ISSN 2176-5480

22nd International Congress of Mechanical Engineering (COBEM 2013)
November 3-7, 2013, Ribeirão Preto, SP, Brazil

A DETERMINISTIC APPROACH FOR NEUTRON TRANSPORT IN TWO-DIMENSIONAL DOMAINS

Anderson Tres
Graduate Program in Applied Mathematics - Federal University of Rio Grande do Sul - Av. Bento Gonçalves, 9500 - 91509-900 - Porto Alegre, RS - Brasil
andersontres.mat@gmail.com

Camila Becker
Graduate Program in Mechanical Engineering - Federal University of Rio Grande do Sul - Rua Sarmento Leite, 425 - 90050-170 - Porto Alegre, RS - Brasil
camila_becker_87@hotmail.com

Rudnei Dias da Cunha and Liliane Basso Barichello
Institute of Mathematics, Federal University of Rio Grande do Sul - Av. Bento Gonçalves, 9500 - 91509-900 - Porto Alegre, RS - Brasil
rcunha@mat.ufrgs.br - lbaric@mat.ufrgs.br

Abstract. *Transport problems in multidimensional geometry arise in a number of different applications and the difficulties associated with their modeling encourage research and development on this issue. In this work, we investigate such a problem, defined in a rectangular domain R with an isotropic neutron source inside and surrounded by vacuum boundary conditions. The main idea is to reduce the complexity of the classical two-dimensional model through nodal schemes along with the application of the analytical discrete ordinates method (ADO) to solve the derived one-dimensional integrated problems in Cartesian geometry. The integration is performed on the entire domain and the final solutions obtained are analytical in terms of the spatial variables. As usual for nodal schemes, the relationship between average fluxes and unknown fluxes at the contours, is approximated. The technique used leads to reduced order eigenvalue systems, thereby providing the solutions more efficiently. The numerical results obtained were compared to test cases found in the literature and they showed to be in good agreement.*

Keywords: *two-dimensional neutron transport problems, nodal schemes, analytical discrete ordinates method (ADO), fixed source problems.*

1. INTRODUCTION

The physical phenomena of transport of neutral particles is of great interest in many scientific applications, as in nuclear reactors (Duderstadt and Hamilton, 1976; Hu *et al.*, 2013; Su'ud, 2008), nuclear medicine and radiological protection (Wagner and Haghighat, 1998), rarefied gas dynamics (Siewert, 2004; Scherer *et al.*, 2009), among others.

The transport equation represents the balance between the gain and loss of particles in a phase space, mathematically describing neutron transport in material media (Lewis and Miller, 1984). In its general form, it is an integro-differential equation that depends on seven variables: three spatial, two angular, energy and time variables. Its solution is very complex, therefore, many numerical methods have been proposed to solve it (Al-Basheer *et al.*, 2010; Stammes *et al.*, 1988; Zhang *et al.*, 2011). These numerical methods are, in general, based on the discretization of the phase space variables and they make use of various direct or iterative schemes for solving the systems of linear and algebraic equations that arise. In time-independent problems, the energy variable is treated, in general, by a multigroup approximation (Duderstadt and Hamilton, 1976). The two angular variables, which indicate the directions of motion of the particles may be discretized using the conventional discrete ordinates method Sn (Lewis and Miller, 1984). The spatial variables, particularly in multi-dimensional problems, can be treated through nodal methods.

In this work, we give attention to nodal methods (Azmy, 1988; Badruzzaman, 1985; Duo *et al.*, 2009), as these are commonly used in solving multidimensional problems where, by integrating in each spatial variables, we decompose the system of PDEs (arising from the discretization of the angular integral) into a systems of ODEs. The use of nodal schemes reduces the complexity of the model and allows the use of various tools for spatial analysis (Barichello *et al.*, 2009, 2011; Gomes and Barros, 2012; Hauser, 2002; Williams, 2007; Vilhena and Barichello, 1997). Here, we extend the idea introduced in previous works (Barichello *et al.*, 2009, 2011), and we develop closed form solutions for the integrated equations derived from the application of a nodal scheme in a two-dimensional discrete ordinates transport problem. The approach is based on the idea of the ADO method (Barichello and Siewert, 1999a), which has been intensively and successfully used for solving, in a concise and accurate way and at low computational cost, a large variety of transport problems (Barichello and Siewert, 1999b, 2000; Cabrera and Barichello, 2006; Scherer *et al.*, 2009). Therefore, we extend here, the research that has been done with the ADO method to solve two-dimensional problems (Barichello *et al.*, 2009, 2011), aiming to address a variety of additional testing problems in order to obtain a comprehensive analysis of the performance of the proposed formulation.

In the next section, the two-dimensional discrete ordinates neutron transport equation is used to describe a fixed source problem in non-multiplicative media. Also, the equation is integrated in the variable y to provide a system of ODEs, so that the one-dimensional problem becomes independent of the variable y, and the unknown terms that appear in the contours are approximated. In Section 3 the ADO method is applied to the one-dimensional problem in the variable x where it was possible to halve the order of the eigenvalue problem. The solutions of the homogeneous problem are then defined. In Section 3.1, the particular solution is proposed and the final system to establish general solution of the problem is presented. The numerical results are listed and compared to other results found in the literature in Section 4. The conclusions of this work and suggestions for future work are in Section 5.

