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GENERALIZED FINITE ELEMENT METHOD TO APPROACH FORCED AND FREE VIBRATION IN ELASTIC 2D PROBLEMS

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Abstract. *The Generalized Finite Element Method (GFEM) is a methodology that has shown excellent results in addressing linear and nonlinear problems in solid mechanics and thermal diffusion fields. However, in its original form in which the stiffness and mass matrices are singular, due to the use of linearly dependent modes to obtain the shape functions, its application became less efficient to approach the eigenvalues-eigenvectors problems. In this paper, we present an alternative to approach the free and forced vibrations in elastic bi-dimensional problems using the high regularity GFEM to avoid the singularity of stiffness and mass matrices. In this work the approximation space is obtained from explicit enrichment of the partition of unity (PU) of high regularity with complete polynomials functions. The PU functions in 2D are obtained by tensorial product of rational polynomials PU functions of high regularity in 1D. Some examples are presented where we investigate the influence of the regularity of the approximation spaces in obtaining relatively high frequencies (up to ten percent of frequencies). In the forced vibration problems the modal superposition method is employed together with Newmark method for time integration. All results are obtained for rectangular domain under isotropic linear elastic assumption. In addition, the results are compared with those obtained by high order FEM spaces.*

Keywords: *Forced vibration, elastic media, free vibration, GFEM.*

1. INTRODUCTION

The problem of propagation of mechanical waves in solid media has gained significant importance in recent decades in the aerospace and naval areas. The simulation accuracy of mechanical wave propagation problems produced by an impulsive force requires that one can predict a high number of frequencies and natural modes with good precision. Accordingly, the Finite Element Method (FEM), widely used by the industry and researches, has shown limitations in the determination of modes and natural frequencies, specially the relatively high frequencies (more than ten percent of frequencies approximated numerically). The limitations of the FEM in the approach of elliptical eigenvalue/eingenvectors problems have been widely discussed in the literature, see for instance Hughes (1987), Cottrell *et al.* (2007a, 2007b) and Givoli (2008). This limitation is a consequence of the low regularity and higher order of FEM approximation spaces and is described by an *a priori* error estimator in terms of eigenvalues for the elliptic problems (see: Hughes, (1987)).

While studying undamped natural frequencies of a road Conttrell *et al.* (2007a) note the presence of jumps in the spectrum diagram of natural frequencies (acoustic and optics branches), resulting from the low order and high regularity space built using Lagrangian elements. Although the observations have been concerned on the FEM the limitation, such kind of response is some kind of normal to other low regularity approximation spaces obtained by other methods, as described in Linzmayer *et al.* (2011). Alternatives to obtain better results for undamped free vibrations problems have been presented by unconventional numerical methods presented by Gu *et al.* (2001) in addressing free and forced vibration problems in plane elasticity. Ferreira *et al.* (2005) analyses the problem of free vibrations in laminated

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composite plates by using a multi-quadric radial basis function (MQRBF). Liew *et al.* (2003), addresses the problem of free vibrations in composite plates with kinematics defined by first shear deformation theory (FSDT). Chen *et al.* (2003), uses the radial basis function method in addressing the problem of free vibrations in the frequency domain of circular plates. Liu *et al.* (2001) employs an approximation space based on the Moving Least Square Method (MLSM) in the approach of elastostatic and free vibration problems in thin plates of complicated shapes. Liu *et al.* (2002) uses the Element-Free Galerkin Method (EFGM) to approach the static and natural frequencies problems of thin shells. In the works cited above the authors show the performance of different methods in addressing eigenvalues/eigenvectors of elliptic problems, but with incipient results involving at most the first ten percent of modes and natural frequencies of the spectrum. Recently, some research has presented alternatives to improve the ability to obtain numerical modes for relatively high frequencies with acceptable accuracy for undamped free vibrations problems. The work presented by Cottrell *et al.* (2007a-2007b) employs the so called *k*-method to build approximation spaces of desired regularity and order. The free vibrations results for a rod problem are very accurate when compared with the analytical solution. In the same way Garcia and Rossi (2012) advocate the use of the Generalized Finite Element Method (GFEM) to obtain the natural frequency associated with axis symmetric modes for thick plates and shells of revolution.

