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NONLINEAR MECHANICAL SYSTEMS IDENTIFICATION THROUGH WIENER/VOLTERRA SERIES

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Abstract. *Mathematical modeling of mechanical systems is an important research area in structural dynamics. The main goal is to obtain an accurate model that predicts the dynamics of the system. However, nonlinear effects caused by gaps, backlash, joints, as well as large displacements are not as well understood as the linear counterpart. In this sense, the Volterra series is an interesting tool for the analysis of nonlinear systems since it is a generalization of the linear model based on the impulse response function. This paper applies the discrete-time Volterra series expanded in orthogonal Kautz functions, named Wiener/Volterra series, to identify a nonparametric model of a nonlinear benchmark system. The input and output data are used to identify the Volterra kernels of the structure. Since the kernels are non-parametric representations of the system, a parameter identification technique based on the Volterra model are proposed for future applications in nonlinear inverse problems. The paper concludes by indicating the main advantages and drawbacks of the technique for modeling the dynamic behavior of nonlinear structures with suggestions for further studies.*

Keywords: *System identification, nonlinear systems, Volterra series, Kautz functions*

1. INTRODUCTION

Mathematical modeling of mechanical structures is an important research topic in structural dynamics. One of the goals of this area is to obtain a reliable model that can accurately predict the dynamic behavior of the system (Kerschen *et al.*, 2006). Model updating strategies for linear structures were extensively investigated in literature (Mottershead and Friswell, 1993). These procedures are generally based on the maximization of the correlation between the mathematical model and experimental vibration data measured on the structure. The objective functions usually depend upon modal parameters of the structure as for example, mode shapes, natural frequencies and frequency response functions. However, these concepts are based in the superposition principle, so that they are not valid in the case of nonlinear systems (Worden *et al.*, 2009). In this context, Volterra series are an attractive technique since they are a generalization of the linear model based on the impulse response function (Schetzen, 1980). The Volterra series represent the output of a nonlinear system through a multidimensional convolution between the input signal and the Volterra kernels (Rugh, 1981).

In the area of structural dynamics, most of the studies applying Volterra series consider their continuous time formulation of Volterra series, while the use of the discrete-time formulation is often disregarded. Cafferty and Tomlinson (1997) identified analytic expressions for the Volterra kernels of automotive dampers in the frequency domain, termed high order frequency response functions (HOFRF) via the harmonic probing method. An experimental test is applied in the structure to identify the main diagonals of the nonlinear kernels and to calculate the nonlinear parameters of the damper. Silva (2005) studied the identification of nonlinear aeroelastic systems using, among other techniques, Volterra models. Input and output signals of the aileron and the vibration measured in a active aeroelastic wing under test in a wind tunnel were used to identify the first three Volterra kernels of this system. da Silva *et al.* (2010) applied the Volterra model in the problem of identification of a cantilever beam with a local nonlinearity in the free end of the beam. The authors expanded the Volterra model with orthogonal Kautz functions, which substantially decreased the number of terms required to represent the kernels. Following the same idea, da Silva (2011) utilized the Volterra model to create a metric for nonlinear model updating, and treated a frame structure with local nonlinearity. The poles of the Kautz basis functions were selected by analyzing the predominant dynamics in the response of the nonlinear system. The results showed the possibility of application of the Volterra model in systems with polynomial and bi-linear nonlinearities.

In this work, the Volterra kernels expanded in Kautz functions of a single degree of freedom benchmark are identified through a simple least squares approximation. A parameter identification is performed by using an objective function based on the deviation of the kernels of the model in comparison to reference kernels of the structure to be identified. The shape of the proposed functions are analyzed and compared aiming future applications of the Volterra series to inverse problems. The paper is summarized as follows. Section 2 gives a background on discrete-time Volterra series and the orthonormal expansion of this model (Wiener/Volterra series). Section 3 describes the benchmark structure used in this study. Section 4 shows the methodology used in the parameter identification procedure. Section 5 shows the identification of the Volterra kernels of the system and the identification of the parameters of the model. Finally, in the section 6 the main advantages and drawbacks of the proposed methodology are exposed and further steps of this research are outlined.

ISSN 2176-5480

2. WIENER/VOLTERRA SERIES

The Volterra series is a direct generalization of the concept of impulse response function of linear systems (Rugh, 1981). In the discrete-time formulation, the Volterra series express the response of the system $x(k)$ to an input $u(k)$ as:

$$x(k) = x_{lin}(k) + x_{quad}(k) + x_{cub}(k) + ... \tag{1}$$

where $x_{lin}(k)$ is the linear contribution of the response given by the convolution of the input $u(k)$ with the linear impulse response function, and $x_{quad}(k) + x_{cub}(k) + ...$ are the non-linear contributions of the output $x(k)$. The application of the discrete Volterra series give the following equation:

$$x(k) = \sum_{m=1}^{+\infty} \sum_{n_1=-\infty}^{+\infty} \dots \sum_{n_m=-\infty}^{+\infty} h_m(n_1, n_2, \dots, n_m) \prod_{i=1}^{m} u(k - n_i) \tag{2}$$

