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MODELING THE TRANSITION SATURATED-UNSATURATED IN FLOWS THROUGH RIGID POROUS MEDIA

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Abstract. *This work proposes a mathematical model to study the filling up of an unsaturated rigid porous medium by a liquid identifying the transition from unsaturated to saturated flow, considering a new relationship between pressure and fluid fraction. The problem is modeled under a mixture theory approach – the mixture consisting of three overlapping continuous constituents, representing the porous matrix (solid constituent), the incompressible fluid (liquid constituent) and an inert gas constituent included to account for its compressibility. This problem is mathematically represented by a nonlinear hyperbolic system. In order to be physically realistic, the fluid fraction must satisfy an inequality – an upper bound. The model presented in this work takes into account this upper bound to be satisfied by the fluid fraction (and the saturation) that depends on the volume of the pores. The complete solution of a Riemann problem associated to the system of conservation laws satisfying the constraint given by the saturation upper bound is presented.*

Keywords: *Flow through unsaturated porous media, transition saturated-unsaturated, constrained unknown, Riemann problem.*

1. INTRODUCTION)

Although the study of transport in porous media dates from the 1920s, an adequate description of the transition from saturated to unsaturated flows through porous media remains an open subject. The relevance of constrained hyperbolic systems may be noted by its presence in different applications, such as two-phase flows, compressible plasticity with shocks and description of traffic of vehicles or crowds, including traffic jams, presence of tollgates and prediction of traffic accidents on roads. (See Saldanha da Gama et al., 2012 and references therein.)

Saldanha da Gama (1986) proposed a constitutive relation accounting for a geometrical bound, which arises from the rigidity of the porous matrix and the incompressibility of the fluid, avoiding solutions without physical meaning. These solutions would correspond to an infinite value for the pressure in saturated flows – when the fluid fraction tends to the porosity.

Saldanha da Gama et al. (2012) imposed mathematically a unilateral geometrical constraint, ensuring that the maximum value that the fluid fraction can reach is the porosity value. This imposes an intermediate state arising from a geometrical restriction. So in the work of Saldanha da Gama et al. (2012) the porous medium can actually be saturated by the fluid, while the equation proposed by Saldanha da Gama (1986) imposes a physical behavior for the fluid preventing it to saturate the porous matrix.

Martins-Costa and Saldanha da Gama (2011) proposed an improvement to the constitutive relation for the partial pressure (1986) in which the unilateral geometrical constraint for the fluid fraction, instead of being assumed in the whole domain, is considered only in a convenient neighborhood of the porosity (provided that the fluid fraction is smaller than the porosity), besides assuring continuity for the pressure and for its first derivative, thus allowing the analytical computation of the Riemann invariants associated to the problem.

This work may be viewed as a sequence of the work of Saldanha da Gama et al. (2012). Although the same principles are followed, a new relationship between pressure and fluid fraction is considered, modifying the problem.

In short, this work uses a physically realistic mathematical model to represent the filling up of an unsaturated rigid porous matrix by an incompressible fluid, identifying the transition from unsaturated to saturated flow. The mechanical model employs a mixture theory approach (Atkin and Craine, 1976; Rajagopal and Tao, 1995) – a convenient method for modeling multicomponent systems – supported by a local theory with thermodynamic consistency. The unsaturated porous medium is modeled as a mixture of three overlapping continuous constituents: a solid (a rigid, homogeneous and isotropic porous matrix), a liquid (an incompressible fluid) and an inert gas, assumed with very low mass density; which is included to account for the compressibility of the system as a whole.

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The problem mathematical description consists of a nonlinear hyperbolic system of two partial differential equations representing the mass and momentum conservation for the liquid constituent, since the solid constituent is supposed rigid and at rest and the gas constituent is assumed to be inert and to have a very low density.

The transition from unsaturated to saturated flow takes place when the upper bound for the fluid fraction is reached. This upper bound represents a physical constraint that ensures that the obtained generalized solution for the non-linear hyperbolic system is within the physically realistic range, regardless the considered initial data. The complete solution of the associated Riemann problem accounting for the physical constraint is presented and some representative problems are simulated allowing comparing the behavior of the generalized solution whether or not subjected to the imposed restriction.