In the present work high order and high regularity approximation spaces are build by the GFEM to address the forced and free vibration problems under plane elasticity assumption. The high regularity and high order approximation spaces are build by explicit enrichment using partition of unity (PU) polynomials with regularity C^2 and C^4. The 2D high regularity PU is build by tensor product of 1D high regularity PU obtained by rational polynomial functions. This work is presented in six sections as follows: introduction, approximation space by GFEM, dynamic approach of 2D elastic problems; numerical results; conclusions, and references.

2. ENRICHED PU APPROXIMATION SPACE

Enrichment of approximation spaces with PU properties has been studied by several authors over the last fifteen years. Different names are used to such methods, for instance, one can find the Generalized Finite Element Method (GFEM) proposed by Duarte *et al.* (2000); eXtended Finite Element Method (XFEM) proposed by Merle and Dolbow, (2002), Element Free Galerkin Method (EFG) proposed by Belytschko *et al.* (1994). These methods build the approximation space using extrinsic enrichment of the PU functions.

The enrichment procedure employed in this work consists of the multiplication of a rational polynomial based PU shape function, defined on a nodal position of the element of the integration mesh, by a set of complete monomials of order *p*. The nodes to be enriched can be either selectively selected, by means of an error estimator, or simply homogeneously selected.

The enriched approximation space is composed of all possible linear combinations of a finite dimension space generated by the product of functions ϕ_α, which defines the PU, by a set of functions Q_α^p. Here, α is the node number. Some important definitions are presented in order to aid the presentation of the global approximation space.

2.1 Partition of Unity of regularity $C^k(\Omega),\ k = 0,2,4,...$

In this work, for construction of the approximation space are used the set functions $\{\phi_\alpha\}_{\alpha\in\Upsilon}$ with Υ a set index functions, which represent a partition of unity (PU) subordinate to an open cover $\Omega \subseteq \bigcup \omega_\alpha$ such that $\exists M \in \mathbb{N}\ \ \forall \boldsymbol{x} \in \Omega\ \text{card}\{\alpha \mid x \in \omega_\alpha\} \le M$, thus a partition of unity, of *Lipschitz* type, has the following properties according Melenk and Babuska (1996):

$$\text{supp}(\phi_\alpha) \subset \omega_\alpha,\ \ \forall \alpha\,; \tag{1}$$

$$\sum_{\alpha\in\Upsilon} \phi_\alpha(x) = 1,\ \ \forall \boldsymbol{x} \in \Omega\,; \tag{2}$$

$$\|\phi_\alpha\|_{L^\infty(\mathbb{R}^n)} \le C_\infty; \tag{3}$$

$$\|\nabla \phi_\alpha\|_{L^\infty(\mathbb{R}^n)} \le \frac{C_G}{diam(\Omega_\alpha)}; \tag{4}$$

where $\|\cdot\|_{L^\infty(\mathbb{R}^n)}$ is the infinite norm, C_∞ and C_G are constants.

In this work the high regularity PU functions ($C^k(\Omega),\ k = 0,2,4,...$**),** are built by tensor product of PU constructed in 1D domain as shown in Garcia and Rossi (2012). They are

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$$\mathbf{P} = \mathbf{\Phi}_{\xi} \otimes \mathbf{\Phi}_{\eta} ; \tag{5}$$

In Eq. (5)

$$\mathbf{\Phi}_{\xi} = \lfloor \varphi_1(\xi) \quad \varphi_2(\xi) \rfloor ; \tag{6}$$

$$\mathbf{\Phi}_{\eta} = \lfloor \varphi_1(\eta) \quad \varphi_2(\eta) \rfloor ; \tag{7}$$

In Eq. (6)-(7), $\{\varphi_i(\xi)\}_{i=1}^{2}$ and $\{\varphi_i(\eta)\}_{i=1}^{2}$ are the PU functions on 1D domain defined in Garcia and Rossi (2012). $\mathbf{P}$ is a matrix defined as:

$$\mathbf{P} = \begin{bmatrix} \varphi_1(\xi,\eta) & \varphi_2(\xi,\eta) \\ \varphi_4(\xi,\eta) & \varphi_3(\xi,\eta) \end{bmatrix} ; \tag{8}$$

In Eq. (8) the PU are $C^2(\Omega_e)$ functions and are defined by Eq. (9)-(12).