where m is the number of non-linear terms, generally, truncated in a low-order values M and $h_m(n_1, n_2, \dots, n_m)$ is the $m-th$ order Volterra kernel. Despite of this truncation, the most common nonlinearities can be well represented by this model, except Coulomb damping, hysteresis, backlash and other (da Silva *et al.*, 2010; da Silva, 2011). Considering the first three terms ($x_{lin}(k)$, $x_{quad}(k)$ and $x_{cub}(k)$) it is possible to obtain:

$$\begin{aligned} x(k) = &\sum_{n_1=0}^{N_1} h_1(n_1) u(k-n_1) + \sum_{n_1=0}^{N_2}\sum_{n_2=0}^{N_2} h_2(n_1,n_2) u(k-n_1) u(k-n_2) + \\ &+ \sum_{n_1=0}^{N_3}\sum_{n_2=0}^{N_3}\sum_{n_3=0}^{N_3} h_3(n_1,n_2,n_3) u(k-n_1) u(k-n_2) u(k-n_3) \end{aligned} \tag{3}$$

where $h_1(n_1)$ is the first Volterra kernel (linear impulse response function), $h_2(n_1, n_2)$ is the second Volterra kernel and $h_3(n_1, n_2, n_3)$ is the third Volterra kernel. Unfortunately, the number of parameters to be estimated in eq. 3 can be large, limiting its use in practical applications. One way to overcome this drawback is to expand the Volterra kernels using orthonormal basis functions. In the case of oscillating underdamped systems, as is the main interest of structural dynamics, Kautz functions can be applied (Kautz, 1954). The expansion of the Volterra kernels can be represented by:

$$h_1(n_1) = \sum_{i_1}^{M_1} \alpha(i_1) \psi_{i_1}(n_1) \tag{4}$$

$$h_2(n_1, n_2) = \sum_{i_1=1}^{M_2} \sum_{i_2=1}^{i_1} \alpha(i_1, i_2) \psi_{i_1}(n_1) \psi_{i_2}(n_2) \tag{5}$$

$$h_3(n_1, n_2, n_3) = \sum_{i_1=1}^{M_3} \sum_{i_2=i_1}^{M_3} \sum_{i_3=i_2}^{M_3} \alpha(i_1, i_2, i_3) \psi_{i_1}(n_1) \psi_{i_2}(n_2) \psi_{i_3}(n_3) \tag{6}$$

where M_1, M_2 and M_3 are the number of orthonormal functions used to describe the first, second and third kernels, respectively, $\alpha(i_1)$, $\alpha(i_1, i_2)$ and $\alpha(i_1, i_2, i_3)$ denotes the orthonormal Volterra kernels, and ψ_{i_1}, ψ_{i_2} and ψ_{i_3} are the orthonormal functions. Substituting Eqs. (4), (5) and (6) in the Eq. (3), it is possible to obtain:

$$\begin{aligned} x(k) = &\sum_{i_1=1}^{M_1} \alpha(i_1) l_{i_1}(k) + \sum_{i_1=1}^{M_2}\sum_{i_2=1}^{i_1} \alpha(i_1,i_2) l_{i_1}(k) l_{i_2}(k) + \\ &+ \sum_{i_1=1}^{M_3}\sum_{i_2=i_1}^{M_3}\sum_{i_3=i_2}^{M_3} \alpha(i_1,i_2,i_3) l_{i_1}(k) l_{i_2}(k) l_{i_3}(k) \end{aligned} \tag{7}$$

where $l_{i_1}(k)$, $l_{i_2}(k)$ and $l_{i_3}(k)$ are the convolution between the input and the impulse response of the orthonormal functions. As the Volterra systems are linear in the parameters, it is possible to write the vector of output signal $\mathbf{X}$ as a multiplication between the regression matrix $\mathbf{\Lambda}$ with the input signal filtered by the orthonormal functions and the parameter vector $\mathbf{\Theta}$ with the orthonormal kernels:

$$\mathbf{X} = \mathbf{\Lambda}\mathbf{\Theta} \tag{8}$$

The Volterra kernels in the vector θ can be estimated by applying the classical least squares method and obtain this vector as (Aguirre, 2007):

$$\mathbf{\Theta} = \mathbf{\Lambda}^{-1}\mathbf{X} = \left(\mathbf{\Lambda}^T\mathbf{\Lambda}\right)^{-1}\mathbf{\Lambda}^T\mathbf{X} \tag{9}$$

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However, the orthonormal expansion of the Volterra kernels requires a suitable function that represents the linear and nonlinear dynamics of the system. The aforementioned Kautz orthonormal functions turn out to be appropriate for the representation of oscillatory systems since it has complex conjugate poles represented by $\beta_{2g-1} = \sigma + j\omega$ and $\beta_{2g} = \sigma - j\omega$ such as $|\beta_{2g-1}|$,$|\beta_{2g}| < 1$ for a stable system. The pair of Kautz functions are given by (Kautz, 1954):

$$\Psi_{2n-1}(z) = \frac{\sqrt{1-b^2}\sqrt{1-c^2}}{z^2 + b(c-1)z - c}[H_{b,c}(z)]^{n-1} \tag{10}$$

$$\Psi_{2n}(z) = \Psi_{2j-1}(z)\frac{z-b}{\sqrt{1-b^2}} \tag{11}$$

where z is the complex variable in the discrete domain, $H_{b,c}(z) = \frac{-cz^2+b(c-1)z+1}{z^2+b(c-1)z-c}$ and the scalar values b and c, relative to the poles β_{2g-1}, β_{2g} considered are given by:

$$b = \frac{\beta_{2g-1} + \beta_{2g}}{1 + \beta_{2g-1}\beta_{2g}} \text{ and } c = -\beta_{2g-1}\beta_{2g} \tag{12}$$

