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SUN PERTURBATIONS ON OPTIMAL TRAJECTORIES FOR EARTH - MOON FLIGHT

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Abstract. *In this work, the problem of transferring a space vehicle from a circular low Earth orbit (LEO) to a circular low Moon orbit (LMO) with minimum fuel consumption is presented. The optimization criterion is the total characteristic velocity. The optimization problem has been formulated using the classic planar circular restricted three-body problem (PCR3BP) and the planar bi-circular restricted four-body problem (PBR4BP). In both cases, the optimization problem has been solved using a gradient algorithm in conjunction with Newton-Raphson method. Numerical results are obtained for several final altitudes of a clockwise or counterclockwise circular low Moon orbit for a specified altitude of a counterclockwise circular low Earth orbit*

Keywords: *Earth-Moon trajectories, optimal trajectories, Sun perturbations*

1. INTRODUCTION

In the last decades, new types of trajectories have been proposed to transfer a spacecraft from an orbit around Earth to an orbit around Moon, which reduce the cost of the traditional transfers based on the two-body dynamics (Chobotov, 2005). The new trajectories are designed using more realistic models of the motion of the spacecraft such as the planar circular restricted three-body problem or the planar bi-circular restricted four-body problem (Belbruno, 2004; Koon et al, 2007). In these models, the motion of the spacecraft exhibits very complex dynamics that are used to design new Earth-to-Moon trajectories (Conley, 1968; Belbruno et al, 2010). New trajectories with large time of flight (about 80 to 150 days) are calculated using the concept of weak stability boundary introduced by Belbruno (2004). These new trajectories are usually referred as low energy transfers (Conley, 1968; Koon et al, 2001; Belbruno, 2004).

Low energy Earth-Moon transfers can be classified into exterior or interior, according to the geometry (Topputo, 2013). In the exterior transfers the spacecraft is injected into an orbit with large apogee which crosses the Moon orbit. The apogee distance is approximately four times the Earth-Moon distance. This kind of trajectories exploits the Sun's gravitational attraction (Yamakawa et al, 1992, 1993). In the interior transfers most part of the trajectory occurs within the Moon orbit. Although the new approaches reduce the cost of the mission, only few works consider the problem of minimizing the total cost (Yagasaki, 2004a,b; Da Silva Fernandes and Marinho, 2012; Topputo, 2013).

In this work, a preliminary analysis about the perturbation of the Sun on the problem of transferring a space vehicle from a circular low Earth orbit (LEO) to a circular low Moon orbit (LMO) with minimum fuel consumption is presented. It is assumed that the velocity changes are instantaneous, that is, the propulsion system is capable of delivering impulses. Trajectories with two impulses are considered in the analysis: a first accelerating velocity impulse tangential to the space vehicle velocity relative to Earth is applied at a circular low Earth orbit and a second braking velocity impulse tangential to the space vehicle velocity relative to Moon is applied at a circular low Moon orbit (Miele and Mancuso, 2001). The minimization of fuel consumption is equivalent to the minimization of the total characteristic velocity which is defined by the arithmetic sum of velocity changes (Marec, 1979). The optimization problem has been formulated using the classic planar circular restricted three-body problem (PCR3BP) and the planar bi-circular restricted four-body problem (PBR4BP). Numerical results are obtained for several final altitudes of a clockwise or counterclockwise circular low Moon orbit for a specified altitude of a counterclockwise circular low Earth orbit. Direct ascent trajectories, with time of flight of approximately 4.5 days, and multiple revolution trajectories, with time of flight of approximately 41.0 days, are considered in this study. The results for mission with multiple revolutions show that fuel can be saved if a lunar swing-by occurs.

2. OPTIMIZATION PROBLEM BASED ON THE PCR3BP

In this section, the optimization problem based on the PCR3BP is formulated. The following assumptions are employed:

1. Earth and Moon move around the center of mass of the Earth-Moon system;
2. The eccentricity of the Moon orbit around Earth is neglected;
3. The flight of the space vehicle takes place in the Moon orbital plane;
4. The space vehicle is subject to only the gravitational fields of Earth and Moon;
5. The gravitational fields of Earth and Moon are central and obey the inverse square law;

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6. The class of two impulse trajectories is considered. The impulses are applied tangentially to the space vehicle velocity relative to Earth (first impulse) and Moon (second impulse).