$$\varphi_1(\xi,\eta) = ((\xi - 1)^2 (\xi + 3)^2 (\eta - 1)^2 (\eta + 3)^2)/(4(\xi^4 - 2\xi^2 + 9)(\eta^4 - 2\eta^2 + 9)) ; \tag{9}$$

$$\varphi_2(\xi,\eta) = ((\xi + 1)^2 (\xi - 3)^2 (\eta - 1)^2 (\eta + 3)^2)/(4(\xi^4 - 2\xi^2 + 9)(\eta^4 - 2\eta^2 + 9)) ; \tag{10}$$

$$\varphi_3(\xi,\eta) = ((\xi + 1)^2 (\xi - 3)^2 (\xi + 1)^2 (\xi - 3)^2)/(4(\xi^4 - 2\xi^2 + 9)(\eta^4 - 2\eta^2 + 9)) ; \tag{11}$$

$$\varphi_4(\xi,\eta) = ((\xi - 1)^2 (\xi + 3)^2 (\eta + 1)^2 (\eta - 3)^2)/(4(\xi^4 - 2\xi^2 + 9)(\eta^4 - 2\eta^2 + 9)) ; \tag{12}$$

The function of Eq. (9)-(12) over the natural domain Ω_e of finite element, are shown in the Fig.1c.

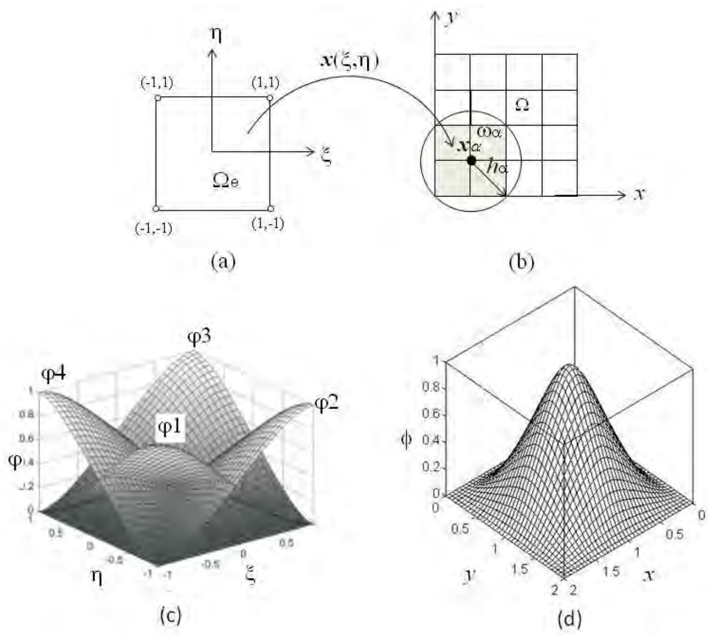

Figure 1. PU global function $\phi_{\alpha}(x,y)$ obtained by geometric mapping $\Omega_e \rightarrow \omega_{\alpha}$. Fig. 1(a) and (b): Mapping defined by Eq.(13)-(14). Fig. 2(c) and (d) PU functions defined by Eq. (9)-(12).

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The $\phi_\alpha(x,y)$ is the global function of PU obtained by geometric mapping to $\Omega_e \to \omega_\alpha$, see Fig.1(a)-(b), defined by Eq.(13)-(14) of the PU functions defined by Eq. (9)-(12).

$$x(\xi,\eta) = \sum_{i=1}^{4} x_i N_i(\xi,\eta); \tag{13}$$

$$y(\xi,\eta) = \sum_{i=1}^{4} y_i N_i(\xi,\eta); \tag{14}$$

In Eq. (13)-(14) $N_i(\xi,\eta)$ are the shape functions of the bilinear element.

2.2 Local Approximation Space Q_α^p

The local approximation space of order *p* associated with the α PU is defined by

$$Q_\alpha^p = span\left[\{\rho_{k\alpha}\}_{k=1}^{p}\right]; \tag{15}$$

where $\rho_{k\alpha}$ are the complete monomials, defined in the Pascal triangle, of order *k* with origin set at the α^{th} node of the mesh.

