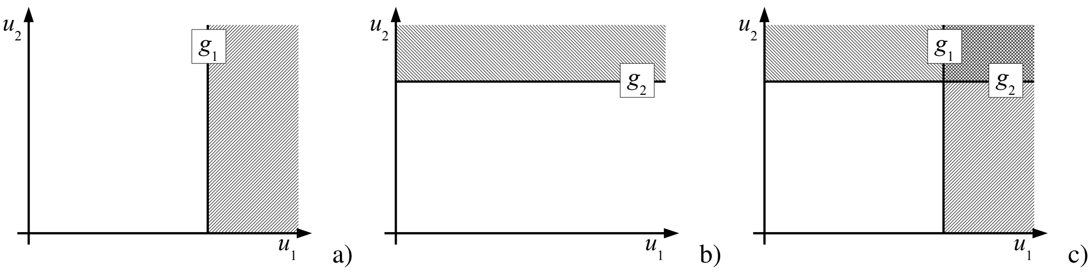

Figure 1. Example of failure regions (hatched) defined by two limit state functions

Since Ω_i are subsets of $\Omega_{\max}$, the integrals from Eq. (14) must be smaller than the one from Eq. (15), since they are obtained integrating the same non negative function (the PDF $\phi(\mathbf{x})$) over smaller regions $\Omega_i \subset \Omega_{\max}$. This results in

$$P(g_i > 0) \leq P(G > 0), \ \forall i = 1, 2, ..., 3n \tag{21}$$

and consequently

$$\beta_i \geq \beta, \ \forall i = 1, 2, ..., 3n. \tag{22}$$

ISSN 2176-5480

From Eq. (22) we then have

$$\beta \leq \min_{i=1,\ldots,3n} \beta_i. \tag{23}$$

This results is extremely important in the context of RBDO. It shows that the reliability index considering the global limit state function is always smaller than or equal to the ones obtained considering a single limit state function at a time. Consequently, assuming that the true reliability index of the structure is the minimum value between the β_i may not result in the reliability index β, obtained with the global limit state function $G = \max g_i$.

This puts reliability index based approaches in a difficult position, since when applied to cases where several limit state functions are defined, they can overestimate the true reliability index of the problem. In other words, standard RBDO approaches based on reliability indexes may be against security. This can occur even in the case where reliability analysis for each limit state function is exact, since no approximation was assumed when Eqs. (22) and (23) were obtained.

3. NUMERICAL EXAMPLE

The example studied in this paper is presented in Fig. 2. The parameters and random variables used in this example are presented in Tab. 1. All random variables have gaussian distributions. We assume $\beta_{\min} = 2.0$, that corresponds to a maximum allowable failure probability $P_{\max} = 0.0228$. The Euler buckling stresses are evaluated by

$$\overline{\sigma}_{b,i} = \frac{1.5162\pi^2 E A_i}{L_i^2}, \tag{24}$$

that is obtained assuming that the cross sections are tubular, with outside diameter d_o and inside diameter $d_i = 0.9d_o$.

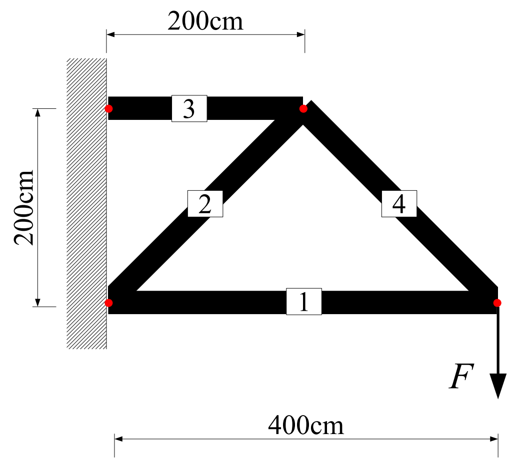

Figure 2. Truss subject to RBDO

Table 1. Parameters and random variables.

Parameter	Mean	Standard deviation
Allowable stress in tension and compression (MPa)	250.0	20.0
Elastic modulus (GPa)	200.0	10.0
Applied load F (kN)	10.0	2.0
Cross sectional areas (cm^2)	A_i	$0.025A_i$

The optimum solution obtained in this case is presented in Fig. 3. The optimum cross sectional areas are: 2.7819cm^2, 2.3393cm^2, 1.1751cm^2 and 0.8309cm^2. The volume of the optimum solution is 2.2444x10^{3}cm^2.