Consider an inertial reference frame G*xy* contained in the Moon orbital plane: its origin is the barycenter of Earth-Moon system; the *x*-axis points towards the Moon position at the initial time $t_0 = 0$ and the *y*-axis is perpendicular to the *x*-axis. In this reference frame, the motion of the space vehicle is described by the following set of differential equations:

$$\frac{dx_P}{dt} = u_P \qquad \frac{du_P}{dt} = -\frac{\mu_E}{r_{EP}^3}(x_P - x_E) - \frac{\mu_M}{r_{MP}^3}(x_P - x_M)$$
$$\frac{dy_P}{dt} = v_P \qquad \frac{dv_P}{dt} = -\frac{\mu_E}{r_{EP}^3}(y_P - y_E) - \frac{\mu_M}{r_{MP}^3}(y_P - y_M) \tag{1}$$

where μ_E is the Earth gravitational parameter, μ_M is the Moon gravitational parameter, r_{EP} and r_{MP} are, respectively, the distances of the space vehicle from Earth (*E*) and Moon (*M*); that is, $r_{EP}^2 = (x_P - x_E)^2 + (y_P - y_E)^2$ and $r_{MP}^2 = (x_P - x_M)^2 + (y_P - y_M)^2$. The position vectors of Earth and Moon are defined in the reference frame G*xy* by the equations

$$x_M(t) = \frac{D}{1+\mu}\cos\theta_M(t) \qquad y_M(t) = \frac{D}{1+\mu}\sin\theta_M(t), \tag{2}$$

$$x_E(t) = -\mu x_M(t) \qquad y_E(t) = -\mu y_M(t), \tag{3}$$

where $\theta_M(t) = \sqrt{(\mu_E + \mu_M)/D^3}\, t$, $\mu = \mu_M / \mu_E$ and *D* is the distance from the Earth to the Moon.

The initial conditions of the system of differential equations (1) correspond to the position and velocity vectors of the space vehicle after the application of the first impulse. The initial conditions $(t_0 = 0)$ can be written as follows

$$x_P(0) = x_E(0) + r_{EP}(0)\cos\theta_{EP}(0) \qquad u_P(0) = -\left[\sqrt{\frac{\mu_E}{r_{EP}(0)}} + \Delta v_{LEO}\right]\sin\theta_{EP}(0) + \dot{x}_E(0)$$

$$y_P(0) = y_E(0) + r_{EP}(0)\sin\theta_{EP}(0) \qquad v_P(0) = \left[\sqrt{\frac{\mu_E}{r_{EP}(0)}} + \Delta v_{LEO}\right]\cos\theta_{EP}(0) + \dot{y}_E(0), \tag{4}$$

where Δv_{LEO} is the velocity change at the first impulse, $r_{EP}(0) = h_0 + a_E$ and $\theta_{EP}(t)$ is the angle which the position vector $\boldsymbol{r}_{EP}$ forms with *x*-axis. h_0 is the altitude of LEO and a_E is the Earth radius. It should be noted that $\boldsymbol{r}_{EP}(0)$ and $\boldsymbol{v}_{EP}(0)$ are orthogonal (impulse is applied tangentially to LEO). From Eqs (2) and (3), one finds

$$x_E(0) = -\mu\frac{D}{1+\mu} \qquad y_E(0) = 0 \qquad \dot{x}_E(0) = 0 \qquad \dot{y}_E(0) = -\mu\frac{D}{1+\mu}\dot{\theta}_M\,. \tag{5}$$

The final conditions of the system of differential equations (1) correspond to the position and velocity vectors of the space vehicle before the application of the second impulse. The final conditions $(t_f = T)$ can be put in the form (da Silva Fernandes and Marinho, 2012),

$$(x_P(T) - x_M(T))^2 + (y_P(T) - y_M(T))^2 = (r_{MP}(T))^2, \tag{6}$$

$$(u_P(T) - \dot{x}_M(T))^2 + (v_P(T) - \dot{y}_M(T))^2 = \left[\sqrt{\frac{\mu_M}{r_{MP}(T)}} + \Delta v_{LMO}\right]^2, \tag{7}$$