:

where β and $\bar{\beta}$ are the Kautz pole and the conjugate pole, respectively. The Kautz poles in the continuous-time domain can be related to the dynamics of the structure through:

$$\beta_g = -\zeta_g\omega_g + j\omega_g\sqrt{1-{\zeta_g}^2} \tag{13}$$

where ζ_g is the damping ratio of the pole, and ω_g is the $g-th$ frequency. For the application of the poles in the model, it is necessary to make a transformation of the poles to the discrete domain:

$$z_g = e^{\beta_g/F_s} \tag{14}$$

where F_s is the sampling frequency. Figure 1 summarizes the main steps in the identification of the orthonormal Volterra kernels. After the acquisition of the input and output signals of the system, the Kautz poles must be defined through a previous knowledge of the dynamics of the system or using an optimization algorithm. The Kautz functions are then used to filter the input signal and construct the matrices to solve the least squares problem and find the orthonormal kernels.

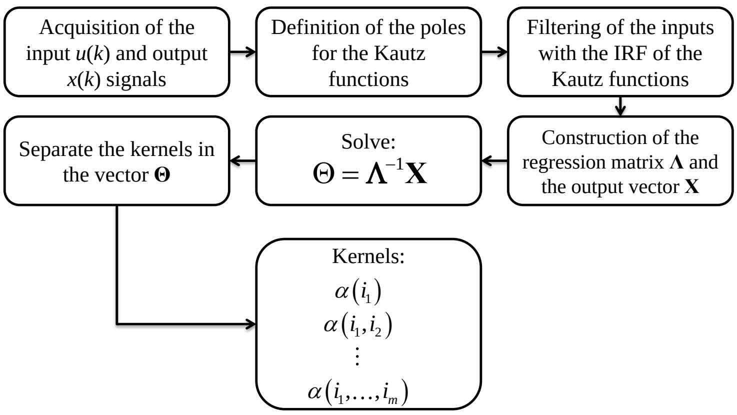

Figure 1. Flowchart summarizing the identification methodology.

In this process, the linear and nonlinear kernels of the model are identified in a single step by using broadband input and output signals where the nonlinearities of the system are relevant. Another approach that can be applied is to identify the kernels in a two step procedure. In the first step the first kernel is identified using a linear data set where the behavior of the system is mainly linear (e.g. a low-input response of the system with polynomial stiffness). In the second step the nonlinear kernels can then be estimated with a nonlinear data set. This two step approach can avoid that the higher order information of the system be assigned to the first kernel. In this paper, both methodologies are applied in order to see the main advantages and drawbacks of each one. Figure 2 present the flowchart of the two step identification methodology.

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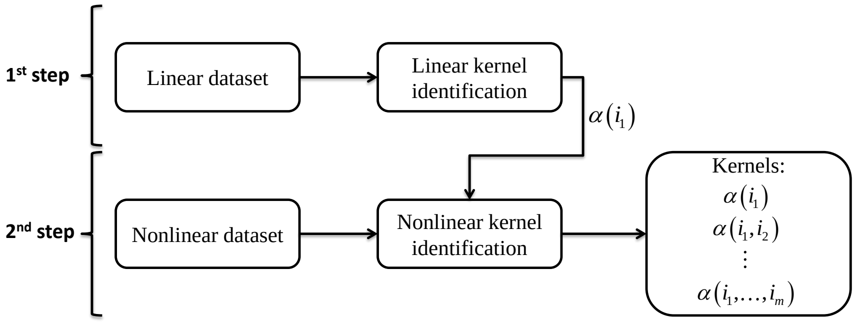


Figure 2. Flowchart summarizing the two step identification methodology.

3. NONLINEAR BENCHMARK

The benchmark used in this work was first studied by Storer (1991) and used also as example in the book of Worden and Tomlinson (2001). This structure consists of a clamped-clamped beam with the application of a preload Δ in the center of the beam by means of an elastic element as shown in Fig. 3.

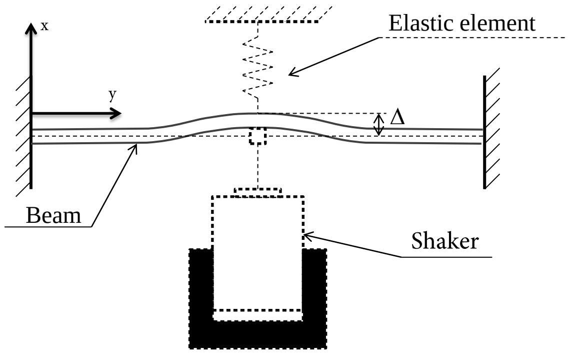


Figure 3. Benchmark structure used in the study.