2. FORMULATION OF THE PROBLEM

The transport problem considered here, is defined in a rectangular domain R, such that $x \in [0, a]$ e $y \in [0, b]$. In the smaller region R_s, defined as $[0, a_s] \times [0, b_s]$ enclosed in R, there is an isotropic neutron source as can be seen in Fig. 1.

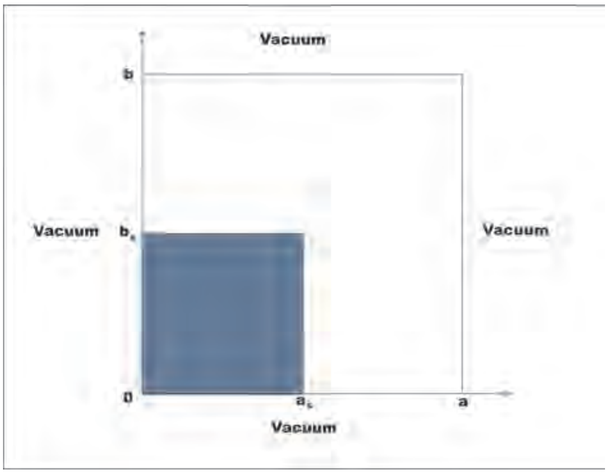

Figure 1. Domain R.

In this context, we start with the discrete ordinates equation for the angular flux, written, for the isotropic case (Lewis and Miller, 1984), as

$$\mu_m \frac{\partial}{\partial x}\Psi(x, y, \mathbf{\Omega}_m) + \eta_m \frac{\partial}{\partial y}\Psi(x, y, \mathbf{\Omega}_m) + \sigma_t \Psi(x, y, \mathbf{\Omega}_m) = Q(x, y) + \frac{\sigma_s}{4}\sum_{k=1}^{M} w_k \Psi(x, y, \mathbf{\Omega}_k), \tag{1}$$

for $m = 1, \ldots, M$ with $M = N(N + 2)/2$, where the w_m are the weights associated to the $\mathbf{\Omega}_m = (\mu_m, \eta_m)$ directions, according to the level-symmetric quadrature scheme. Also, σ_t and σ_s are, respectively, the total and scattering cross sections and $Q(x, y)$ is the isotropic neutron source term.

We proceed to develop what we call the nodal scheme. For the first step, we choose to obtain the one-dimensional nodal equation for the x direction and we associate the directions $\mathbf{\Omega}_m = (\mu_m, \eta_m)$ defined by $\mu_m > 0$ to indexes $m = 1, \ldots, M/2$ and $\mu_m < 0$ to indexes $m = M/2 + 1, \ldots, M$ then we rewrite the Eq. (1) for $m = 1, \ldots, M/2$ in the form

$$\mu_m \frac{\partial}{\partial x}\Psi(x, y, \mathbf{\Omega}_m) + \eta_m \frac{\partial}{\partial y}\Psi(x, y, \mathbf{\Omega}_m) + \sigma_t \Psi(x, y, \mathbf{\Omega}_m) = Q(x, y) + \frac{\sigma_s}{4}\sum_{k=1}^{M/2} w_k [\Psi(x, y, \mathbf{\Omega}_k) + \Psi(x, y, \mathbf{\Omega}_{k+M/2})] \tag{2}$$

and

$$-\mu_m \frac{\partial}{\partial x}\Psi(x, y, \mathbf{\Omega}_{m+M/2}) + \eta_m \frac{\partial}{\partial y}\Psi(x, y, \mathbf{\Omega}_{m+M/2}) + \sigma_t \Psi(x, y, \mathbf{\Omega}_{m+M/2}) = Q(x, y) + \frac{\sigma_s}{4}\sum_{k=1}^{M/2} w_k [\Psi(x, y, \mathbf{\Omega}_k) + \Psi(x, y, \mathbf{\Omega}_{k+M/2})]. \tag{3}$$

Now, we integrate Eqs. (2) and (3), for all y, to obtain the one-dimensional nodal equations for the x variable

$$\mu_m \frac{d}{dx}\Psi_y(x,\mathbf{\Omega}_m) + \frac{\eta_m}{b}[\Psi(x,b,\mathbf{\Omega}_m) - \Psi(x,0,\mathbf{\Omega}_m)] + \sigma_t \Psi_y(x,\mathbf{\Omega}_m) = Q_y(x) + \frac{\sigma_s}{4}\sum_{k=1}^{M/2} w_k[\Psi_y(x,\mathbf{\Omega}_k) + \Psi_y(x,\mathbf{\Omega}_{k+M/2})] \tag{4}$$

and

$$-\mu_m \frac{d}{dx}\Psi_y(x,\mathbf{\Omega}_{m+M/2}) + \frac{\eta_{m+M/2}}{b}[\Psi(x,b,\mathbf{\Omega}_{m+M/2}) - \Psi(x,0,\mathbf{\Omega}_{m+M/2})] + \sigma_t \Psi_y(x,\mathbf{\Omega}_{m+M/2}) = Q_y(x) + \frac{\sigma_s}{4}\sum_{k=1}^{M/2} w_k[\Psi_y(x,\mathbf{\Omega}_k) + \Psi_y(x,\mathbf{\Omega}_{k+M/2})], \tag{5}$$