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$$\left(x_P(T)-x_M(T)\right)\left(v_P(T)-\dot{y}_M(T)\right)-\left(y_P(T)-y_M(T)\right)\left(u_P(T)-\dot{x}_M(T)\right)=\mp r_{MP}(T)\left[\sqrt{\frac{\mu_M}{r_{MP}(T)}}+\Delta v_{LMO}\right]. \quad (8)$$

where Δv_{LMO} is the velocity change at the second impulse, $r_{MP}(T)=a_M+h_f$, h_f is the altitude of LMO and a_M is the Moon radius. The upper sign refers to clockwise arrival to LMO and the lower sign refers to counterclockwise arrival to LMO. From Eq. (2), one finds

$$x_M(T)=\frac{D}{1+\mu}\cos\theta_M(T) \qquad y_M(T)=\frac{D}{1+\mu}\sin\theta_M(T)$$

$$\dot{x}_M(T)=-\frac{D\dot{\theta}_M}{1+\mu}\sin\theta_M(T) \qquad \dot{y}_M(T)=\frac{D\dot{\theta}_M}{1+\mu}\cos\theta_M(T). \quad (9)$$

The problem defined by Eqs (1) – (9) involves four unknowns Δv_{LEO}, Δv_{LMO}, T and $\theta_{EP}(0)$ that must be determined in order to satisfy the three final conditions. Since this problem has one degree of freedom, an optimization problem can be formulated as follows: Determine Δv_{LEO}, Δv_{LMO}, T and $\theta_{EP}(0)$ which satisfy the final constraints (6) – (8) and minimize the total characteristic velocity $\Delta v_{Total}=\Delta v_{LEO}+\Delta v_{LMO}$. This problem has been solved by da Silva Fernandes and Marinho (2012) using an algorithm based on gradient method (Miele et al, 1969) in conjunction with Newton-Raphson method (Stoer and Bulirsch, 2002).

3. OPTIMIZATION PROBLEM BASED ON THE PBR4BP

In this section, the optimization problem based on the PBR4BP is formulated. The following assumptions are employed:

1. Earth and Moon move in circular orbits around the center of mass of the Earth-Moon system;
2. Earth-Moon system barycenter moves in circular orbit around the center of mass of the Sun-Earth-Moon system;
3. The flight of the space vehicle takes place in the Moon orbital plane;
4. The space vehicle is subject to the gravitational fields of Earth, Moon and Sun;
5. The gravitational fields of Earth, Moon and Sun are central and obey the inverse square law;
6. The class of two impulse trajectories is considered. The impulses are applied tangentially to the space vehicle velocity relative to Earth (first impulse) and Moon (second impulse).

Consider a moving reference frame G*xy* contained in the Moon orbital plane: its origin is the barycenter of Earth-Moon system; the *x*-axis points towards the Moon position at the initial time $t_0=0$ and the *y*-axis is perpendicular to the *x*-axis. In this reference frame, the motion of the space vehicle is described by the following set of differential equations:

$$\frac{dx_P}{dt}=u_P \qquad \frac{du_P}{dt}=-\frac{\mu_S}{r_{SP}^3}(x_P-x_S)-\frac{\mu_E}{r_{EP}^3}(x_P-x_E)-\frac{\mu_M}{r_{MP}^3}(x_P-x_M)-\frac{\mu_S}{r_S^2}\cos(\omega_S t+\theta_{S0})$$

$$\frac{dy_P}{dt}=v_P \qquad \frac{dv_P}{dt}=-\frac{\mu_S}{r_{SP}^3}(y_P-y_S)-\frac{\mu_E}{r_{EP}^3}(y_P-y_E)-\frac{\mu_M}{r_{MP}^3}(y_P-y_M)-\frac{\mu_S}{r_S^2}\sin(\omega_S t+\theta_{S0}), \quad (10)$$

where μ_E, μ_M, r_{EP} and r_{MP} are the same ones defined in the preceding section, μ_S is the Sun gravitational parameter, r_{SP} is the distance of the space vehicle from the Sun $r_{SP}^2=(x_P-x_S)^2+(y_P-y_S)^2$. The distance from *G* to Earth, Moon and Sun are denoted by r_E, r_M and r_S, respectively. So, the position vectors of Earth, Moon and Sun are defined in the reference frame G*xy* by the equations

$$x_E(t)=-r_E\cos\theta_E(t) \qquad y_E(t)=-r_E\sin\theta_E(t), \quad (11)$$

$$x_M(t)=r_M\cos\theta_M(t) \qquad y_M(t)=r_M\sin\theta_M(t), \quad (12)$$

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$$x_S(t) = r_S \cos\theta_S(t) \qquad\qquad y_S(t) = r_S \sin\theta_M(t), \tag{13}$$

where $\theta_E(t) = \theta_M(t) = \theta_0 + \dot{\theta}_M t$, and $\theta_S(t) = \theta_{S0} + \sqrt{\mu_S / r_S^3}\, t$, $\theta_0 = 0$ is the initial phase of the Moon, θ_{S0} is the initial phase of the Sun, $r_E = \mu D/(1+\mu)$ and $r_M = D/(1+\mu)$. Note that Eq. (1) can be obtained directly from Eq. (10) by taking $\mu_S = 0$.