4. RESULTS AND DISCUSSION

The reliability index of each constraint of the problem, evaluated using FORM and Monte Carlo Simulation (MCS), is presented in Tab. 2. MCS was made using 10^5 sampling points, only for constraints with reliability index close to 2.0. Here we assume that the reliability indexes found by MCS are exact. Constraints 1-3 are related to bar 1, constraints 4-6 are related to bar 2, constraints 7-9 are related to bar 3 and constraints 10-12 are related to bar 4. Besides, the constraints for each bar are ordered as: tension, compression and buckling. Consequently, from Tab. 2 we note that bars 1 and 2 critical state is buckling, while bars 3 and 4 critical state is tension, as expected.

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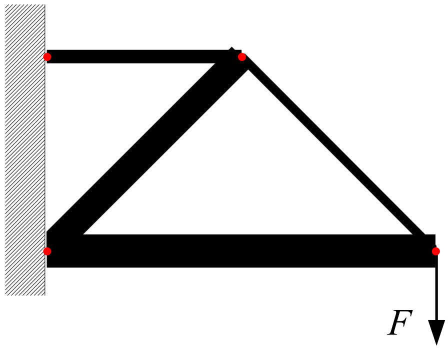

Figure 3. Optimum solution obtained with RBDO using SORA and PMA

Table 2. Reliability index related to each stress constraint.

Limit state function	β (FORM)	β (MCS)
g_1	13.4544	–
g_2	10.0453	–
g_3	2.0001	1.9950
g_4	13.2799	–
g_5	8.0511	–
g_6	2.0002	2.0052
g_7	2.0002	1.9877
g_8	10.5992	–
g_9	7.4662	–
g_{10}	2.0001	1.9969
g_{11}	10.5991	–
g_{12}	5.9011	–

From Tab. 2 we note that the SORA indeed found a solution for which all FORM reliability indexes respect the minimum allowable value $\beta_{\min} = 2.0$. Besides,the definition of reliability index used in FORM (also used in SORA) agrees with the reliability indexes found by MCS. Consequently, from the point of view of each constraint individually, the results are very accurate and nothing is wrong. The results obtained with SORA respect the minimum reliability index and these results agree with MCS.

The reliability indexes obtained considering the global limit state function from Eq. (10) is presented in Tab. 3. In this case, we note that the reliability index found by FORM agrees with the smallest reliability index presented in Tab. 2. However, the reliability index found by MCS is much smaller, showing that the structure is actually less reliable than can be concluded from Tab. 2.

Table 3. Reliability index evaluated assuming a single global limit state function.

Limit state function	β (FORM)	β (MCS)
$G = \max g_i$	2.0001	1.6897

These results agree with Eq. (23). The actual reliability index of the structure is overestimated when each limit state function is considered separately. The FORM approach is not able to capture this fact, even when the global limit state function $G = \max g_i$ is used, because it only takes into account the limit state function that is closest to the origin. However, we note that even the results obtained with MCS for each g_i does not agree with the true reliability index of the problem (found with MCS and the global limit state function), confirming that this error can occur even in the case where reliability analysis for each limit state function is exact. We note that enforcing bounds on reliability indexes of every constraint does not necessarily enforces a bound on the true reliability index of the structure.

We put in evidence the fact that this example is a very simple one. Only gaussian distributions were used, the structure is simple and presents linear behavior. From Tab. 2 and Tab. 3 we note that the loss of accuracy cannot be attributed to the reliability analysis carried for each limit state function. Errors related to non gaussian distribution or highly non linear limit state functions play no role in this example. In this case, it seems that the main reason for loss of accuracy is the fact pointed out in Eq. (23) and is related to the existence of several limit state functions.

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5. CONCLUDING REMARKS

The results presented in this paper address the loss of accuracy of reliability index based approaches for RBDO. Several reasons for loss of accuracy have already been pointed by other authors, such as non gaussian distributions, nonlinear limit sate functions and dependent random variables. Here we showed that the existence of several limit state functions can also degenerate the results obtained with FORM (and even MCS).

Equation (22) shows that the reliability indexes of each individual limit state function will always be greater than or equal to the true reliability index of the problem. Taking the minimum value between them can result in an overestimated reliability index, as shown in Eq. (23). In other words, standard reliability index based approaches can overestimate the true reliability index of the system when several limit state functions are defined. This is true even if the reliability indexes are evaluated exactly, and the result can be very poor.

This is a very delicate issue, since the results obtained can be against security. The numerical example presented confirms the result from Eq. (23) and shows that the error can be significant, even for very simple examples. Besides, this kind of error is different from the ones commonly appointed in literature.

In this context, we conclude that more research is necessary in order to increase the accuracy of reliability assessment in RBDO methods. Strategies that evaluate better approximations for the reliability index (when several limit state functions are defined) and alternative approaches not based on reliability indexes at all should be subject of further research.

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