for $m = 1, \ldots, M/2$. Here we have defined the moments of the angular flux as

$$\Psi_y(x,\mathbf{\Omega}_m) = \frac{1}{b}\int_0^b \Psi(x,y,\mathbf{\Omega}_m)dy \tag{6}$$

and the integrated source as

$$Q_y(x) = \frac{1}{b}\int_0^b Q(x,y)dy. \tag{7}$$

If one considers the vacuum boundary conditions in the problem, then the neutron flux is zero in the incidence directions in $y = b$ and $y = 0$, in other words, the term $\Psi(x,b,\Omega_m)$ and $\Psi(x,0,\Omega_m)$ in Eqs. (4) and (5) in the incidence directions are equals to zero

$$\Psi(x,b,\mathbf{\Omega}_m) = 0 \qquad m = M/4+1,\ldots,M/2 \quad \text{and} \quad m = 3M/4+1,\ldots,M \tag{8}$$

and

$$\Psi(x,0,\mathbf{\Omega}_m) = 0 \qquad m = 1,\ldots,M/4 \quad \text{and} \quad m = M/2+1,\ldots,3M/4. \tag{9}$$

In regard to the usual unknown terms, fluxes at the boundaries, raised in the derivation of the nodal scheme, as the flow in emerging directions, we propose an approximation in the form

$$\Psi(x,0,\mathbf{\Omega}_m) \approx \widehat{k}_1 \Psi_y(x,\mathbf{\Omega}_m) \qquad m = M/4+1,\ldots,M/2 \quad \text{and} \quad m = 3M/4+1,\ldots,M \tag{10}$$

and

$$\Psi(x,b,\mathbf{\Omega}_m) \approx \widehat{k}_2 \Psi_y(x,\mathbf{\Omega}_m) \qquad m = 1,\ldots,M/4 \quad \text{and} \quad m = M/2+1,\ldots,3M/4, \tag{11}$$

in order to derive auxiliary conditions for solving the system. At this point, we leave the constants $\widehat{k}_1$ and $\widehat{k}_2$ arbitrary.

We now substitute Eqs. (8) and (9), as well as the approximations defined in Eqs. (10)-(11) into Eqs. (4) and (5), to obtain, the following set of one-dimensional ordinary differential equations in the x direction

$$\mu_m \frac{d}{dx}\Psi_y(x,\mathbf{\Omega}_m) + [\sigma_t + k_2\eta_m]\Psi_y(x,\mathbf{\Omega}_m) = Q_y(x) + \frac{\sigma_s}{4}\sum_{k=1}^{M/2} w_k[\Psi_y(x,\mathbf{\Omega}_k) + \Psi_y(x,\mathbf{\Omega}_{k+M/2})], \tag{12}$$

$$\mu_{m+M/4} \frac{d}{dx}\Psi_y(x,\mathbf{\Omega}_{m+M/4}) + [\sigma_t - k_1\eta_{m+M/4}]\Psi_y(x,\mathbf{\Omega}_{m+M/4}) = Q_y(x) + \frac{\sigma_s}{4}\sum_{k=1}^{M/2} w_k[\Psi_y(x,\mathbf{\Omega}_k) + \Psi_y(x,\mathbf{\Omega}_{k+M/2})], \tag{13}$$

$$-\mu_m \frac{d}{dx}\Psi_y(x,\mathbf{\Omega}_{m+M/2}) + [\sigma_t + k_2\eta_{m+M/2}]\Psi_y(x,\mathbf{\Omega}_{m+M/2}) = Q_y(x) + \frac{\sigma_s}{4}\sum_{k=1}^{M/2} w_k[\Psi_y(x,\mathbf{\Omega}_k) + \Psi_y(x,\mathbf{\Omega}_{k+M/2})], \tag{14}$$

and

$$-\mu_{m+M/4}\frac{d}{dx}\Psi_y(x,\mathbf{\Omega}_{m+3M/4}) + [\sigma_t - k_1\eta_{m+3M/4}]\Psi_y(x,\mathbf{\Omega}_{m+3M/4}) = Q_y(x) + \frac{\sigma_s}{4}\sum_{k=1}^{M/2} w_k[\Psi_y(x,\mathbf{\Omega}_k) + \Psi_y(x,\mathbf{\Omega}_{k+M/2})], \tag{15}$$

for $m = 1,\ldots,M/4$. Here, $k_1 = \widehat{k}_1/b$ and $k_2 = \widehat{k}_2/b$.

We note that, in the proposed scheme, the terms derived from the unknown fluxes at the boundary are not introduced as modifications on the source term of the original problem, as may be usual in nodal schemes (Azmy, 1988; Gomes and Barros, 2012; Mello and Barros, 2002). As a consequence, the problem derived for the x-direction remains uncoupled from the y-direction.