The initial conditions of the system of differential equations (10) correspond to the position and velocity vectors of the space vehicle after the application of the first impulse and are given by Eqs (4) and (5). The final conditions correspond to the position and velocity vectors of the space vehicle before the application of the second impulse and are given by Eqs (6) – (9). The problem defined by Eqs (6) – (9) and (10) involves five unknowns Δv_{LEO}, Δv_{LMO}, T, $\theta_{EP}(0)$ and θ_{S0} that must be determined in order to satisfy the three final conditions. This problem has two degrees of freedom, so an optimization problem can be formulated as follows: Determine Δv_{LEO}, Δv_{LMO}, T, $\theta_{EP}(0)$ and θ_{S0} which satisfy the final constraints (6) – (9) and minimize the total characteristic velocity $\Delta v_{Total} = \Delta v_{LEO} + \Delta v_{LMO}$. This optimization problem has been solved using the same algorithm described in the preceding section.

4. RESULTS

In this section, results are presented for lunar missions using the optimization problems described above. The following data are used:

$\mu_S = 1.327\times10^{11}\ \text{km}^3/\text{s}^2$ $\qquad$ $\mu_E = 3.986\times10^{5}\ \text{km}^3/\text{s}^2$
$\mu_M = 4.903\times10^{3}\ \text{km}^3/\text{s}^2$ $\qquad$ $r_S = 1.496\times10^{8}$ km
$r_E = 4.678\times10^{3}$ km $\qquad$ $r_M = 3.803\times10^{5}$ km
$D = 384400$ km (distance from the Earth to the Moon),
$a_E = 6378$ km (Earth radius) $\qquad$ $a_M = 1738$ km (Moon radius)
$h_0 = 167$ km $\qquad$ $h_f = 100, 200, 300$ km.

Tables 1 and 2 show the major parameters for the lunar missions involving direct ascent trajectories with time of flight of approximately 4.5 days, considering clockwise or counterclockwise arrival at the Moon. Tables 3 and 4 show similar results for trajectories with time of flight of approximately 41.0 days. Figures 1 and 2 depict a maneuver with three revolutions for the two dynamical models, PCR3BP and PBR4BP, respectively, considering counterclockwise arrival at the Moon.

Table 1 – Lunar missions, major parameters - $h_{LEO} = 167\ km$

Model	***Maneuver***	***LMO altitude*** *km*	Δv_{Total} *km/s*	Δv_{LEO} *km/s*	Δv_{LMO} *km/s*	T *days*	$\theta_{EP}(0)$ *degree*
PCR3BP	*Clockwise*	100	3.9570	3.1413	0.8157	4.762	–113.84
		200	3.9429	3.1413	0.8016	4.766	–113.81
		300	3.9300	3.1413	0.7887	4.771	–113.77
	Counterclockwise	100	3.9519	3.1386	0.8133	4.571	–116.47
		200	3.9378	3.1385	0.7991	4.569	–116.51
		300	3.9245	3.1385	0.7860	4.567	–116.55

From the results presented in Tables 1 and 2, and, Figures 1 and 2, the major comments are:

1. Sun perturbation effects are too small for direct ascent trajectories.

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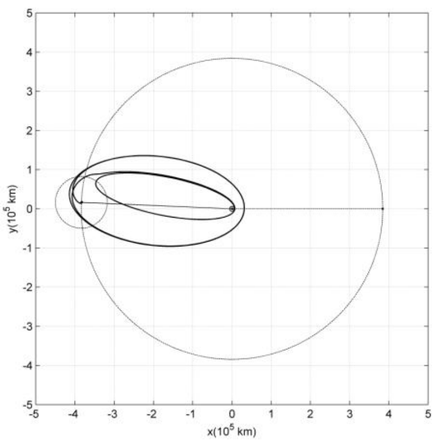
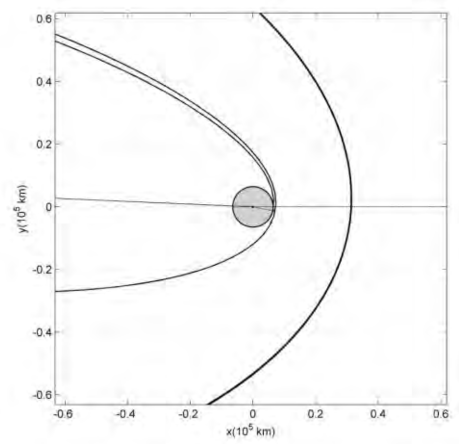
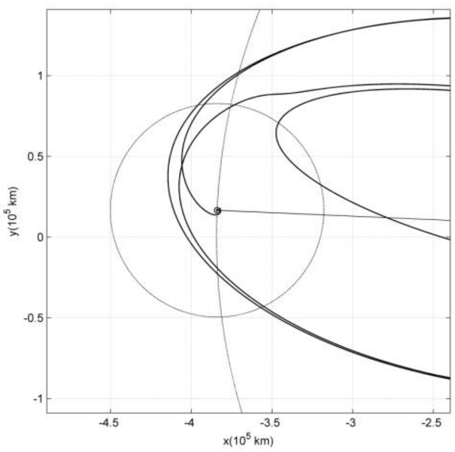
2. The differences in Δv_{LMO} between the two models are of order of 2 m/s for direct ascent trajectories.
3. The optimum initial position of the Sun is almost the same for direct ascent trajectories regardless the altitude of LMO. For the trajectories with clockwise arrival at the Moon, θ_{s0} is approximately 97 degree. For the trajectories with counterclockwise arrival at the Moon, θ_{s0} is approximately 95.4 degree.
4. Sun perturbation effects are significant for trajectories with three revolutions. Fuel consumption can vary significantly according the initial position of the Sun.
5. A swing-by maneuver with the Moon is made in the trajectories with three revolutions, for the both dynamical models.
6. The velocity increment at the second impulse - Δv_{LMO} - is significantly affected by the presence of the Sun for trajectories with three revolutions.



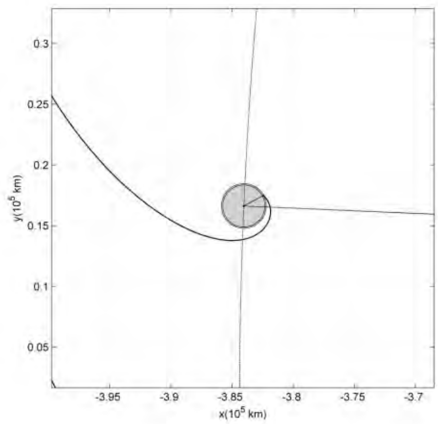
(a) Earth-Moon trajectory



(b) LEO departure



(c) Swing-by



(d) LMO arrival

Figure 1 – Trajectory with three revolutions – PCR3BP

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Table 2 – Lunar missions, major parameters - $h_{LEO} = 167\ km$

Model	*Maneuver*	*LMO altitude km*	Δv_{Total} *km/s*	Δv_{LEO} *km/s*	Δv_{LMO} *km/s*	*T days*	$\theta_{EP}(0)$ *degree*	$\theta_S(0)$ *degree*
PBR4BP	*Clockwise*	100	3.9547	3.1410	0.8137	4.795	−113.65	96.84
		200	3.9406	3.1411	0.7995	4.799	−113.62	96.98
		300	3.9277	3.1412	0.7865	4.804	−113.57	97.13
	Counterclockwise	100	3.9498	3.1384	0.8114	4.597	−116.33	95.28
		200	3.9354	3.1383	0.7971	4.595	−116.37	95.39
		300	3.9223	3.1383	0.7840	4.594	−116.41	95.60

Table 3 – Lunar missions, major parameters - $h_{LEO} = 167\ km$

Model	*Maneuver*	*LMO altitude km*	Δv_{Total} *km/s*	Δv_{LEO} *km/s*	Δv_{LMO} *km/s*	*T days*	$\theta_{EP}(0)$ *degree*
PCR3BP	*Clockwise*	100	3.9036	3.1315	0.7720	40.66	348.64
	Counterclockwise		3.9127	3.1314	0.7812	40.70	348.35