2.3 Enriched approximation space $\Im_N^p$

Let $\{\phi_i\}_{i=1}^{N}$ be a Partition of Unity subordinated to an open covering $\{\omega_\alpha\}_{\alpha=1}^{N}$, then, the global approximation space of order *p* is defined as,

$$\Im_N^p = span\left[\left\{\phi_\alpha Q_\alpha^p\right\}_{\alpha=1}^{N}\right]; \tag{16}$$

where for $p=2$ one has

$$Q_\alpha^{p=2} = \left\{1, \bar{x}, \bar{y}, \bar{x}^2, \overline{xy}, \bar{y}^2\right\}. \tag{17}$$

In Eq. (17) $\bar{x}, \bar{y}, \ldots$ are the normalized coordinates value defined by

$$\begin{cases} \bar{x} = \dfrac{x - x_\alpha}{h_\alpha} \\ \bar{y} = \dfrac{y - y_\alpha}{h_\alpha} \end{cases}. \tag{18}$$

The (x,y) is the point overlapped for α^{th} particle (x_α, y_α), with h_α radius, see: Fig. 1(b).

For the uniform local enrichment of PU functions with $Q_\alpha^{p=2}$ the global space is such as

$$\Im_\alpha^{p=2} = \left\{\psi_1^\alpha \quad \psi_2^\alpha \quad \psi_3^\alpha \quad \psi_4^\alpha \quad \psi_5^\alpha \quad \psi_6^\alpha\right\} = \left\{\phi_\alpha \quad \phi_\alpha\bar{x} \quad \phi_\alpha\bar{y} \quad \phi_\alpha\bar{x}^2 \quad \phi_\alpha\overline{xy} \quad \phi_\alpha\bar{y}^2\right\}. \tag{19}$$

One feature of the spaces $\Im_N^p$ constructed with the PU defined above is that the functions of the local approximation spaces are weakly ill-conditioned. This feature result in non-singular mass and stiffness matrices, see Garcia and Rossi (2012).

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3. DYNAMIC APPROACH OF 2D ELASTIC PROBLEMS

In this section we approach the free vibrations/modes and the forced vibration problem of 2D elastic plane problems using the Galerkin discrete and semi-discrete formulation.

3.1 Natural frequencies and modes

The elliptic eigenvalue/eigenvector problems in solid mechanics results in the natural frequencies and modes of the structural component. The Bubnov-Galerkin method applied for the free vibration problem can be stated as: Find $\boldsymbol{U}_i \in \mathbb{R}^{2n}$, $\boldsymbol{U}_i \neq \boldsymbol{0}$ and $\omega_i \in \mathbb{R}$, $0 < \omega_i \leq \omega_{i+1}$, $i = 1,...,n$, such that,

$$\left(\int_{\Omega} \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} d\Omega - \omega_i^2 \int_{\Omega} \rho \boldsymbol{N}^T \boldsymbol{N} d\Omega \right) \boldsymbol{U}_i = \boldsymbol{0}. \tag{20}$$

In Eq.(20), $\boldsymbol{D}$ is the constitutive matrix for 2D elastostatic problem, and $\boldsymbol{N}(\xi,\eta)$, $\boldsymbol{B}(\xi,\eta)$ and $\boldsymbol{U}_i$ are the shape function matrix, deformation matrix and the vector displacement parameters, defined by Eq.(21)-(23) as follow:

$$\boldsymbol{N}(\xi,\eta) = \begin{bmatrix} \psi_1^1(\xi,\eta) & 0 & \cdots & \psi_\alpha^i(\xi,\eta) & 0 & \cdots & \psi_N^p(\xi,\eta) & 0 \\ 0 & \psi_1^1(\xi,\eta) & \cdots & 0 & \psi_\alpha^i(\xi,\eta) & \cdots & 0 & \psi_N^p(\xi,\eta) \end{bmatrix}. \tag{21}$$

$$\boldsymbol{B} = \boldsymbol{HJ}\partial\boldsymbol{N}; \tag{22}$$

$$\boldsymbol{U}_i^T = \left\{ u_x^1 \quad u_y^1 \quad \cdots \quad u_x^n \quad u_y^n \right\}; \tag{23}$$