2. MECHANICAL MODEL

Since the chemically non reacting mixture consists of a rigid solid constituent at rest, a liquid constituent – from now on denoted as fluid constituent and an inert gas, playing the role of the third constituent, the mechanical model for an isothermal flow is obtained by combining suitable constitutive assumptions with the following mass and momentum balance equations for the fluid constituent (Atkin and Craine, 1976; Rajagopal and Tao, 1995)

$$\begin{aligned} &\frac{\partial \rho_F}{\partial t} + \nabla \cdot (\rho_F \mathbf{v}_F) = 0 \\ &\rho_F \left[\frac{\partial \mathbf{v}_F}{\partial t} + (\nabla \mathbf{v}_F)\mathbf{v}_F \right] = \nabla \cdot \mathbf{T}_F + \mathbf{m}_F + \rho_F \mathbf{b}_F \end{aligned} \tag{1}$$

where ρ_F stands for the fluid constituent mass density – representing the local ratio between the fluid constituent mass and the corresponding volume of mixture, $\mathbf{v}_F$ is the fluid constituent velocity in the mixture, $\mathbf{T}_F$ represents the partial stress tensor – analogous to Cauchy stress tensor in Continuum Mechanics – associated with the fluid constituent, $\mathbf{b}_F$ stands for the body force (per unit mass) and $\mathbf{m}_F$ for the momentum supply acting on the fluid constituent due to its interaction with the remaining constituents of the mixture. The ratio between the fluid fraction ϕ and the porous matrix porosity ε is defined as the saturation ψ, so that

$$\psi = \frac{\phi}{\varepsilon} = \frac{\rho_F}{\varepsilon \rho_f} \qquad 0 < \psi \le 1 \quad \text{everywhere} \tag{2}$$

in which ρ_f is the actual mass density of the fluid – regarded from a Continuum Mechanics viewpoint, in contrast to ρ_F defined as the fluid constituent mass density.

Constitutive relations are now presented for the partial stress tensor associated with the fluid constituent and for the momentum supply acting on the fluid constituent. The former is modeled under the simplifying assumption proposed by Allen (see Saldanha da Gama et al., 2012 and references therein) considering the normal fluid stresses dominant over shear stresses and interphase tractions. The momentum source $\mathbf{m}_F$ is given by a term related to the fluid constituent velocity, usually called Darcian term and a term related to the saturation gradient. (Martins-Costa and Saldanha da Gama (2001) presented a detailed discussion on this issue.)

$$\begin{aligned} &\mathbf{m}_F = -\frac{\mu_f}{K}\phi^2 \mathbf{v}_F - \frac{\mu_f \mathrm{D}}{K} \nabla \phi \\ &\mathbf{T}_F = -\phi \bar{p}\, \mathbf{I} \end{aligned} \tag{3}$$

where μ_f represents the fluid viscosity, K the porous matrix specific permeability (both measured considering a Continuum Mechanics viewpoint), D a diffusion coefficient (analogous to the usual mass diffusion coefficient), $\bar{p}$ is a pressure (assumed constant while the flow is unsaturated) and $\mathbf{I}$ is the identity tensor. Since unsaturated flows through porous media are characterized by a strong dependence of the motion on the saturation, another simplifying assumption is considered in this work: the first term of the momentum source $\mathbf{m}_F$, the Darcian term, is neglected.

The mechanical model is obtained combining the balance equations (1) with the constitutive relations (3). Assuming all the quantities depending only on the time t and on the position x, absence of body forces effect and that v is the only non-vanishing component of the fluid constituent velocity $\mathbf{v}_F$, the problem may be represented by

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$$\begin{aligned} &\frac{\partial \phi}{\partial t}+\frac{\partial}{\partial x}(\phi v)=0 \\ &\rho_f\left[\phi \frac{\partial v}{\partial t}+\phi v \frac{\partial v}{\partial x}\right]=-\frac{\partial}{\partial x}(\phi \bar{p})-\frac{D\mu_f}{K}\frac{\partial \phi}{\partial x} \end{aligned} \tag{4}$$

The nonlinear system presented in equation (4) may be rewritten in a more convenient form by redefining a pressure $p=\hat{p}(\phi)$ as $p=\bar{p}\phi+(\mu_f D/K)\phi$, so that the following nonlinear hyperbolic system represents mathematically a mixture theory description of a one dimensional flow of a fluid through an unsaturated rigid porous matrix at rest

$$\begin{aligned} &\frac{\partial \phi}{\partial t}+\frac{\partial}{\partial x}(\phi v)=0 \\ &\frac{\partial}{\partial t}(\phi v)+\frac{\partial}{\partial x}\left(p+\phi v^2\right)=0 \end{aligned} \tag{5}$$

This hyperbolic system may not admit continuous solutions, requiring an enlargement of the space of admissible functions in order to admit generalized (discontinuous) solutions. The system presented in equation (5) may also represent other relevant Engineering problems such as the dynamical behavior of an elastic rod in the current configuration [20] or the dynamics of an ideal gas [21].