This system can be modeled considering the first mode shape of the beam by the Lagrange equations. After a careful manipulation of the terms it is possible to obtain:

$$\frac{3m'L}{8}\ddot{x} + \left[\frac{3\Delta^2 EA\pi^4}{8L^3} + \frac{2EI\pi^4}{L^3} + k\right]x + \frac{3\Delta EA\pi^4}{8L^3}x^2 + \frac{EA\pi^4}{8L^3}x^3 = F(t) \tag{15}$$

where $\ddot{x}$, $\dot{x}$ and x are the acceleration, velocity and displacement of the center of the beam, m' is the mass per unit length of the beam, L is the total length of the beam, E is the Young modulus, A is the cross-section area, I is the inertia moment, F is the input force in the system, and X the initial displacement of the beam due to the preload.

It is possible to write Eq. (15) in a more friendly way, obtaining the classical Duffing oscillator (Brennan and Kovacic, 2011):

$$m\ddot{x} + k_1 x + k_2 x^2 + k_3 x^3 = F(t) \tag{16}$$

in which:

$$m = \frac{3m'L}{8}, \quad k_1 = \frac{3\Delta^2 EA\pi^4}{8L^3} + \frac{2EI\pi^4}{L^3} + k, \quad k_2 = \frac{3\Delta EA\pi^4}{8L^3}, \quad k_3 = \frac{EA\pi^4}{8L^3} \tag{17}$$

where m is the equivalent mass of the system, k_1, k_2 and k_3 are the linear, quadratic and cubic stiffness respectively. Considering proportional viscous damping to take into account the energy dissipation, it is possible to obtain:

$$m\ddot{x} + c\dot{x} + k_1 x + k_2 x^2 + k_3 x^3 = F(t) \tag{18}$$

where c is the proportional viscous damping. In this work, it was considered an aluminum beam with the physical and geometrical properties described in the Tab. 1.

ISSN 2176-5480

Table 1. Physical and geometrical properties of the benchmark.

Property	Value
Specific mass (ρ)	$2700\ \mathrm{kg/m^3}$
Young modulus (E)	$62\ \mathrm{GPa}$
Stiffness of the elastic element (k)	$50\ \mathrm{N/m}$
Length of the beam (L)	$0.92\ \mathrm{m}$
Width of the beam (b)	$25.4\ \mathrm{mm}$
Thickness of the beam (h)	$3.3\ \mathrm{mm}$
Preload (Δ)	$2.0\ \mathrm{mm}$

Table 2. Parameters of the Duffing oscillator.

	m [kg]	c [Ns/m]	k_1 [N/m]	k_2 [N/m^2]	k_3 [N/m^3]
Value	0.078	0.50	2.2×10^3	4.9×10^5	8.1×10^7

For these properties, and considering the damping coefficient with the value of $0.50\ \mathrm{Ns/m}$, the parameters of the motion equation can be calculated using Eq. (17). The constant values are shown in Tab. 2.

Although simple, the Duffing equation showed in Eq. (18) can exhibit interesting nonlinear features like the jump phenomenon (Brennan *et al.*, 2008). This nonlinear system was used to generate the responses to a given input signal for the identification of the Volterra model.

4. PARAMETER IDENTIFICATION METHODOLOGY

To identify the parameters of the motion equation a objective function (OF) is proposed based on the deviation of the orthonormal Volterra kernels of the motion equation to experimentally identified kernels with the methodologies presented in section 2. The linear and nonlinear OFs applied in the study are:

$$J_{lin}\left(p\right) = \frac{\left\|\alpha_{1,ref} - \alpha_1\left(p\right)\right\|}{\left\|\alpha_{1,ref} - mean\left(\alpha_{1,ref}\right)\right\|} \tag{19}$$

$$J_{nlin}\left(p\right) = \frac{\left\|\alpha_{2,ref} - \alpha_2\left(p\right)\right\|}{\left\|\alpha_{2,ref} - mean\left(\alpha_{2,ref}\right)\right\|} + \frac{\left\|\alpha_{3,ref} - \alpha_3\left(p\right)\right\|}{\left\|\alpha_{3,ref} - mean\left(\alpha_{3,ref}\right)\right\|} \tag{20}$$

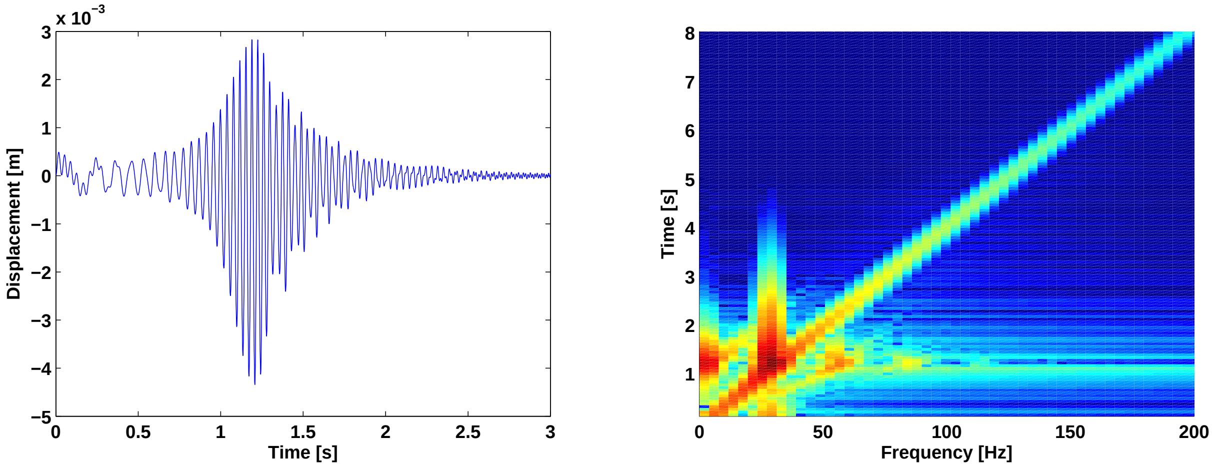
where p is the parameter to be identified, and the index ref denotes the reference kernels identified with the experimental data of the nonlinear system. In this paper the shape of the OFs are investigated through a mapping of the values of the motion equation parameters. Since the Volterra kernels are dependent of the parameters of the system, these OFs can be used to verify structural variations or to quantify these variations in both linear and nonlinear parameters as damage indexes.