In Eq. (21), $i = 1,..,p$ and $\alpha = 1,...,N$. For homogeneous "*p*" enrichment the number of degree of freedom is $ndof = 2n$, and $n = pN$. In Eq. (22), ∂, $\boldsymbol{J}$ and $\boldsymbol{H}$ are the differential operator, de Jacobian operators and the Boolean matrix respectively. They are defined by Eq. (24)-(26)

$$\partial = \begin{bmatrix} \dfrac{\partial(\cdot)}{\partial\xi} & 0 \\ \dfrac{\partial(\cdot)}{\partial\eta} & 0 \\ 0 & \dfrac{\partial(\cdot)}{\partial\xi} \\ 0 & \dfrac{\partial(\cdot)}{\partial\eta} \end{bmatrix}; \tag{24}$$

$$\boldsymbol{J} = \begin{bmatrix} J^{-1} & 0 \\ 0 & J^{-1} \end{bmatrix}; \tag{25}$$

In Eq. (25), $\boldsymbol{J}$ is the Jacobian matrix. The Eq. (20) is frequently written as

$$\left(\boldsymbol{K} - \omega_i^2 \boldsymbol{M} \right) \boldsymbol{U}_i. \tag{26}$$

In Eq. (26) one has

$$\boldsymbol{K} = \int_{\Omega} \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} d\Omega \tag{27}$$

and

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$$\boldsymbol{M} = \int_{\Omega} \rho \boldsymbol{N}^T \boldsymbol{N} d\Omega . \tag{28}$$

The Eq. (28) and (29) define the Stiffness and Mass matrices respectively.

3.2 Forced vibration problem

To approach the forced vibration problem is employed the Galerkin semi-discrete formulation. The variational formulation for undamped forced vibration problem, shown in the Fig. 2, for plane elasticity is defined as: Find $\boldsymbol{U}(x,y,t)$ such that

$$\int_{\Omega} \rho \boldsymbol{N}^T \boldsymbol{N} d\Omega \ddot{\boldsymbol{U}} + \int_{\Omega} \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} d\Omega \boldsymbol{U} - \int_{\Gamma_N} \boldsymbol{N}^T \boldsymbol{q}(t) d\Gamma = 0 ; \tag{29}$$

The Eq. (29) is frequently mentioned in the matrix form of dynamic equilibrium equation as

$$\boldsymbol{M}\ddot{\boldsymbol{U}} + \boldsymbol{K}\boldsymbol{U} = \boldsymbol{F} ; \tag{30}$$

In Eq. (30), $\boldsymbol{M}$ and $\boldsymbol{K}$ are de Mass and the Stiffness matrix shown in Eq. (27)-(28) and $\boldsymbol{F}$ is a time-dependent force vector defined by

$$\boldsymbol{F} = \int_{\Gamma_N} \boldsymbol{N}^T \boldsymbol{q}(t) d\Gamma ; \tag{31}$$

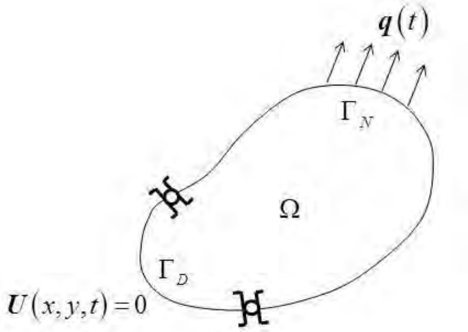

Figure 2. Fixed body at the border Γ_D and excited by $\boldsymbol{q}(t)$ force at the border Γ_N .

In this work the time integration of Eq. (30) is performed using the Newmark Method together with the Modal Decomposition Method. When using the implicit Newmark Method the equilibrium equation Eq. (30) is described in the time $t + \Delta t$, as follows

$$\boldsymbol{M}\ddot{\boldsymbol{U}}(t+\Delta t) + \boldsymbol{K}\boldsymbol{U}(t+\Delta t) = \boldsymbol{F}(t+\Delta t) . \tag{32}$$

In this methodology the vectors $\ddot{\boldsymbol{U}}(t+\Delta t)$ and $\boldsymbol{U}(t+\Delta t)$ are obtained from the vectors $\ddot{\boldsymbol{U}}(t)$, $\boldsymbol{U}(t)$ as

$$\dot{\boldsymbol{U}}(t+\Delta t) = \dot{\boldsymbol{U}}(t) + \left[(1-\delta)\ddot{\boldsymbol{U}}(t) + \delta\ddot{\boldsymbol{U}}(t+\Delta t)\right]\Delta t ; \tag{33}$$

$$\boldsymbol{U}(t+\Delta t) = \boldsymbol{U}(t) + \dot{\boldsymbol{U}}(t)\Delta t + \left[\left(\frac{1}{2} - \alpha\right)\ddot{\boldsymbol{U}}(t) + \alpha\ddot{\boldsymbol{U}}(t+\Delta t)\right]\Delta t^2 . \tag{34}$$