At this point it should be noticed that the unknowns ϕ and v, represent, respectively, the fluid fraction and the fluid constituent velocity in the description of the flow through an unsaturated porous medium presented in equation (5), and the function p is a ratio between pressure and density, from now on simply referred to as "pressure". Both ϕ and v depend on the position x and on the time t. While the velocity can assume any real value, the fluid fraction ϕ must be positive valued and smaller than (or equal to) the porosity ε, in order to be physically meaningful. In other words,

$$0<\phi \leq \varepsilon, \quad \forall\, t>0, \forall\, x \tag{6}$$

The inequality presented in equation (6) clearly shows that the volume of the fluid cannot exceed the volume of the pores and the positiveness assures the existence of fluid in the pores.

Since a rigid homogeneous porous matrix is considered, the pressure p is assumed as a linear increasing function of the fluid fraction ϕ, provided that the porous medium is not saturated. However, the (physical) upper bound for the fluid fraction must be imposed during any simulation, in order to avoid a physically inadmissible situation – namely to allow the fluid fraction to be greater than the porosity. This constraint also gives rise to an adequate description of the transition from unsaturated flow ($\phi<\varepsilon$) to saturated flow ($\phi=\varepsilon$). The constraint $\phi \leq \varepsilon$ must be verified for all position and time, otherwise, depending on the initial data, the results may present regions without physical meaning – characterized by $\phi>\varepsilon$. Some of these cases are illustrated in this work.

It is important to observe that initial data may be conveniently chosen in order to automatically ensure the inequality (6) (see, for instance [19, 22]). On the other hand, some initial data may give rise to mathematical descriptions without physical meaning (in which Eq. (6) is not always satisfied), except when the constraint is imposed during the simulation. This is the case of the saturation process that cannot be simulated without employing the constraint $\phi \leq \varepsilon$.

This work main subject is to adequately model the one-dimensional flow of a fluid through a rigid porous medium, with uniform porosity ε, which is mathematically represented by the following system

$$\left.\begin{aligned} \frac{\partial \phi}{\partial t}+\frac{\partial}{\partial x}(\phi v)=0 \\ \frac{\partial}{\partial t}(\phi v)+\frac{\partial}{\partial x}\left(p+\phi v^2\right)=0 \end{aligned}\right\} \text{ with } \left\langle\begin{aligned} &0<\phi \leq \varepsilon \\ &\forall t>0,\ \forall x \end{aligned}\right. \tag{7}$$

2.1 Relationship between pressure and fluid fraction

In order to clearly identify the transition from unsaturated to saturated flow, the relationship between pressure and fluid fraction is now analyzed. As stated before, the unsaturated flow of a liquid through a rigid porous medium

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may be regarded as a mixture of three overlapping continuous constituents: a liquid (incompressible fluid), a solid (porous matrix) and a very low mass density gas (providing the compressibility of the system). The existence of this gas can be assumed while the fluid (liquid) fraction ϕ, is less than the porosity ε – in other words while the flow is unsaturated. In such cases it is possible to assume that the (partial) pressure p is a constitutive function of the fluid fraction. In this work, assuming that c is a positive constant, the following relation is assumed

$$p = \frac{1}{3} c^2 \phi^3 \quad \text{provided} \quad 0 \le \phi < \varepsilon \tag{8}$$

It worth noting that the above constitutive relation is distinct to the relation employed by Saldanha da Gama et al. (2012), characterizing a distinct problem. Any constitutive choice for the pressure p only makes sense for ϕ within the open interval $(0,\varepsilon)$. In fact, p cannot be evaluated from (8) when $\phi = \varepsilon$ – in other words, when the flow is saturated, since there exists a geometrical bound (rigid porous medium) that allows a pressure increasing with a fixed fluid fraction ϕ. For a rigid and homogeneous porous medium, the following must hold

$$\begin{array}{lll} p = \hat{p}(\phi) & \text{for} \quad 0 < \phi < \varepsilon & \to \text{unsaturated flow} \\ \hat{p}(\varepsilon) \le p < \infty & \text{for} \quad \phi = \varepsilon & \to \text{saturated flow} \end{array} \tag{9}$$

It should be remarked that relation (9) must be taken into account in order to adequately perform the the simulation of transition from unsaturated flow to saturated flow.