3. A DISCRETE-ORDINATES SOLUTION

Through the two-dimensional transport equation, we can generate a one-dimensional equations system, allowing the use of ADO method for its resolution. The homogeneous solution obtained by this method is constructed in terms of the eigenvalues and eigenfunctions (Barichello and Siewert, 1999a).

We proceed to solve the one-dimensional equations in the variable x as follows. We propose a homogeneous solution as

$$\Psi_y(x,\mathbf{\Omega}_m) = \Phi(\nu,\mathbf{\Omega}_m)e^{-x/\nu}, \tag{16}$$

and then substitute Eq. (16) into Eqs. (12)-(15) obtaining,

$$-\frac{\mu_m}{\nu}\Phi(\nu,\mathbf{\Omega}_m) + [\sigma_t + k_2\eta_m]\Phi(\nu,\mathbf{\Omega}_m) = \frac{\sigma_s}{4}\sum_{k=1}^{M/2} w_k[\Phi(\nu,\mathbf{\Omega}_k) + \Phi(\nu,\mathbf{\Omega}_{k+M/2})], \tag{17}$$

$$-\frac{\mu_{m+M/4}}{\nu}\Phi(\nu,\mathbf{\Omega}_{m+M/4}) + [\sigma_t - k_1\eta_{m+M/4}]\Phi(\nu,\mathbf{\Omega}_{m+M/4}) = \frac{\sigma_s}{4}\sum_{k=1}^{M/2} w_k[\Phi(\nu,\mathbf{\Omega}_k) + \Phi(\nu,\mathbf{\Omega}_{k+M/2})], \tag{18}$$

$$\frac{\mu_m}{\nu}\Phi(\nu,\mathbf{\Omega}_{m+M/2}) + [\sigma_t + k_2\eta_{m+M/2}]\Phi(\nu,\mathbf{\Omega}_{m+M/2}) = \frac{\sigma_s}{4}\sum_{k=1}^{M/2} w_k[\Phi(\nu,\mathbf{\Omega}_k) + \Phi(\nu,\mathbf{\Omega}_{k+M/2})], \tag{19}$$

and

$$\frac{\mu_{m+M/4}}{\nu}\Phi(\nu,\mathbf{\Omega}_{m+3M/4}) + [\sigma_t - k_1\eta_{m+3M/4}]\Phi(\nu,\mathbf{\Omega}_{m+3M/4}) = \frac{\sigma_s}{4}\sum_{k=1}^{M/2} w_k[\Phi(\nu,\mathbf{\Omega}_k) + \Phi(\nu,\mathbf{\Omega}_{k+M/2})], \tag{20}$$

for $m = 1,\ldots,M/4$.

Adding Eqs. (17) to (19) we obtain

$$\frac{\mu_m}{\nu}[\Phi(\nu,\mathbf{\Omega}_{m+M/2}) - \Phi(\nu,\mathbf{\Omega}_m)] + [\sigma_t + k_2\eta_m](\Phi(\nu,\mathbf{\Omega}_m) + \Phi(\nu,\mathbf{\Omega}_{m+M/2})) = \frac{\sigma_s}{2}\sum_{k=1}^{M/2} w_k[\Phi(\nu,\mathbf{\Omega}_k) + \Phi(\nu,\mathbf{\Omega}_{k+M/2})]. \tag{21}$$

Noting that (Barichello *et al.*, 2011)

$$\eta_m = \eta_{m+M/2}, \qquad m = 1,\ldots,M/2, \tag{22}$$

and defining

$$U(\nu,\mathbf{\Omega}_m) = \Phi(\nu,\mathbf{\Omega}_m) + \Phi(\nu,\mathbf{\Omega}_{m+M/2}) \tag{23}$$

and

$$V(\nu,\mathbf{\Omega}_m) = \Phi(\nu,\mathbf{\Omega}_m) - \Phi(\nu,\mathbf{\Omega}_{m+M/2}) \tag{24}$$

ISSN 2176-5480

we write

$$V(\nu,\mathbf{\Omega}_m)=\frac{\nu}{\mu_m}[\sigma_t+k_2\eta_m]U(\nu,\mathbf{\Omega}_m)-\frac{\sigma_s\nu}{2\mu_m}\sum_{k=1}^{M/2}w_kU(\nu,\mathbf{\Omega}_k). \tag{25}$$

We add Eqs. (18) to (20) to obtain

$$V(\nu,\mathbf{\Omega}_{m+M/4})=\frac{\nu}{\mu_{m+M/4}}[\sigma_t-k_1\eta_{m+M/4}]U(\nu,\mathbf{\Omega}_{m+M/4})-\frac{\sigma_s\nu}{2\mu_{m+M/4}}\sum_{k=1}^{M/2}w_kU(\nu,\mathbf{\Omega}_k). \tag{26}$$

Now, subtracting Eq. (17) from Eq. (19) and Eq. (18) from Eq. (20),

$$-\frac{\mu_m}{\nu}U(\nu,\mathbf{\Omega}_m)+(\sigma_t+k_2\eta_m)V(\nu,\mathbf{\Omega}_m)=0 \tag{27}$$

and

$$-\frac{\mu_{m+M/4}}{\nu}U(\nu,\mathbf{\Omega}_{m+M/4})+(\sigma_t-k_1\eta_{m+M/4})V(\nu,\mathbf{\Omega}_{m+M/4})=0. \tag{28}$$