5. RESULTS

The input and output signals were obtained through the Newmark integration procedure (Newmark, 1959) solved the Newton-Raphson, considering 8192 samples and a sampling frequency of 1 kHz. The excitation used in the simulation was a linear chirp signal sweeping the frequency range of 1 to 200 Hz with a amplitude of 0.6 N. For the linear data set to identify the first kernel in the two step procedure, the same kind of signal was applies as input but with a amplitude of 0.06 N where the response of the system is mainly linear. The output signal in the time domain and the spectrogram of this signal for the high input data set are shown in the Fig. 5.

The response of the system increases near the linear natural frequency of the equivalent linear system (26.7 Hz). Also it is possible to observe some even and odd harmonics of the response when the input frequency reaches the natural frequency, and a low frequency response due to asymmetry. These phenomena happens due to the quadratic and cubic stiffness in the Duffing oscillator. To identify the Kautz poles for the representation of the nonlinear dynamics of the system, a optimization procedure based in the sequential quadratic programming algorithm (SQP) (Luenberguer and Ye, 2008) to minimize the prediction error of the model. The linear pole was calculated using the known natural frequency and the damping ratio of the system, since these values can be easily obtained by well-established linear techniques (Inman, 1996). Table 3 shows the search ranges of the parameters to be found and the initial guess used in the algorithm that was set to the values of the linear behavior of the system.

Figure 5 illustrates the iterations of the optimization procedure, while the Tab. 4 shows the values found by the



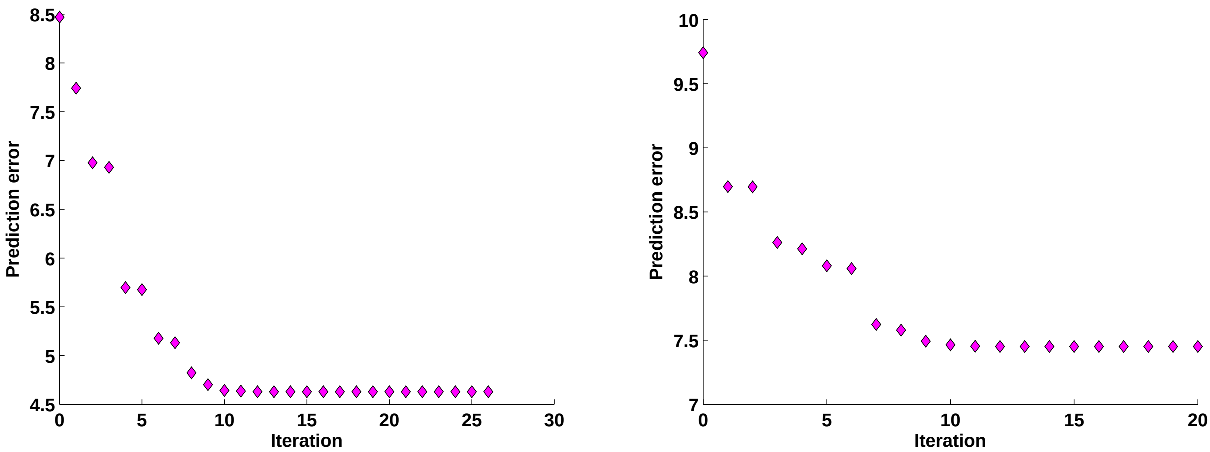
(a) Output signal in time domain. (b) Spectrogram of the output signal.

Figure 4. Response of the Duffing oscillator to a chirp input.

Table 3. Search ranges used in the optimization algorithm.

	ω_2 [Hz]	ζ_2	ω_3 [Hz]	ζ_3
Search range	$[10; 40]$	$[0.001; 0.5]$	$[10; 40]$	$[0.001; 0.5]$
Initial guess	26.73	0.019	26.73	0.019

algorithm for each kind of identification methodology. It is possible to observe that the nonlinear poles usually remains in the neighborhood of the dominant linear dynamics.



(a) Iterations of the one step identification. (b) Iterations of the two step identification.

Figure 5. Iterations of the optimization procedure to find the Kautz poles.