3. THE RIEMANN PROBLEM ASSOCIATED TO SYSTEM (5) – UNCONSTRAINED

The Riemann problem associated to system (5) is built by assuming a step function as its initial data.

The solution (in a generalized sense) of this Riemann problem depends only on the ratio x/t being obtained by connecting the left state (ϕ_L, v_L) and the right state (ϕ_R, v_R) to an intermediate state (ϕ_*, v_*) by means of rarefactions and shocks (Smoller, 1983; Toro, 1999).

The two eigenvalues of system (5) are given, in increasing order, by $\lambda_1 = \hat{\lambda}_1(\phi, v) = v - \sqrt{p'} = v - c\phi$ and $\lambda_2 = \hat{\lambda}_2(\phi, v) = v + \sqrt{p'} = v + c\phi$, where p' represents the first derivative of p with respect to ϕ, given by $c^2\phi^2$.

The left state is connected to the intermediate state by a 1-rarefaction if, and only if, between these two states, $\lambda_1 = x/t$. Analogously, the right state is connected to the intermediate state by a 2-rarefaction if, and only if, between these two states, $\lambda_2 = x/t$.

When two states are connected by a discontinuity, they must satisfy the Rankine-Hugoniot jump conditions (Smoller, 1983; Godlewski and Raviart, 1991, Dafermos, 2010) given by

$$\frac{[\phi v]}{[\phi]} = \frac{[\phi v^2 + p]}{[\phi v]} = s \tag{10}$$

where "[ ]" denotes the jump and s denotes the shock (discontinuity) speed. Also, the entropy conditions must be satisfied – ensuring that the states cannot be connected by a rarefaction (Smoller, 1983; Toro, 1999).

The states (ϕ_L, v_L) and (ϕ_*, v_*) are connected by a 1-shock while the states (ϕ_*, v_*) and (ϕ_R, v_R) are connected by a 2-shock, if they satisfy the jump conditions (10) and the entropy conditions given, respectively, by

$$\left.\begin{array}{l} s_1 < \hat{\lambda}_1(\phi_L, v_L) \\ \hat{\lambda}_1(\phi_*, v_*) < s_1 < \hat{\lambda}_2(\phi_*, v_*) \end{array}\right\} \text{for 1-shock}$$
$$\text{and} \quad \left.\begin{array}{l} s_2 > \hat{\lambda}_2(\phi_R, v_R) \\ \hat{\lambda}_2(\phi_*, v_*) > s_2 > \hat{\lambda}_1(\phi_*, v_*) \end{array}\right\} \text{for 2-shock} \tag{11}$$

So, if $c|\phi_L - \phi_R| < v_R - v_L$, the solution is 1-rarefaction/2-rarefaction, with the intermediate state (ϕ_*, v_*) given by

$$v_* = c\left(\frac{\phi_L - \phi_R}{2}\right) + \frac{v_L + v_R}{2}, \qquad \phi_* = \frac{\phi_L + \phi_R}{2} + \frac{v_L - v_R}{2c} \tag{12}$$

For a solution 1-shock/2-shock, it is sufficient that $-\left|\phi_R - \phi_L\right| \sqrt{\frac{c^2\phi_L^2}{3\phi_R\left(\phi_L - \phi_R\right)} - \frac{c^2\phi_R^2}{3\phi_L\left(\phi_L - \phi_R\right)}} < v_L - v_R$, being the intermediate state obtained from

$$\begin{aligned} v_* - v_L &= -\left(\phi_* - \phi_L\right)\sqrt{\left(\frac{c^2\phi_*^2 - c^2\phi_L^2}{3\left(\phi_* - \phi_L\right)}\right)\frac{1}{\phi_*\phi_L}} \\ v_R - v_* &= -\left(\phi_R - \phi_*\right)\sqrt{\left(\frac{c^2\phi_R^2 - c^2\phi_*^2}{3\left(\phi_R - \phi_*\right)}\right)\frac{1}{\phi_R\phi_*}} \end{aligned} \tag{13}$$

When the solution is neither 1-rarefaction/2-rarefaction nor 1-shock/2-shock, then it will be 1-rarefaction/2-shock (provided $\phi_L > \phi_R$) or 1-shock/2-rarefaction (provided $\phi_L < \phi_R$).