Finally, substituting Eq. (25) into Eq. (27) and Eq. (26) into Eq. (28), for $m=1,\dots,M/4$, the following eigenvalue problem arises,

$$\frac{1}{\mu_m^2}[\sigma_t+k_2\eta_m]^2U(\nu,\mathbf{\Omega}_m)-\frac{\sigma_s}{2\mu_m^2}[\sigma_t+k_2\eta_m]\sum_{k=1}^{M/2}w_kU(\nu,\mathbf{\Omega}_k)=\lambda U(\nu,\mathbf{\Omega}_m) \tag{29}$$

and

$$\frac{1}{\mu_{m+M/4}^2}[\sigma_t-k_1\eta_{m+M/4}]^2U(\nu,\mathbf{\Omega}_{m+M/4})-\frac{\sigma_s}{2\mu_{m+M/4}^2}[\sigma_t-k_1\eta_{m+M/4}]\sum_{k=1}^{M/2}w_kU(\nu,\mathbf{\Omega}_k)=\lambda U(\nu,\mathbf{\Omega}_{m+M/4}) \tag{30}$$

with

$$\lambda=\frac{1}{\nu^2}. \tag{31}$$

We evaluate Eqs. (29) and (30) in the directions associated with $m=1,\dots,M/4$, such that, if we define the vector **U** with dimension $M/2\times 1$, which components are $U(\nu,\mathbf{\Omega}_m)$, we can write the eigenvalue problem in matrix form as

$$[\mathbf{D}-\mathbf{A}]\mathbf{U}=\lambda\mathbf{U}, \tag{32}$$

where **D** and **A** are matrices $M/2\times M/2$. In fact, we define **D** as

$$\mathbf{D}=diag\left\{\left(\frac{\sigma_t+k_2\eta_1}{\mu_1}\right)^2,\dots,\left(\frac{\sigma_t+k_2\eta_{M/4}}{\mu_{M/4}}\right)^2,\left(\frac{\sigma_t-k_1\eta_{M/4+1}}{\mu_{M/4+1}}\right)^2,\dots,\left(\frac{\sigma_t-k_1\eta_{M/2}}{\mu_{M/2}}\right)^2\right\} \tag{33}$$

and the entries of **A** are expressed, for $j=1,\dots,M/2$, as

$$a(i,j)=\frac{w_j\sigma_s[\sigma_t+k_2\eta_i]}{2\mu_i^2},\qquad \text{for}\qquad i=1,\dots,M/4 \tag{34}$$

$$\tag{35}$$

and

$$a(i,j)=\frac{w_j\sigma_s[\sigma_t-k_1\eta_i]}{2\mu_i^2},\qquad \text{for}\qquad i=M/4+1,\dots,M/2. \tag{36}$$

From the solution of the eigenvalue problem, Eq. (32), we obtain $\{\lambda_j,\mathbf{U}_j\}$ for $j=1,\dots,M/2$ such that, we find the separation constants ν_j, in Eq. (31). We use Eqs. (27) and (28) to define the function V, thus determining the eigenfunctions Φ from the expressions

$$\Phi(\nu_j,\mathbf{\Omega}_m)=\frac{U(\nu,\mathbf{\Omega}_m)+V(\nu,\mathbf{\Omega}_m)}{2} \tag{37}$$

and

$$\Phi(\nu_j,\mathbf{\Omega}_{m+M/2})=\frac{U(\nu,\Omega_m)-V(\nu,\mathbf{\Omega}_m)}{2}. \tag{38}$$

Since the separation constants occurs in pairs, $\pm\nu_j$, we write the homogeneous solution of Eq. (4) and (5), for $m = 1, \dots, M/2$ in the general form

$$\Psi_y^h(x, \mathbf{\Omega}_m) = \sum_{j=1}^{M/2} A_j \Phi(\nu_j, \mathbf{\Omega}_m) e^{-x/\nu_j} + B_j \Phi(\nu_j, \mathbf{\Omega}_{m+M/2}) e^{-(a-x)/\nu_j} \tag{39}$$

and

$$\Psi_y^h(x, \mathbf{\Omega}_{m+M/2}) = \sum_{j=1}^{M/2} A_j \Phi(\nu_j, \mathbf{\Omega}_{m+M/2}) e^{-x/\nu_j} + B_j \Phi(\nu_j, \mathbf{\Omega}_m) e^{-(a-x)/\nu_j}, \tag{40}$$

where the coefficients A_j and B_j are to be determined.