The variable $\boldsymbol{U}(t+\Delta t)$ is obtained from Eq. (32)-(34) as follow

$$\bar{\boldsymbol{K}}\boldsymbol{U}(t+\Delta t) = \bar{\boldsymbol{F}}(t+\Delta t) . \tag{35}$$

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For $\delta = 0.5$ and $\alpha = 0.25$, the time integration in the Newmark Method is unconditional stable, that is, independent of the Δt magnitude. If the problem of eigenvalue/eigenvector defined in Eq. (30) is symmetric then, some advantages can be obtained by the diagonalization of the linear system in Eq. (30) using modal decomposition. The modal decomposition consist in write the Eq. (30) in the modal space as follow

$$\boldsymbol{U}(t) = \Phi \boldsymbol{X}(t); \tag{36}$$

$$\ddot{\boldsymbol{U}}(t) = \Phi \ddot{\boldsymbol{X}}(t). \tag{37}$$

Substituting Eq. (36)-(37) into Eq. (30) and multiplying both sides by $\boldsymbol{\Phi}^T$ result in

$$\boldsymbol{\Phi}^T \boldsymbol{M} \boldsymbol{\Phi} \ddot{\boldsymbol{X}}(t) + \boldsymbol{\Phi}^T \boldsymbol{K} \boldsymbol{\Phi} \boldsymbol{X}(t) = \boldsymbol{\Phi}^T \boldsymbol{F}(t). \tag{38}$$

In Eq. (38), $\boldsymbol{\Phi}$ is the mass-orthonormal eigenvectors matrix of Eq. (26). Considering mass-orthonormal eigenvectors Eq. (38) can be written as

$$\boldsymbol{I}\ddot{\boldsymbol{X}}(t) + \Lambda \boldsymbol{X}(t) = \tilde{\boldsymbol{F}}(t). \tag{39}$$

In Eq.(39) one has

$$\boldsymbol{\Phi}^T \boldsymbol{M} \boldsymbol{\Phi} = \boldsymbol{I} \tag{40}$$

$$\boldsymbol{\Phi}^T \boldsymbol{K} \boldsymbol{\Phi} = \Lambda \tag{41}$$

$$\boldsymbol{\Phi}^T \boldsymbol{F} = \tilde{\boldsymbol{F}} \tag{42}$$

Equation (39) describes Eq. (30) in the modal space, where $\boldsymbol{I}$ is the identity matrix and Λ is the diagonal matrix of eigenvalues. In the next section will show the results of free and forced vibration problems for plane elasticity models.

4. NUMERICAL RESULTS

The numerical result aims to show the performance of the high regularity GFEM and the high order FEM to approach the undamped free vibrations and the forced vibrations problems for the plane elasticity problem. The results for all case studied relate to a beam with geometric properties, mechanical and contour conditions shown in Fig.3.