Table 4 – Lunar missions, major parameters - $h_{LEO} = 167\ km$

Model	*Maneuver*	*LMO altitude km*	Δv_{Total} *km/s*	Δv_{LEO} *km/s*	Δv_{LMO} *km/s*	*T days*	$\theta_{EP}(0)$ *degree*	$\theta_S(0)$ *degree*
PBR4BP	*Clockwise*	100	3.8876	3.1312	0.7564	40.71	348.59	0.0
			3.8923	3.1317	0.7606	40.81	352.43	45.0
			3.9202	3.1317	0.7885	40.78	350.70	90.0
	Counterclockwise		3.8964	3.1311	0.7654	40.77	348.16	0.0
			3.9017	3.1316	0.7701	40.86	351.93	45.0
			3.9296	3.1316	0.7980	40.78	350.26	90.0

5. CONCLUSION

In this work, a preliminary study about the perturbation of the Sun on optimal trajectories for Earth-Moon flight of a space vehicle is presented. The optimization problem has been formulated using the classic planar circular restricted three-body problem (PCR3BP) and the planar bi-circular restricted four-body problem (PBR4BP). Results presented for some lunar missions with time of flight of approximately 4.5 days show that the presence of the Sun causes small perturbations in the main parameters defining the optimal solutions and some fuel can be saved if the duration of the transfer becomes larger (approximately 41.0 days).

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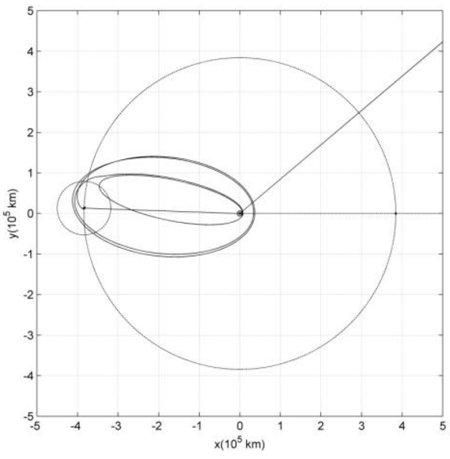
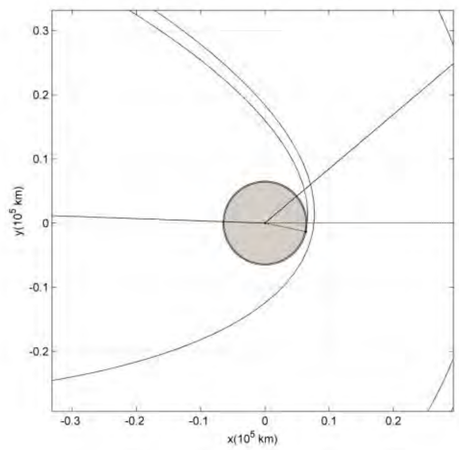
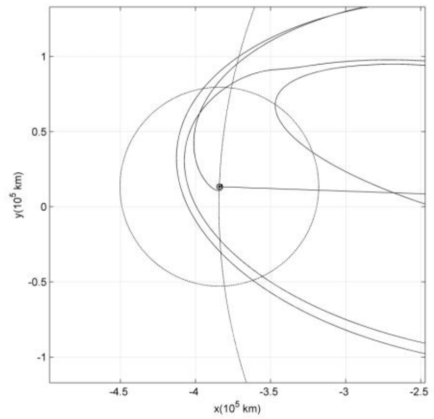


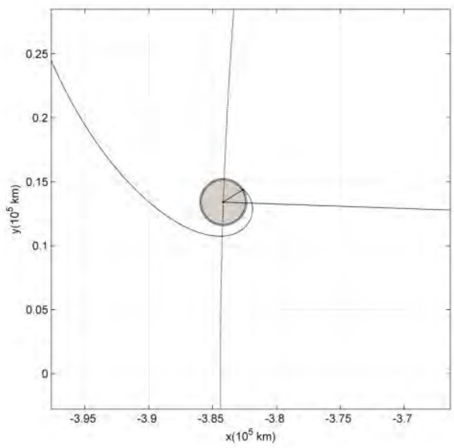
(c) Earth-Moon trajectory



(d) LEO departure



(c) Swing-by



(d) LMO arrival

Figure 2 – Trajectory with three revolutions – PBR4BP – $\theta_S(0) = 0°$

6. ACKNOWLEDGEMENTS

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