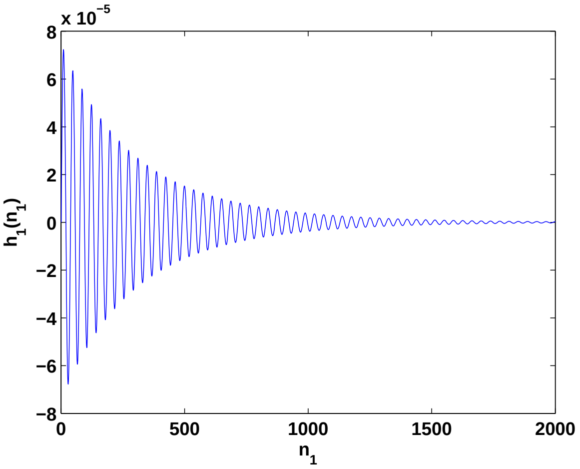
With the poles defined, it was then possible to proceed with the least squares estimation of the Volterra kernels. The first two Volterra kernels, and a partial representation of the third kernel are shown in the physical basis in Fig. 6 for the one step identification procedure. The kernels identified in the two step procedure are suppressed here since their shape are very similar to the kernels identified with the one step methodology.

With the kernels representing the structure in the reference condition, it was then possible to map the OFs showed in Eq. (19) and Eq. (20) for many values of the motion equation parameters. In this study, the stiffness values k_1, k_2 and k_3 were mapped in a region containing the reference values used to calculate the Volterra kernels. However, in a real physical experiment it would be complicated to make a individual variation in each of these parameters as it is possible to observe in Eq. (17). Considering this fact, the preload Δ in the nonlinear beam was also modified to check the behavior of the OFs since it can be easily changed in a real laboratory test. For each of these new parameter values, the motion equation was integrated and the input and output signal were used to identify the new orthonormal kernels. Also, in order to check

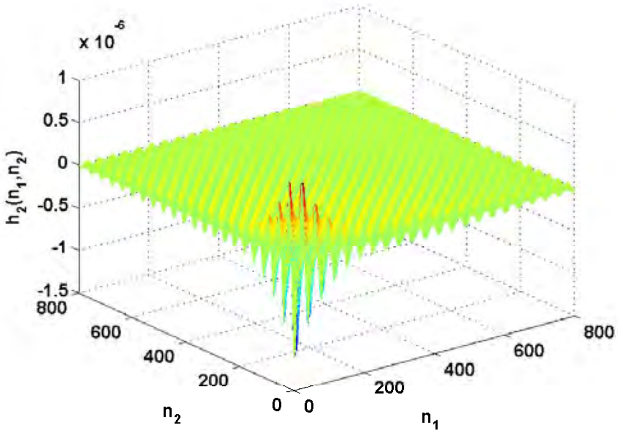
ISSN 2176-5480

Table 4. Information for the identification of the Kautz poles.

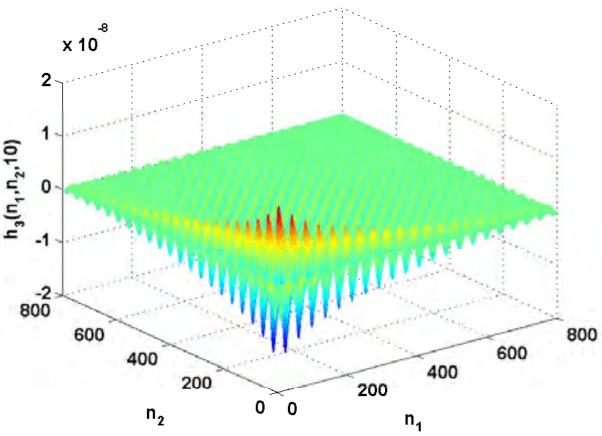
	ω_1 [Hz]	ζ_1	ω_2 [Hz]	ζ_2	ω_3 [Hz]	ζ_3
One step identification	26.73	0.019	26.04	0.033	26.12	0.033
Two step identification	26.73	0.019	26.03	0.033	25.70	0.037



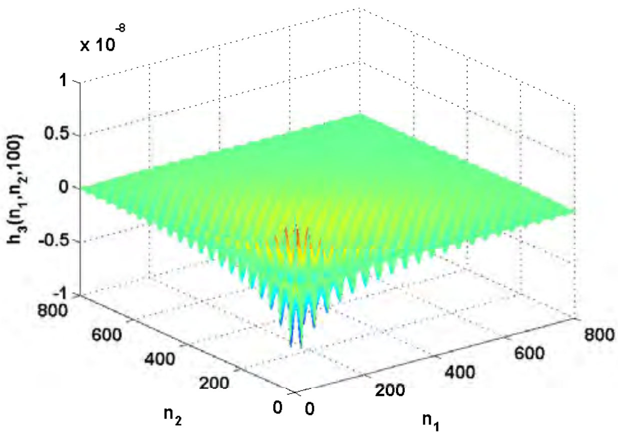
(a) First kernel.



(b) Second kernel.



(c) Third kernel (cut in 10).



(d) Third kernel (cut in 100).

Figure 6. Volterra kernels in the physical basis.

the robustness of the proposed OF, 10% RMS of Gaussian noise was added to the reference input and output signals to simulate measurement noise. The curves representing the OFs for these cases are shown normalized in the range of $[0; 1]$ for the one step and two step methodology in Fig. 7 and Fig. 8 respectively.