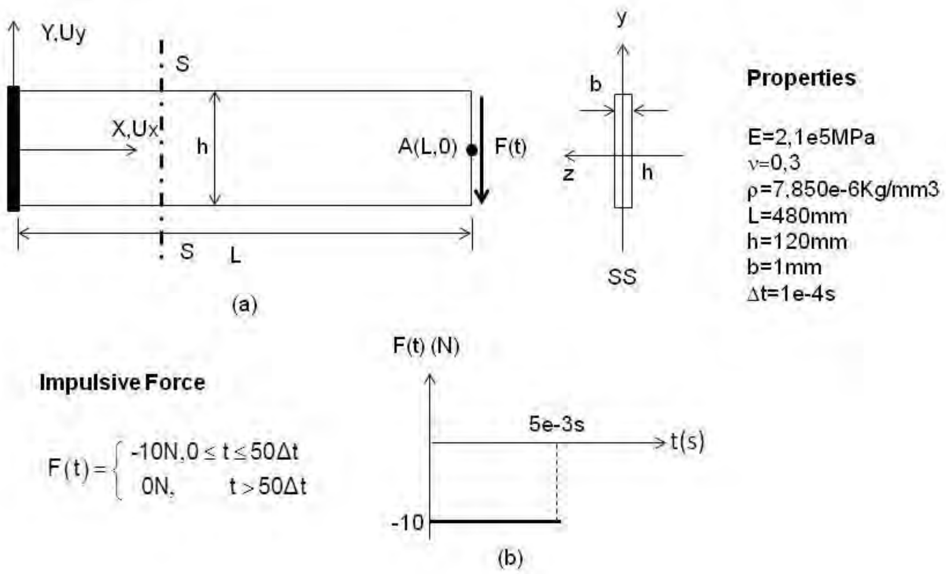

Figure 3: Beam model: a) Elastic beam; b) impulsive force.

4.1 Free vibrations

The results of this study are obtained using the relative error, see Eq. (44), related to reference solution for the first ω_{140} natural frequencies. The reference solution is built using the rule of thumb proposed by Dan Givoli (2008). This solution is obtained using 20x1 forth order Lagrangian finite element (25 nodes element), what results in an approximation with 800 degrees of freedom.

The rule presented by Dan Givoli, 2009, defined in Eq. (43), is obtained for a pre-established error of $\varepsilon = 0,001$, and provides the first $M \cong 142$ natural frequencies with error bellow the pre-established error.

$$M = r\varepsilon^{\frac{1}{p}} N; \tag{43}$$

$$E_r = \frac{|\omega_h - \omega_r|}{|\omega_r|}; \tag{44}$$

In Eq. (43) $r = 1$, $N = 800$, $p = 4$ resulting in $M \cong 142$. In Eq. (44) E_r is the relative error, ω_h the natural frequency obtained by the proposed strategies and ω_r the natural frequency obtained by the reference solution.

Figure 4 present the results on relative error and are obtained for the following strategies:

A. Numerical model built using FEM with a uniform mesh 20x1 Lagrangian quadrilateral quadratic element (9 nodes), approximating the problem with 240 degree of freedom;

B. Numerical model obtained by GFEM with 11x2 particles and approximation space obtained by explicit and homogeneous enrichment of PU ($C^2(\Omega)$) by polynomial function with order p = 2. The numerical model approaches the problem with 254 degree of freedom;

C. Numerical model obtained by GFEM with 11x2 particles and approximation space obtained by explicit and homogeneous enrichment of PU($C^4(\Omega)$) by polynomial function with order p = 2. The numerical model approaches the problem with 254 degree of freedom.

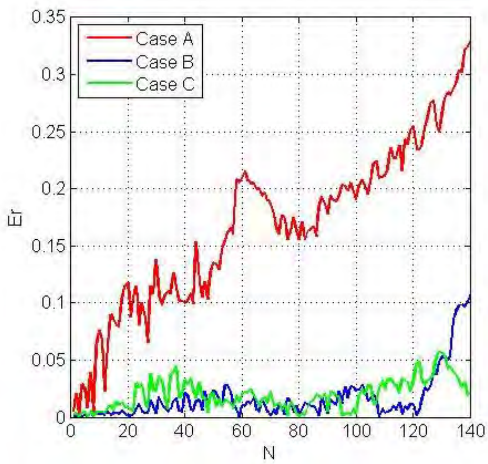

Figure 4: Relative error of natural frequencies.

The results in Fig.4 show the excellent performance of the approximation spaces obtained by Case B-C resulting in relative errors below five percent for almost all 140 natural frequencies. In contrast, Case A shows a significant increase in the relative error being more than thirty percent of error for the natural frequency ω_{140}.

These results confirm those obtained by Cottrell *et al.*(2007a, 2007b) for the natural frequency problem for a rod problem using the *k*-method and those obtained by Garcia and Rossi (2012) using the high regularity GFEM in the determination of natural frequencies associated with axisymmetric modes of plates and shells of revolution.

In the present work, as in the works cited in the last paragraph, there is a significant influence of the regularity of the approximation spaces in obtaining accurate frequencies of high-order (up to ten percent of the frequencies

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approximate by the numerical model). This fact has a direct influence on the accuracy of the impulse response as will be shown in the next case studied.