3.1 Particular solution

Since our problem has an inhomogeneous source term, a particular solution has to be defined. We consider the special definition of the source term for this specific problem

$$Q(x, y) = 1, \qquad x \in [0, a_s], \qquad y \in [0, b_s] \tag{41}$$

and zero otherwise (Loyalka and Tsai, 1975). We then return to Eq. (7), to obtain

$$Q_y(x) = \begin{cases} b_s/b, & x \in [0, a_s] \\ 0, & a_s < x \le a. \end{cases} \tag{42}$$

In this way, for $m = 1, \dots, M$, we seek a particular solution of the form

$$\Psi_y^p(x, \mathbf{\Omega}_m) = C_m. \tag{43}$$

If we substitute Eq. (43) into Eqs. (12) to (15) we find that C_m must satisfy the following $M \times M$ linear system for $x \in [0, a_s]$ (Barichello *et al.*, 2011),

$$K_m C_m - \frac{\sigma_s}{4} \sum_{k=1}^{M} w_k C_k = \frac{b_s}{b}, \tag{44}$$

with

$$K_m = \sigma_t + k_2 \eta_m \qquad m = 1, \dots, M/4 \quad \text{and} \quad m = M/2 + 1, \dots, 3M/4, \tag{45}$$

and

$$K_m = \sigma_t - k_1 \eta_m \qquad m = M/4 + 1, \dots, M/2 \quad \text{and} \quad m = 3M/4 + 1, \dots, M. \tag{46}$$

At this point we have the homogeneous and particular solutions for Eqs. (12)-(15) established. We are now able to determine the general solution in terms of the unknown arbitrary coefficients A_j and B_j. As the source term is defined for $x \in [0, a_s]$, we write the general solution, for $m = 1, \dots, M/2$, in this case as

$$\Psi_y(x, \mathbf{\Omega}_m) = \sum_{j=1}^{M/2} A_j \Phi(\nu_j, \mathbf{\Omega}_m) e^{-x/\nu_j} + B_j \Phi(\nu_j, \mathbf{\Omega}_{m+M/2}) e^{-(a_s-x)/\nu_j} + C_m, \tag{47}$$

$$\Psi_y(x, \mathbf{\Omega}_{m+M/2}) = \sum_{j=1}^{M/2} A_j \Phi(\nu_j, \mathbf{\Omega}_{m+M/2}) e^{-x/\nu_j} + B_j \Phi(\nu_j, \mathbf{\Omega}_m) e^{-(a_s-x)/\nu_j} + C_{m+M/2} \tag{48}$$

for $x \in (0, a_s)$ and

$$\Psi_y(x, \mathbf{\Omega}_m) = \sum_{j=1}^{M/2} C_j \Phi(\nu_j, \mathbf{\Omega}_m) e^{-(x-a_s)/\nu_j} + D_j \Phi(\nu_j, \mathbf{\Omega}_{m+M/2}) e^{-(a-x)/\nu_j}, \tag{49}$$

$$\Psi_y(x, \mathbf{\Omega}_{m+M/2}) = \sum_{j=1}^{M/2} C_j \Phi(\nu_j, \mathbf{\Omega}_{m+M/2}) e^{-(x-a_s)/\nu_j} + D_j \Phi(\nu_j, \mathbf{\Omega}_m) e^{-(a-x)/\nu_j} \tag{50}$$

ISSN 2176-5480

for $x \in (a_s, a)$.

To obtain the complete solution we will use boundary and interface conditions to define the arbitrary coefficients A_j, B_j, C_j and D_j, for $j = 1, \ldots, M/2$. We first consider the boundary conditions already written in nodal version as

$$\Psi_y(0, \mathbf{\Omega}_m) = 0 \qquad m = 1, \ldots, M/2 \tag{51}$$

and

$$\Psi_y(a, \mathbf{\Omega}_m) = 0 \qquad m = M/2 + 1, \ldots, M. \tag{52}$$

Thus, if we substitute Eqs. (47)-(50) into the boundary conditions Eqs. (51) and (52), and take into account the continuity interface conditions, we obtain, for $m = 1, \ldots, M/2$,

$$\sum_{j=1}^{M/2} A_j \Phi(\nu_j, \mathbf{\Omega}_m) + B_j \Phi(\nu_j, \mathbf{\Omega}_{m+M/2}) e^{-a_s/\nu_j} = -C_m, \tag{53}$$

$$\sum_{j=1}^{M/2} \Phi(\nu_j, \mathbf{\Omega}_m)[C_j - A_j e^{-a_s/\nu_j}] + \Phi(\nu_j, \mathbf{\Omega}_{m+M/2})[D_j e^{-(a-a_s)/\nu_j} - B_j] = C_m, \tag{54}$$

$$\sum_{j=1}^{M/2} \Phi(\nu_j, \mathbf{\Omega}_m)[D_j e^{-(a-a_s)/\nu_j} - B_j] + \Phi(\nu_j, \mathbf{\Omega}_{m+M/2})[C_j - A_j e^{-a_s/\nu_j}] = C_{m+M/2} \tag{55}$$

and

$$\sum_{j=1}^{M/2} C_j \Phi(\nu_j, \mathbf{\Omega}_{m+M/2}) e^{-(a-a_s)/\nu_j} + D_j \Phi(\nu_j, \mathbf{\Omega}_m) = 0, \tag{56}$$

which turns out to be a $2M \times 2M$ linear system. Solving this system we obtain the 2M coefficients of the homogeneous solution, therefore fully defining the solution. Once these are obtained, we find the general solution of the integrated problem in the x-direction as

$$\Psi_y(x, \mathbf{\Omega}_m) = \Psi_y^h(x, \mathbf{\Omega}_m) + \Psi_y^p(x, \mathbf{\Omega}_m), \qquad \text{for} \qquad m = 1, \ldots, M. \tag{57}$$

We remind the reader that the definition and solution of the problem formulated in the x-direction is decoupled from that in the y-direction. It follows that the formulation does not depend of in which direction it is made. To obtain a solution of the problem in terms of the y-variable, we follow similar steps as presented before.