It can be observed that the nonlinear kernels are considerably more sensitive to variations both in the linear and nonlinear parameters. Since the linear dynamics of the system is dominant, the nonlinear kernels are sensitive also to linear parameter modifications. This can be clearly seen by computing analytic expressions of the kernels using the classical harmonic probing technique (Worden *et al.*, 2009). By the inspection of these expressions it is possible to observe that the nonlinear kernels are, in fact, combinations of the linear kernels. Although harmonic probing seems to be a attractive technique, its practical application can be very difficult to achieve for complex analytic models (da Silva *et al.*, 2010).

Another interesting feature in the mapping of the OFs is that the shape of the curves was not greatly affected even for a considerably high amount of noise added in the reference data (10% RMS). This can be interesting in experimental applications since noise is always present in vibration testing. Also, the OFs calculated using the two step methodology (Fig. 8) showed to have a smoother behavior than the ones identified in a single step (Fig. 7). This feature can make easier the solution of inverse problems based on these OFs (i.e parameter identification or model updating) since local search algorithms are often used in these problems. The reason for the more well behaved shape of the OF in the two step

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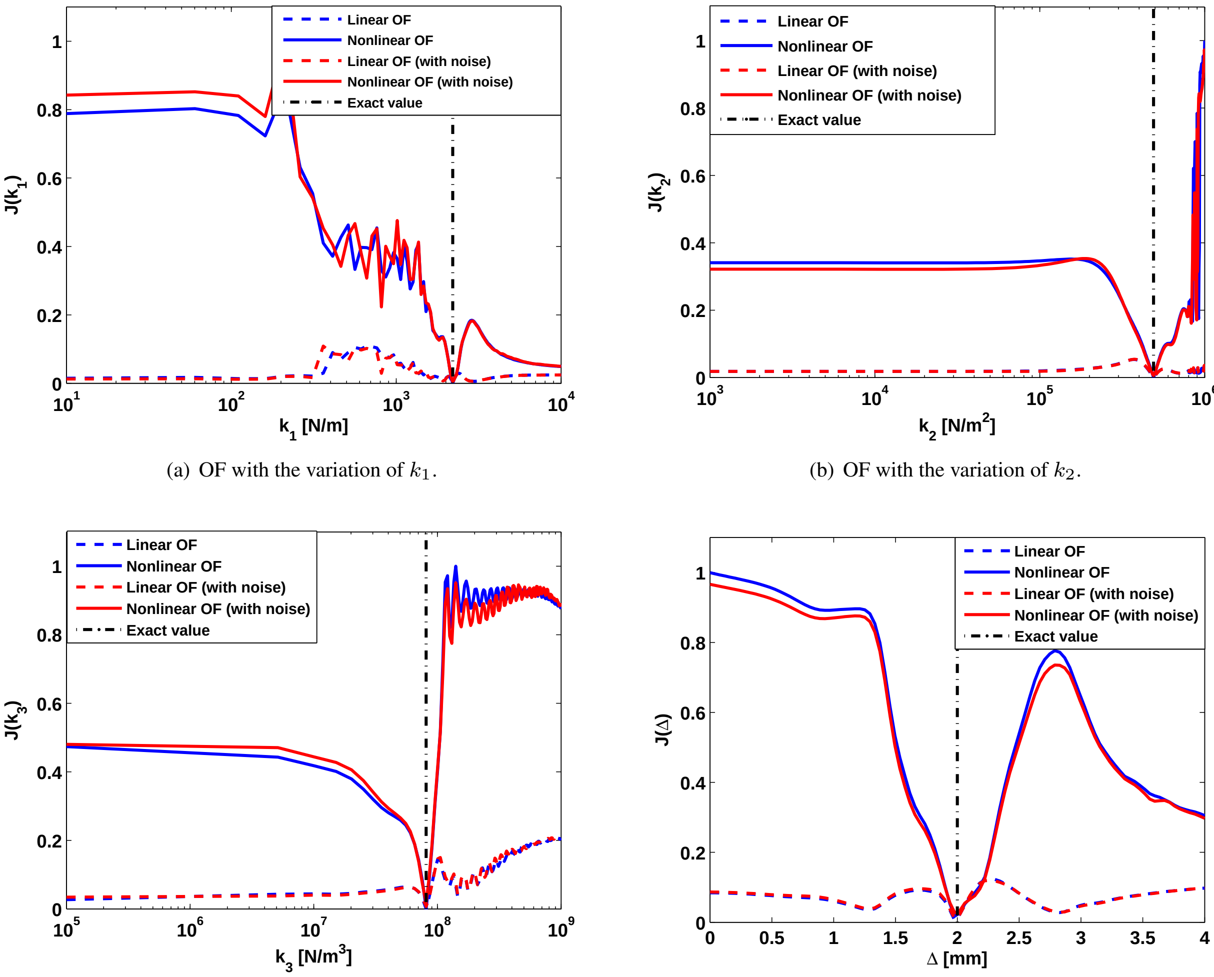

(a) OF with the variation of k_1.

(b) OF with the variation of k_2.

(c) OF with the variation of k_3.

(d) OF with the variation of Δ.

Figure 7. OFs of the parameter identification problem applying the one step identification.

procedure is probably related to the fact that in this kind of methodology the linear portion of the response of the system is improved since a linear data set is exclusively used for the identification of the first kernel.