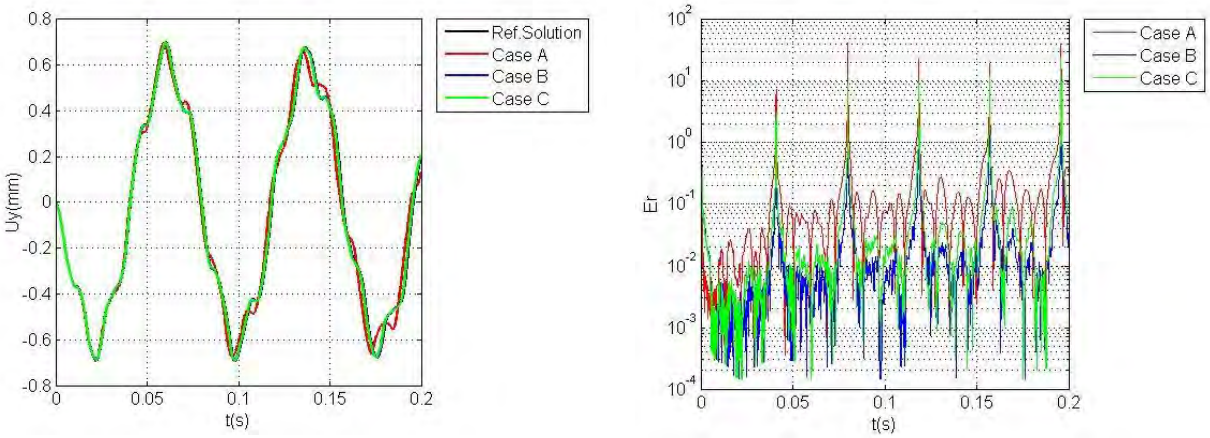
4.2 Forced vibrations

The following results approximate the displacement at point A, using the Newmark method with modal decomposition, due to an impulsive force as shown in Fig. 1(b). A time-step $\Delta t = 1\times10^{-4}\,s$ is employed and observed time is $t\in[0,0.2]s$.

The results are for the strategies mentioned above, where we record the $U_y(t)$ in the analyzed time interval, see Fig. 5(a), and the relative error, see Fig. 5(b), given by Eq. (45).

$$E_r = \frac{\left|U_y^h - U_y^r\right|}{\left|U_y^r\right|} \tag{45}$$

In Eq. (45), U_y^h is the displacement component obtained using the strategies A to C and U_y^r is the displacement component obtained using the reference solution.



(a) (b)

Figure 5: Numerical results: a) U_y in A for $t=[0,2s]$; b) E_r in A for $t=[0,2s]$.

The results observed in Fig. 5(a) shows a behavior very close to the reference solution for the strategies B and C, while for the strategy A a deviation is presented. In this example it is obvious the influence of the high regularity of the approximation spaces used in strategies B and C causing improved accuracy for relatively high order frequencies (more than ten percent of the modes and frequencies numerically approximated).

The low regularity effect of the approximation space used in strategy A has impacts in the capture of impulsive response, as shown in Fig. 5(a)-(b). The main cause for the low precision results obtained in strategy A lies in a poor estimation of the eigenvalues, due to approximation space features, what results in an inaccurate modal decomposition. Notice that, in such cases, a few number of eigenvalues are accurately obtained resulting in an imprecise or deviated transformation matrix $\boldsymbol{\Phi}$ being the procedure presented in Eq.(36)-(42) fatally compromised.

5. CONCLUSION

In this work it is numerically evidenced the significant influence of the regularity of the approximation spaces to obtain accuracy in natural frequencies and modes in undamped free and forced vibration problems. Although not yet published, the authors of this work have been found very accurate results for relatively high frequencies in thick plates problems modeled by high regularity GFEM. This fact points to a little explored research topic but with implications that may be relevant in the structural dynamics area. The results observed for the problems of free and forced vibration explored in this paper confirm those found by Cottrell *et. al.* (2007a, 2007b) and Garcia and Rossi (2010-2012) among others.

6. ACKNOWLEDGEMENTS

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8. RESPONSIBILITY NOTICE

The authors Rudimar Mazzochi, Oscar Alfredo Garcia de Suarez, Rodrigo Rossi and João Morais da Silva Neto are the only responsible for the printed material included in this paper.