4. NUMERICAL RESULTS AND COMPUTATIONAL ASPECTS

We consider here, a test case described by Loyalka and Tsai (1975), where the rectangular region is defined by $a = b = 1.0$, and an unitary source is located in the region $[0, 0.52] \times [0, 0.52]$. In that work, a method is proposed for solving the integral form of the neutron transport equation. Other parameters used in this case are $\sigma_t = 1.0cm^{-1}$ and $\sigma_s = 0.5cm^{-1}$, as described there.

We provide numerical results for the scalar flux evaluated as

$$\phi(x) = \frac{1}{4} \sum_{k=1}^{M/2} w_k [\Psi_y(x, \mathbf{\Omega}_k) + \Psi_y(x, \mathbf{\Omega}_{k+M/2})], \tag{58}$$

to be compared with those available in Loyalka and Tsai (1975), for $x = y = 0.5$, $x = y = 0.7$ and $x = y = 0.98$.

The results listed in Table 1 were generated for $N = 2$ to $N = 16$, with different values of $\widehat{k}_1$ and $\widehat{k}_2$. From the analysis of the results for the scalar flow shown in Table 1, it was observed that depending on where the scalar flux of neutrons is measured, different values of $\widehat{k}_1$ and $\widehat{k}_2$ should be taken, which is physically reasonable. In this work, the relative errors for the three cases discussed ($x = 0.5$; $x = 0.7$ and $x = 0.98$) not exceed 5% when compared to those listed in (Lathrop and Brinkley, 1973; Loyalka and Tsai, 1975). Moreover, it is possible to note that the relative errors between the results obtained by Loyalka (Loyalka and Tsai, 1975) and the code TWOTRAN-II (Lathrop and Brinkley, 1973) comes to 6%.

Table 1. Scalar flux $\phi(x, y)$, for $\sigma_s = 0.5cm^{-1}$.

$x = y$	Loyalka and Tsai (1975) N = 5,7,9,11,15	TWOTRAN-II (Loyalka and Tsai 1975) N = 4,8,16	This Work N = 2,4,6,8,12,16
			$\widehat{k}_1 = 0.7$ e $\widehat{k}_2 = 0.35$
0.5	0.231990	0.217527	0.188
	0.231219	0.215225	0.215
	0.230473	0.216245	0.221
	0.229927		0.223
	0.229296		0.225
			0.226
			$\widehat{k}_1 = 2.3$ e $\widehat{k}_2 = 1.8$
0.7	0.075402	0.077642	0.069
	0.066100	0.062200	0.068
	0.065768	0.063407	0.067
	0.065733		0.067
	0.064714		0.066
			0.066
			$\widehat{k}_1 = 4.1$ e $\widehat{k}_2 = 2.5$
0.98	0.022529	0.023717	0.013
	0.022294	0.023662	0.017
	0.022165	0.021915	0.019
	0.022108		0.020
	0.022084		0.021
			0.021

The results obtained in this work through the ADO method, keeps in general 1 to 2 digits of agreement as N increases, and the computational time to generate the results was $1 - 2$ seconds, using the Fortran language. Also, we note that same results are obtained for the scalar flux solving the problem in the y direction.

5. CONCLUDING REMARKS

We found that the method ADO along with the nodal equations are a good alternative in the approach of two-dimensional transport problems. The method proved to be efficient in the sense that was not used iterative schemes, in addition, the derived associated eigenvalue problem is of half-order, when compared with other discrete ordinates available approaches. Even that in this proposed approach the relation between the integrated flux and the fluxes at the boundaries is arbitrary, we have obtained results comparable to the literature at low computational cost, and further, this approach facilitates the determination of particular solutions. For future work, we wish to apply the method to neutron transport problems in heterogeneous media and we keep investigating appropriate relations between the average flows and fluxes at the boundary.

6. ACKNOWLEDGEMENTS

The authors would like to thank to CNPq and CAPES for financial support to this work.

7. REFERENCES

Al-Basheer, A.K., Sjoden, G.E. and Ghita, M., 2010. “Critical discretization issues in 3-D Sn simulations relevant to dosimetry and medical physics”. *Nuclear Technology*, Vol. 169, pp. 252–262.

Azmy, Y.Y., 1988. “Comparison of three approximations to the linear-linear nodal transport method in weighted diamond-difference form”. *Nuclear Science and Engineering*, Vol. 100, pp. 190–200.

Badruzzaman, A., 1985. “An efficient algorithm for nodal-transport solutions in multidimensional geometry”. *Nuclear Science and Engineering*, Vol. 89, pp. 281–290.

Barichello, L.B., Cabrera, L.C. and Filho, J.F.P., 2009. “An analytical discrete ordinates solution for two dimensional problems based on nodal schemes”. In *Proceedings of the International Nuclear Atlantic Conference - INAC2009*. Rio de Janeiro, Brazil.