For a 1-rarefaction/2-shock solution, the intermediate state (ϕ_*, v_*) is obtained from

$$\begin{aligned} c\phi_L + v_L &= c\phi_* + v_* \\ v_R - v_* &= -\left(\phi_R - \phi_*\right)\sqrt{\left(\frac{c^2\phi_R^2 - c^2\phi_*^2}{3\left(\phi_R - \phi_*\right)}\right)\frac{1}{\phi_R\phi_*}} \end{aligned} \tag{14}$$

while for a 1-shock/2-rarefaction case, the intermediate state (ϕ_*, v_*) is obtained from

$$\begin{aligned} v_* - v_L &= -\left(\phi_* - \phi_L\right)\sqrt{\left(\frac{c^2\phi_*^2 - c^2\phi_L^2}{3\left(\phi_* - \phi_L\right)}\right)\frac{1}{\phi_*\phi_L}} \\ -c\phi_R + v_R &= -c\phi_* + v_* \end{aligned} \tag{15}$$

Table 1 states the conditions for the four possible Riemann problem solutions, provided that the intermediate state differs from the left and from the right ones.

It is remarkable that the inequality $\phi > 0$ is automatically satisfied for the solutions 1-shock/2-shock, 1-rarefaction/2-shock and 1-shock/2-rarefaction, since $\phi_* \geq MIN\left(\phi_L, \phi_R\right) > 0$. When the solution is 1-rarefaction/2-rarefaction, it comes that $\phi_* \leq MIN\left(\phi_L, \phi_R\right)$. Nevertheless, taking into account equation (12), the positiveness of the fluid fraction is ensured.

4. AN EXAMPLE – UNCONSTRAINED

Considering the flow through a rigid porous medium with constant porosity ε, fluid fraction ϕ and fluid constituent velocity *v*, the following associated Riemann problem may be stated

$$\begin{aligned} &\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x}(\phi v) = 0 \\ &\frac{\partial}{\partial t}(\phi v) + \frac{\partial}{\partial x}\left(\frac{c^2\phi^3}{3} + \phi v^2\right) = 0 \\ &(\phi, v) = \begin{cases} \left(\bar{\phi}, \bar{v}\right) & \text{for} \quad t = 0, \quad -\infty < x < 0 \\ \left(\bar{\phi}, -\bar{v}\right) & \text{for} \quad t = 0, \quad 0 < x < \infty \end{cases} \end{aligned} \tag{16}$$

where $\bar{\phi}$ and $\bar{v}$ are positive constants, such that $\bar{\phi} < \varepsilon$.

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In this case, it is easy to see that the solution (in a generalized sense) is 1-shock/2-shock. So,

$$(\phi, v) = \begin{cases} (\phi_L, v_L) & \text{if} \quad -\infty < x/t < s_1 \\ (\phi_*, v_*) & \text{if} \quad s_1 < x/t < s_2 \\ (\phi_R, v_R) & \text{if} \quad s_2 < x/t < \infty \end{cases} \tag{17}$$

in which the intermediate state, the pressure p and the shock speeds are obtained from

$$v_* = 0 \quad \text{and} \quad \phi_* \text{ is such that } \bar{v} = -(\bar{\phi} - \phi_*)\sqrt{\left(\frac{(c\bar{\phi})^2 - c^2\phi_*^2}{3(\bar{\phi} - \phi_*)}\right)\frac{1}{\bar{\phi}\phi_*}} \tag{18}$$

$$p = \begin{cases} c^2\bar{\phi} & \text{if} \quad -\infty < x/t < s_1 \\ \frac{1}{3}c^2\phi_*^3 & \text{if} \quad s_1 < x/t < s_2 \\ c^2\bar{\phi} & \text{if} \quad s_2 < x/t < \infty \end{cases} \tag{19}$$

$$s_2 = -s_1 = \frac{\bar{v}}{\left(\frac{\phi_*}{\bar{\phi}}\right) - 1} \tag{20}$$

The solution of the associated Riemann problem (16), given by equations (17)-(20), is physically realistic provided that the intermediate fluid fraction ϕ_* is always smaller than or equal to the porosity ε.

It is important to note that many initial data give rise results without physical meaning. However, if adequate initial data is chosen, assuring $\phi < \varepsilon$, the nonlinear hyperbolic system (7) may be approximated by Glimm's method, implemented by employing the solution of a certain number of associated Riemann problems (Saldanha da Gama and Martins-Costa, 1997; Martins-Costa and Saldanha da Gama, 2001).