6. FINAL REMARKS

This work proposed the application of Volterra series expanded in orthogonal Kautz functions to model the nonlinear multiple convolution between the input and output signals of a benchmark structure modeled by the classical Duffing equation. The orthogonal Volterra kernels were identified using time-domain data generated through numerical integration of the motion equation of the system. With these kernels representing the reference structure, a parameter identification procedure was tested with OFs based on the difference between the reference kernels and the nonlinear kernels extracted with the response of the model.

The results showed a interesting approach to be applied in some inverse problems involving nonlinear systems that can be described by the Volterra model. The OFs presented smooth behavior in most of the mappedd range of the structural parameters even with presence of noise in the time-domain data. Two distinct methodologies for the kernel identification were tested based on one and two step procedures. The two step procedure showed some interesting advantages since it employs a linear time-domain data set for the identification of the first Volterra kernel. The only major drawback of this procedure is the need of having two different data sets and also two separated least squares calculations.

The future steps of this research include the application of this technique to develop structural health monitoring strategies for applications in nonlinear structures. Specifically, this model will be applied to damage detection with a index based on the prediction error of the model or the deviation of the identified kernels. In the cases where the damage are detected, a model updating procedure will be executed to recover the parameters of the system in the new configuration.

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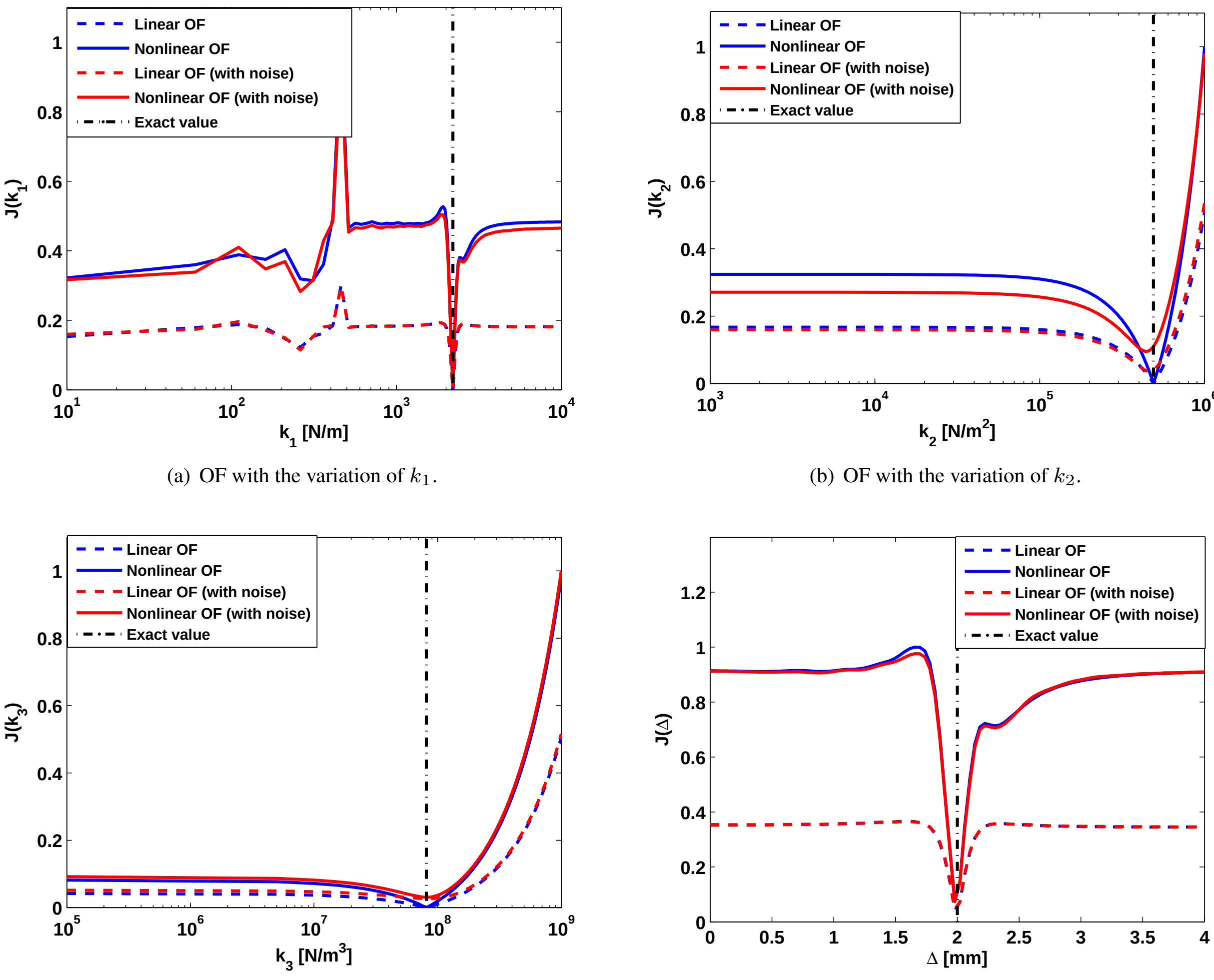

(a) OF with the variation of k_1. (b) OF with the variation of k_2.

(c) OF with the variation of k_3. (d) OF with the variation of Δ.

Figure 8. OFs of the parameter identification problem applying the two step identification.

7. ACKNOWLEDGEMENTS

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