Barichello, L.B., Cabrera, L.C. and Filho, J.F.P., 2011. “An analytical approach for a nodal scheme of two-dimensional neutron transport problems”. *Annals of Nuclear Energy*, Vol. 38, pp. 1310–1317.

Barichello, L.B. and Siewert, C.E., 1999a. “A discrete-ordinates solution for a non-grey model with complete frequency redistribution”. *Journal of Quantitative Spectroscopy and Radiative Transfer*, Vol. 62, pp. 665–675.

Barichello, L.B. and Siewert, C.E., 1999b. “A discrete-ordinates solution for a polarization model with complete frequency redistribution”. *The Astrophysical Journal*, Vol. 513, pp. 370–382.

Barichello, L.B. and Siewert, C.E., 2000. “The searchlight problem for radiative transfer in a finite slab”. *Journal of*

Computational Physics, Vol. 157, pp. 707–726.

Cabrera, L.C. and Barichello, L.B., 2006. "Unified solutions to some classical problems in rarefied gas dynamics based on the one-dimensional linearized S-model equations". *Zeitschrift für angewandte Mathematik und Physik*, Vol. 57, pp. 285–312.

Duderstadt, J.J. and Hamilton, L.J., 1976. *Nuclear reactor analysis*. John Wiley and Sons, New York.

Duo, J.I., Azmy, Y.Y. and Zikatanov, L.T., 2009. "A posteriori error estimator and AMR for discrete ordinates nodal transport methods". *Annals of Nuclear Energy*, Vol. 36, pp. 268–273.

Gomes, R.R. and Barros, R.C., 2012. "Computational modeling of monoenergetic neutral particle inverse transport problems in slab geometry". *American Institute of Physics*, Vol. 1479, pp. 2225–2228.

Hauser, E.B., 2002. *Study and solution of the steady-state two-dimensional transport equation by the LTSN method for high order angular quadrature sets: LTSN2D-Diag and LTSN2D-DiagExp*. Ph.D. thesis, Graduate Program in Mechanical Engineering - Universidade Federal do Rio Grande do Sul, Porto Alegre - Brazil.

Hu, B.X., Wu, Y.W., Tian, W.X., Su, G.H. and Qiu, S.Z., 2013. "Development of a transient thermal-hydraulic code for analysis of China demonstration fast reactor". *Annals of Nuclear Energy*, Vol. 55, pp. 302–311.

Lathrop, K.D. and Brinkley, F.W., 1973. *TWOTRAN-II, two dimensional multigroup discrete ordinates transport code*. Los Alamos Scientific Laboratory, LA-4848-MS.

Lewis, E.E. and Miller, W.F., 1984. *Computational Methods of Neutron Transport*. American Nuclear Society, IL, USA.

Loyalka, S.K. and Tsai, R.W., 1975. "A numerical method for solving the integral equation of neutron transport: II". *Nuclear Science and Engineering*, Vol. 58, pp. 193–202.

Mello, J.A.M. and Barros, R.C., 2002. "An exponential spectral nodal method for one-speed X,Y-geometry deep penetration discrete ordinates problems". *Annals of Nuclear Energy*, Vol. 29, pp. 1855–1869.

Scherer, C.S., Filho, J.F.P. and Barichello, L.B., 2009. "An analytical approach to the unified solution of kinetic equations in the rarefied gas dynamics. II. heat transfer problems". *Zeitschrift für angewandte Mathematik und Physik*, Vol. 60, pp. 651–687.

Siewert, C.E., 2004. "The temperature-jump problem based on the CES model of the linearized boltzmann equation". *Zeitschrift für angewandte Mathematik und Physik*, Vol. 55, pp. 92–104.

Stammes, K., Tsay, S.C., Wiscombe, W.J. and Jayaweera, K., 1988. "Numerical stable algorithm for discrete-ordinates method radiative transfer in multiple scattering and emitting layered media". *Applied Optics*, Vol. 27, pp. 2502–2509.

Su'ud, Z., 2008. "Safety performance comparation of MOX, nitride and metallic fuel based 25-100 MWe Pb-Bi cooled long life fast reactors without on-site refuelling". *Progress in Nuclear Energy*, Vol. 50, pp. 157–162.

Vilhena, J.R.Z.M.T. and Barichello, L.B., 1997. "An analytical solution for the two-dimensional discrete ordinate problem in a convex domain". *Progress in Nuclear Energy*, Vol. 31, pp. 225–228.

Wagner, J.C. and Haghighat, A., 1998. "Automated variance reduction of Monte Carlo shielding calculations using the discrete ordinates adjoint function". *Nuclear Science Engineering*, Vol. 128, pp. 186–208.

Williams, M.M.R., 2007. "Radiation transport in a light duct using a one-dimensional model". *Physica Scripta*, Vol. 76, pp. 303–313.

Zhang, Z., Rahnema, F., Zhang, D., Pounders, J.M. and Ougouag, A.M., 2011. "Simplified two and three dimensional HTTR benchmark problems". *Annals of Nuclear Energy*, Vol. 38, pp. 1172–1185.

8. RESPONSIBILITY NOTICE

The authors are the only responsible for the printed material included in this paper.