5. THE RIEMANN PROBLEM AND THE CONSTRAINT $\phi \leq \varepsilon$

Table 1. Conditions for the four possible Riemann problem solutions.

CONDITION	Solution Type	Relation among States
$c\lvert\phi_L - \phi_R\rvert < v_R - v_L$	1-rarefaction/2-rarefaction	$\phi_L > \phi_* < \phi_R$
$-\lvert\phi_R - \phi_L\rvert\sqrt{\frac{c^2\phi_L^2}{3\phi_R(\phi_L - \phi_R)} - \frac{c^2\phi_R^2}{3\phi_L(\phi_L - \phi_R)}} < v_L - v_R$	1-shock/2-shock	$\phi_L < \phi_* > \phi_R$
$\phi_L > \phi_R$ and none of the above inequalities satisfied	1-rarefaction/2-shock	$\phi_L > \phi_* > \phi_R$
$\phi_L < \phi_R$ and none of the above inequalities satisfied	1-shock/2-rarefaction	$\phi_L < \phi_* < \phi_R$

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Since ϕ_L and ϕ_R satisfy the inequality $\phi \le \varepsilon$ then, taking into account the conditions presented in Table 1, this inequality is ensured everywhere, provided that the solution is 1-rarefaction/2-rarefaction ($\phi_L > \phi_* < \phi_R$), 1-rarefaction/2-shock ($\phi_L > \phi_* > \phi_R$) or 1-shock/2-rarefaction ($\phi_L < \phi_* < \phi_R$).

However, when the solution is 1-shock/2-shock, one may have $\phi > \varepsilon$, for x/t between s_1 and s_2, if p is considered as a function of ϕ. In such cases it must be taken into account that the pressure p for the saturated case ($\phi = \varepsilon$) is not constitutive. In other words, p may assume any value greater than (or equal to) $p = c^2\varepsilon^3/3$, provided the Rankine-Hugoniot jump conditions are satisfied as well as the entropy conditions.

So, if the root of equation (13) is such that $\phi_* > \varepsilon$ the physical meaning of the phenomenon has been lost. In these cases, the intermediate state must satisfy the jump conditions with $\phi_* = \varepsilon$. Hence p_* and v_* are evaluated from

$$\begin{aligned} \frac{\varepsilon v_* - \phi_L v_L}{\varepsilon - \phi_L} &= \frac{\varepsilon v_*^2 - \phi_L v_L^2 + p_* - c^2\phi_L}{\varepsilon v_* - \phi_L v_L} = s_1 \\ \frac{\phi_R v_R - \varepsilon v_*}{\phi_R - \varepsilon} &= \frac{\phi_R v_R^2 - \varepsilon v_*^2 + c^2\phi_R - p_*}{\phi_R v_R - \varepsilon v_*} = s_2 \end{aligned} \tag{21}$$

Thus, the intermediate pressure is obtained from the following equation

$$\sqrt{\left(p_* - \frac{1}{3}c^2\phi_L^3\right)\left(\frac{1}{\phi_L} - \frac{1}{\varepsilon}\right)} + \sqrt{\left(\frac{1}{3}c^2\phi_L^3 - p_*\right)\left(\frac{1}{\varepsilon} - \frac{1}{\phi_R}\right)} = v_L - v_R \tag{22}$$

Since

$$-\left|\phi_R - \phi_L\right| \sqrt{\frac{c^2\phi_L^2}{3\phi_R\left(\phi_L - \phi_R\right)} - \frac{c^2\phi_R^2}{3\phi_L\left(\phi_L - \phi_R\right)}} < v_L - v_R \tag{23}$$

and the intermediate state for the fluid fraction (ϕ_*) must satisfy the Rankine-Hugoniot condition and the constraint, the intermediate pressure must be such that $p_* \ge c^2\varepsilon^3/3$.

6. FINAL REMARKS

In this work a mathematical model for flows through unsaturated porous media, identifying the transition from unsaturated to saturated flow, was proposed by including a constraint that must be satisfied to build physically realistic generalized solutions for any initial data. The complete solution of a constrained nonlinear hyperbolic problem with shock waves – an associated Riemann problem containing a restriction (an upper bound for the fluid fraction, represented by the porosity), was presented as well as its application to flows through porous media.

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